Solving Corrupted Quadratic Equations, Provably

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Data science

New imaging/sensing modalities allow us to probe the nature in unprecedented manners:

but also with a lot of new (and exciting) challenges due to the unconventional manner these data are obtained.
Subspace retrieval using intensity measurements only

- We wish to estimate a subspace \( U \in \mathbb{R}^{n \times r} \) by interrogating it with vectors \( \{a_i\}_{i=1}^m \) and forming backprojections;

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\|U^T a_i\|^2 = a_i^T (UU^T) a_i, \quad i = 1, \ldots, m.
\]

Intensity measurements are much easier to implement by an energy detector for high-frequency and wide-band (THz) applications.
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- We only observe the intensity of the backprojections, namely,

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They are quadratic with respect to \( U \).
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Phase retrieval

How to recover structure of a sample from its diffraction pattern?

- In the important special case of \( r = 1 \), it becomes equivalent to phase retrieval*, namely, recover \( x \in \mathbb{R}^n / \mathbb{C}^n \) from

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where \( \mathcal{F} \) is Fourier transform,

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This has wide applications in X-ray crystallography, electron microscopy and coherent diffractive imaging, and leads to winning of Nobel prize (e.g. discovery of double helix structure).

Covariance sketching for streaming data

Multivariate streaming data: a new data snapshot $x_t \in \mathbb{C}^n / \mathbb{R}^n$ is generated by the sensor platform at each time $t$;
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- **high-dimensional**: the number of variables, $n$, is large;
- **real-time**: data processed “on the fly”;
- **decentralized**: data collected at decentralized locations;
- **resource-constrained**: cannot store and transmit all data;
Covariance sketching

**Observation:** Fortunately, inference requires only statistics of the data stream, not the stream itself; we can “sketch”/compress the data at the hope of directly recovering its statistics!
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**Approach:** distributed data sketching and aggregation to recover the covariance structure or principal components.

- access each data sample via quadratic (energy) sketches;
- aggregate the sketches into linear observations of the covariance matrix.
Quadratic sampling

How to sketch a high-dimensional data stream in order to recover its covariance matrix?

\[ z_t = |\langle a_t, x_t \rangle|^2, \]

- sketching complexity is linear in length of \( x_t \);

network traffic

hyperspectral imagery
Quadratic sampling

How to sketch a high-dimensional data stream in order to recover its covariance matrix?

- To meet resource constraints, we would like to sample in a single pass on the fly: a single quadratic sketch of $x_t$:

$$z_t = |\langle a_t, x_t \rangle|^2,$$

which reduces the dim. of each $x_t$ to merely a scalar.

- sketching complexity is linear in length of $x_t$;
Quadratic sampling for covariance sketching

- Consider a data stream possible distributively observed at \( m \) sensors, each with a sketching vector \( \mathbf{a}_i \in \mathbb{R}^n \), \( i = 1, \ldots, m \):

\[
\langle \mathbf{a}_i, \mathbf{x}_{\ell}^{(t)} \rangle^2 \quad \text{and} \quad y_{i,T} = \frac{1}{T} \sum_{t=1}^{T} \left| \langle \mathbf{a}_i, \mathbf{x}_{\ell}^{(t)} \rangle \right|^2 \rightarrow \mathbf{a}_i^T \mathbf{Xa}_i,
\]

where \( \mathbf{X} = \mathbb{E} [\mathbf{x}\mathbf{x}^T] \) is the covariance matrix.
Quadratic sampling for covariance sketching

- Consider a data stream possible distributively observed at $m$ sensors, each with a sketching vector $a_i \in \mathbb{R}^n$, $i = 1, \ldots, m$:

  \[
  x_1 \ x_2 \ x_3 \ \ldots \ldots \ \ldots \ldots \ x_t
  \]

- Sketch a substream indexed by $\{\ell_i^t\}_{t=1}^T$ with $|\langle a_i, x_{\ell_i^t} \rangle|^2$ and compute the average:

  \[
  y_{i,T} = \frac{1}{T} \sum_{t=1}^T \left| \langle a_i, x_{\ell_i^t} \rangle \right|^2 = a_i^T \left( \frac{1}{T} \sum_{t=1}^T x_{\ell_i^t} x_{\ell_i^t}^T \right) a_i
  \]

  \[
  \overset{T \to \infty}{\longrightarrow} a_i^T \mathbf{X} a_i,
  \]

  where $\mathbf{X} = \mathbb{E}[xx^T]$ is the covariance matrix.
Low-rank covariance estimation

- More generally, quadratic samplers produce the following:

\[ y_i = a_i^T X a_i + \eta_i, \quad i = 1, \ldots, m; \]

where \( \eta \) is an additive noise.

- linear in the covariance matrix \( X \)!
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  - **linear in the covariance matrix** \( X \)!

- **Low-rank covariance matrix**: Many high-dimensional data lie in a low-dimensional subspace, when a small number of components accounts for most of the variability in the data.

\[ X = U U^T = \begin{pmatrix} u_1 & \ldots & u_r \end{pmatrix} \]

- This yields the *subspace retrieval* problem.
Two sides of the same coin: We can recover

- either $X = UU^T \in \mathbb{R}^{n \times n}$ (when $r$ is possibly unknown) or
- the subspace $U \in \mathbb{R}^{n \times r}$ (when $r$ is known);

| measurements | $X$ | $U$
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Reconstruction?

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*We will discuss both convex (for reconstructing \( X \)) and nonconvex methods (for reconstructing \( U \)), possibly with additional corruptions in the measurements.*
Low-rank covariance estimation via convex relaxation

- We would like to seek the covariance matrix satisfying the observations with the minimal rank:

\[ \hat{X} = \arg\min_{M \succeq 0} \text{rank}(M) \quad \text{s.t.} \quad y_i = a_i^T M a_i, \ i = 1, \ldots, m. \]
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- However this is non-convex and NP-hard. Therefore, we replace it by a convex relaxation which is the trace minimization, over all PSD matrices compatible with the measurements:

\[ \hat{X} = \arg\min_{M \succeq 0} \text{Tr}(M) \quad \text{s.t.} \quad y_i = a_i^T M a_i, \ i = 1, \ldots, m. \]
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• Additionally, if \( X \) is Toeplitz, solve:

\[ \hat{X} = \arg\min_{M \succeq 0, \text{Toeplitz}} \text{Tr}(M) \quad \text{s.t.} \quad y_i = a_i^T M a_i, \ i = 1, \ldots, m. \]
Near-optimal recovery via convex programming

**Theorem (Chen, C. and Goldsmith)**

Assuming $a_i$'s are composed of i.i.d. Gaussian entries, with high probability, the solution $\hat{X}$ exactly recovers all rank-$r$ matrices $X$, provided that

$$m \gtrsim nr.$$  

If there exists additional Toeplitz constraint, then similar guarantee holds provided

$$m \gtrsim r \text{polylog} n.$$  

- **Exact recovery** with $m = O(nr)$;
- **Robust** against approximate low-rankness and bounded noise.
- **Under Toeplitz constraint:**

![Information Theoretic Limit](image)
Kaczmarz method for solving quadratic equations

- Goal: reduce the memory and computational cost by directly estimating $U \in \mathbb{R}^{n \times r}$. 
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Kaczmarz method for solving quadratic equations

- Extend Kaczmarz method by, at each iteration, project the current estimate to the closest signal that satisfies a (quadratic) constraint:†

\[
U_k = \arg \min_{V} \| U_{k-1} - V \|_F^2, \quad V: \| V^T a_{\ell(k)} \|_2^2 = y_{\ell(k)}
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which can be solved in **closed form** via a **rank-one** update:

\[
U_k = \left[ I - \left( \frac{\| U_{k-1}^T a_{\ell(k)} \|_2^2 - \sqrt{y_{\ell(k)}}}{\| U_{k-1}^T a_{\ell(k)} \|_2} \right) \frac{a_{\ell(k)} a_{\ell(k)}^T}{\| a_{\ell(k)} \|_2^2} \right] U_{k-1}.
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- The solution is equivalent to

\[
\min_{s: \|s\|_2^2 = 1} \arg\min_{V: \|V^T a_{\ell(k)}\|_2^2 = s \sqrt{y_{\ell(k)}}} \|U_{k-1} - V\|_F^2
\]

which corresponds to projecting the current estimate to the hyperplane with the phase that minimizes the projection.

Performanc e Guarantee of Kaczmarz Method

Consider the phase retrieval case.

**Theorem (Zhang, C., Liang)**

Assume \( \alpha_i \)'s are generated with i.i.d. Gaussian entries, there exist some universal constants \( \rho > 0 \) such that if \( m \gtrsim n \), then with high probability, randomized Kaczmarz update rule yields

\[
\mathbb{E}_{\xi_t} \left[ \text{dist}^2 (z^{(t+1)}, x) \right] \leq \left( 1 - \frac{\rho}{n} \right) \text{dist}^2 (z^{(t)}, x)
\]

where \( z^{(0)} \) is initialized via the spectral method.

- This establishes linear convergence rate *in expectation*, despite the nonlinearity!
- We can obtain similar guarantees for the block Kaczmarz method which is further accelerated.

![Graph showing NMSE (dB) vs. Number of iterations for different values of \( r \).](image)
What about outliers?

- Outliers happen with
  - sensor failures, malicious attacks, ...
  - For covariance sketching, insufficiently aggregated sketches can be regarded as an outlier;

\[ y_i = a^T X a_i + \eta_i + w_i, \quad i = 1, \ldots, m, \]

- Goal: develop algorithms that are oblivious to outliers, and statistically and computationally efficient.
  - small sample size: hopefully \( m \) is linear in \( n \);
  - large fraction of outliers: hopefully \( s \) is a small constant;
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Outlier-robust recovery by convex programming

- To motivate, ideally one would like to look for low-rank matrices that maintain outlier sparsity:

\[ \hat{X} = \text{argmin} \text{cardinality}(\text{outliers}), \quad \text{s.t.} \quad \text{rank}(M) = r \]

- Parameter-free formulation without trace minimization or tuning parameters;
- No prior information is required for the matrix rank, corruption level or bounded noise level.
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• By *relaxing* the objective function to the $\ell_1$-norm minimization, and *dropping* the rank constraint, we propose to solve

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- Parameter-free formulation without trace minimization or tuning parameters;
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Suppose that $\|w\|_1 \leq \epsilon$. Assume the support of $\eta$ is selected uniformly at random with the signs of $\eta$ are generated from a symmetric Bernoulli distribution. Then as long as $m \gtrsim nr^2$, $s \lesssim 1/r$, the solution to the proposed algorithm satisfies

$$\|\hat{X} - X\|_F \lesssim \frac{r\epsilon}{m}$$

with high probability.

- Exact recovery when $w = 0$ as long as $m \gtrsim nr^2$ and $s \lesssim 1/r$.
- When $r = 1$ recovers a previous result for the phase retrieval case‡;
- RHS is phase transition for $m$ vs $r$ with 5% corruptions.

‡P. Hand, “Phaselift is robust to a constant fraction of arbitrary errors”.

[Graph showing percent of outliers vs rank for different values of m and s]
Robust recovery of Toeplitz PSD Matrices

If $X$ is additionally Toeplitz, this can be incorporated:

$$\hat{X} = \arg\min_{M \succeq 0, \text{Toeplitz}} \sum_{i=1}^{m} |y_i - a_i^T M a_i|.$$  

Figure: Phase transitions of low-rank Toeplitz PSD matrix recovery w.r.t. the number of measurements and the rank with 5% of measurements corrupted by standard Gaussian variables, when $n = 64$. 
Non-convex approach based on factored model

Can we reduce the computational complexity?

- Recall $X = UU^T$ where $U \in \mathbb{R}^{n \times r}$, one can directly recover $U$ by attempting:

$$\hat{U} = \arg\min_{U \in \mathbb{R}^{n \times r}} \ell(U) := \arg\min_{U \in \mathbb{R}^{n \times r}} \frac{1}{m} \sum_{i=1}^{m} \ell(y_i; U)$$
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  $$

  for some loss function $\ell(y_i, \mathbf{U})$:

  - quadratic loss of power: $\ell(\mathbf{U}; y_i) = \left( y_i - \|\mathbf{U}^T \mathbf{a}_i\|_2^2 \right)^2$
  - quadratic loss of amplitude: $\ell(\mathbf{U}; y_i) = \left( \sqrt{y_i} - \|\mathbf{U}^T \mathbf{a}_i\|_2 \right)^2$
  - Poisson loss: $\ell(\mathbf{U}; y_i) = \|\mathbf{U}^T \mathbf{a}_i\|_2^2 - y_i \log \|\mathbf{U}^T \mathbf{a}_i\|_2^2$
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- What are the challenges?
  - \( \ell(U) \) can be non-convex and non-smooth.
  - With outliers, we want the loss to sum over only clean samples.
Non-convex phase retrieval

Exciting developments (without outliers) – all following the same recipe (for the phase retrieval or rank-1 case):

\[ \hat{z} = \arg\min_{z \in \mathbb{R}^n} \frac{1}{m} \sum_{i=1}^{m} \ell(y_i; z) \]

- Initialize \( z^{(0)} \) via the (truncated) spectral method to land in the neighborhood of the ground truth;
- Iterative update using (truncated) gradient descent;

\[ \text{Figure credit: Yuxin Chen.} \]
Non-convex phase retrieval

Provable near-optimal performance for Gaussian measurement model:

- Statistically: $m = O(n)$ near-optimal sample complexity
- Computationally: linear convergence with near-linear run time
Non-convex phase retrieval

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Non-convex phase retrieval with outliers

In the presence of *arbitrary outliers*, **existing approaches fail**:

- **Spectral initialization would fail**: the eigenvector of $Y$ can be arbitrarily perturbed

  $$Y = \frac{1}{m} \sum_{i=1}^{m} y_i a_i a_i^T$$

  or

  $$Y = \frac{1}{m} \sum_{i=1}^{m} y_i a_i a_i^T 1\{|y_i| \leq \alpha_y \cdot \text{mean}\{y_i\}\}$$

  \[\text{WF}\]

  \[\text{TWF}\]

  \[\text{with some details hiding}\]
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  \]

  - **WF**
  - **TWF**

- **Gradient descent would fail:** the search direction can be arbitrarily perturbed

  \[
  z^{(t+1)} = z^{(t)} - \frac{\mu}{\|z^{(0)}\|^2} \sum_{i \in \mathcal{T}_t} \nabla \ell(z^{(t)}; y_i)
  \]

  where $\mathcal{T}_t = \{1, \ldots, m\}$ for WF and

  - $\mathcal{T}_t = \left\{ i : |y_i - |a_i^T z^{(t)}|^2| \leq \alpha_h \cdot \text{mean}(\{|y_i - |a_i^T z^{(t)}|^2|\}) \right\}$ for TWF.

*with some details hiding*
Robust phase retrieval via median-truncation

Need better strategy to eliminate outliers!

Key approach: “median-truncation”
• well-known in robust statistics to be outlier-resilient;
• little appearance in high-dimensional estimation;
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Median is more stable than mean and top-k truncation (which truncates a fixed amount of samples) for various levels of outliers.

- no outliers
- small outlier magnitudes
- large outlier magnitudes
Median-Truncated Wirtinger Flow (median-TWF)

We adopt the Poisson loss function (other loss functions work too) and the Gaussian measurement model.

- **Median-truncated spectral initialization:** Set $z^{(0)} := \lambda_0 \tilde{z}$ where
  - **Direction estimation:** $\tilde{z}$ is the leading eigenvector of
    \[
    Y = \frac{1}{m} \sum_{i=1}^{m} y_i a_i a_i^T \mathbb{1}_{\{|y_i| \leq 9/0.455 \cdot \text{median}(\{y_i\})\}}.
    \]
  - **Norm estimation:** $\lambda_0 = \sqrt{\text{median}(\{y_i\})/0.455}$
    \[
    y_i = |a_i^T x|^2 \sim \chi_1^2 \quad \text{and} \quad \mathbb{E}[\text{median}(\chi_1^2)] = 0.455
    \]
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    \]
  - Norm estimation: \( \lambda_0 = \sqrt{\text{median}\{\{y_i\}\}/0.455} \)
    \[
    y_i = |a_i^T x|^2 \sim \chi_1^2 \quad \text{and} \quad \mathbb{E}[\text{median}(\chi_1^2)] = 0.455
    \]
- As long as \( m = O(n \log n) \) and \( s = O(1) \), the initialization is provably close to the ground truth:
  \[
  \text{dist}(z^{(0)}, x) \leq \frac{1}{10} \|x\|,
  \]
  where \( \text{dist}(z^{(0)}, x) = \min\{\|z^{(0)} + x\|, \|z^{(0)} - x\|\} \).
Median-Truncated Wirtinger Flow (median-TWF)

- Median-truncated gradient descent:

\[
\begin{align*}
    z^{(t+1)} &= z^{(t)} - \frac{2\mu}{m} \sum_{i \in E_1 \cap E_2} \frac{|a_i^T z^{(t)}|^2 - y_i}{a_i^T z^{(t)}} a_i,
\end{align*}
\]

where

\[ E_1 = \left\{ i : 0.3 \leq \frac{|a_i^T z^{(t)}|}{\|z^{(t)}\|} \leq 5 \right\}, \quad E_2 = \left\{ i : r_i^{(t)} \leq 12 \frac{|a_i^T z^{(t)}|}{\|z^{(t)}\|} \cdot \text{median}(\{r_i^{(t)}\}) \right\}, \]

with \[ r_i^{(t)} = |y_i - (a_i^T z^{(t)})^2|. \]
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\]

with \( r_i^{(t)} = |y_i - (a_i^T z^{(t)})^2| \).

- As long as \( m = O(n \log n) \) and \( s = O(1) \), \( \nabla \ell_{tr}(z) \) satisfies the Regularity Condition \( \text{RC}(\mu, \lambda) \) for all \( z, h = z - x \):

\[
- \left\langle \frac{1}{m} \nabla \ell_{tr}(z), h \right\rangle \geq \mu \left\| \frac{1}{m} \nabla \ell_{tr}(z) \right\|^2 + \lambda \|h\|^2, \quad \|h\| \leq \frac{1}{10} \|z\|.
\]

which guarantees \( \text{dist}(z^{(t+1)}, x) \leq (1 - \mu \lambda) \text{dist}(z^{(t)}, x) \).
Theorem (Zhang, C. and Liang, 2016)

Assume $\|w\|_\infty \leq c_1 \|x\|^2$. Assume $a_i$’s are generated with i.i.d. Gaussian entries. If $m \gtrsim n \log n$ and $s \lesssim s_0$, then with high probability, median-TWF yields

$$\text{dist}(z^{(t)}, x) \lesssim \frac{\|w\|_\infty}{\|x\|} + (1 - \rho)^t \|x\|, \quad \forall t \in \mathbb{N}$$

simultaneously for all $x \in \mathbb{R}^n \setminus \{0\}$ for some $0 < \rho < 1$.

- **Exact recovery** when $\|w\| = 0$ with slight more samples ($m = O(n \log n)$) but a constant fraction of outliers $s = O(1)$.
- **Stable recovery** with additional bounded noise;
- Resist outliers **obliviously**: no prior knowledge of outliers.
- **First** non-asymptotic robust recovery guarantee using median: much more involved due to the nonlinearity of median.
Proof sketch - preparation

**Definition (Generalized quantile function)**

Let $0 < p < 1$. If $F$ is a CDF, the generalized quantile function is

$$F^{-1}(p) = \inf \{ x \in \mathbb{R} : F(x) \geq p \}.$$

Denote $\theta_p(F) := F^{-1}(p)$ and $\theta_p(\{X_i\}) := \theta_p(\hat{F})$, where $\hat{F}$ is the empirical distribution of the samples $\{X_i\}_{i=1}^m$. 
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![Diagram of the cumulative distribution function (CDF) with points $p_1$, $p_2$, $p_3$ on the y-axis and $q_1$, $q_2$, $q_3$ on the x-axis. The CDF curve increases from 0 to 1 as x increases.]
Proof sketch

**Lemma (Concentration of sample quantile)**

Assume \( \{X_i\}_{i=1}^m \) are i.i.d. drawn from some distribution \( F \). Under some minor assumptions, w.h.p.

\[
|\theta_p(\{X_i\}_{i=1}^m) - \theta_p(F)| < \epsilon
\]

**Lemma (Sandwich median by quantiles of clean samples)**

Consider clean samples \( \{\tilde{X}_i\}_{i=1}^m \) and contaminated samples \( \{X_i\}_{i=1}^m \).

Then

\[
\theta_{\frac{1}{2} - s}(\{\tilde{X}_i\}) \leq \theta_{\frac{1}{2}}(\{X_i\}) \leq \theta_{\frac{1}{2} + s}(\{\tilde{X}_i\}).
\]

**Lemma (Concentration of median)**

If \( m > c_0 n \log n \), then with probability at least \( 1 - c_1 \exp(-c_2 m) \), there exist constants \( \beta \) and \( \beta' \) such that

\[
\beta \|z\| \|h\| \leq \text{median}(\{||a_i^T x||^2 - ||a_i^T z||^2\}_{i=1}^m) \leq \beta' \|z\| \|h\|,
\]

holds for all \( z, h := z - x \) satisfying \( \|h\| < 1/11 \|z\| \).
Numerical experiments with median-TWF

![Graphs showing success rate vs outliers fraction for different norms and TWF methods.]

(a) $\|\eta\|_\infty = 0.1\|x\|^2$

(b) $\|\eta\|_\infty = \|x\|^2$

(c) $\|\eta\|_\infty = 10\|x\|^2$

(d) $\|\eta\|_\infty = 100\|x\|^2$

Figure: Success rate of exact recovery with outliers for median-TWF, trimean-TWF, and TWF at different levels of outlier magnitudes.
Numerical experiments with median-TWF

Recovery with both dense noise and sparse outliers:

- With outliers, median-TWF achieve better accuracy than TWF.
- Moreover, median-TWF with outliers achieves almost the same accuracy of TWF without outliers.

![Relative error of median-TWF vs. TWF](image)

**Figure**: Relative error of median-TWF vs. TWF w.r.t. iteration when $s = 0.1$, $\|w\|_\infty = 0.01\|x\|^2$, and $\|\eta\|_\infty = \|w\|$.
Conclusions

We have discussed how to solve random quadratic systems of equations, possibly corrupted by a constant fraction of outliers, in a provable manner.

<table>
<thead>
<tr>
<th>measurements</th>
<th>$y_i = a_i^T X a_i$</th>
<th>$y_i = |U^T a_i|_2^2$</th>
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<td>loss</td>
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<td>without outliers</td>
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</tr>
<tr>
<td>with outliers</td>
<td>Semidefinite Prog.</td>
<td>median-TWF</td>
</tr>
</tbody>
</table>

- **The class of convex methods** are based on convex relaxation for low-rank matrix completion and sparse recovery. It is easier to design but the computational cost is high;
- **The class of non-convex methods** are based on iterative updates with careful initializations. The computational cost is low but the design is a bit of an art.
References


http://www.ece.osu.edu/~chi/
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