Compressive Harmonic Retrieval via Matrix Completion

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***Abstract*—We study the problem of recovering a mixture of r complex sinusoids at *continuous-valued* frequencies from $[0,1]^K$, when a random subset of its $n_1 \times n_2$ regularly spaced time-domain samples are observed. Addressing this problem is important in many applications such as imaging system, control, radar, and can help overcome the basis mismatch issue in compressed sensing. We develop a new algorithm based on an enhanced Hankel structure and nuclear norm minimization. Under certain incoherence condition, this method admits accurate recovery from the order of $r \log^2(n_1, n_2)$. Additionally, our results extend to the more general problem of low-rank Hankel matrix completion.**

I. Problem Formulation

A large class of signals of interest entails a superposition of spikes in the continuous frequency domain, which arises in numerous applications including radar, imaging systems, wireless networks, etc. The resolution of signal acquisition devices is often limited by physics and hardwares, precluding sampling with desired resolution. It is of great interest to identify the underlying multi-dimensional frequencies of infinite precision from partial observations. This is known as the *harmonic retrieval* problem; its 1-D version has been recently addressed in [1].

For concreteness, we restrict our attention to 2-D frequency models. Consider an $n_1 \times n_2$ data matrix $\boldsymbol{X} = (x_{i,j})_{0 \le i < n_1, 0 \le l < n_2}$ that can be expressed as $x_{k,l} = \sum_{i=1}^{r} d_i y_i^k z_i^l$, where $y_i := \exp(j2\pi f_{1i})$ and $z_i := \exp(j2\pi f_{2i})$ for some set of frequency pairs $\{(f_{1i}, f_{2i}) \mid 1 \le i \le r\}$. Suppose that there exists a location set Ω of size m such that $x_{k,l}$ is observed iff $(k,l) \in \Omega$. We are interested in perfectly recovering $\boldsymbol{X}$ from a small set of observations. Once the data matrix $\boldsymbol{X}$ is recovered, the underlying frequencies can be retrieved using conventional methods such as MEMP [2].

We first convert $\boldsymbol{X}$ to an enhanced form $\boldsymbol{X}_{\mathrm{e}}$ as introduced in [2]

$$\boldsymbol{X}_{\mathrm{e}} := \begin{bmatrix} \boldsymbol{X}_0 & \boldsymbol{X}_1 & \cdots & \boldsymbol{X}_{n_1-k_1} \\ \boldsymbol{X}_1 & \boldsymbol{X}_2 & \cdots & \boldsymbol{X}_{n_1-k_1+1} \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{X}_{k_1-1} & \boldsymbol{X}_{k_1} & \cdots & \boldsymbol{X}_{n_1-1} \end{bmatrix}, \tag{1}$$

where each block $\boldsymbol{X}_l$ is a Hankel matrix defined as

$$\boldsymbol{X}_l := \begin{bmatrix} x_{l,0} & x_{l,1} & \cdots & x_{l,n_2-k_2} \\ x_{l,1} & x_{l,2} & \cdots & x_{l,n_2-k_2+1} \\ \vdots & \vdots & \vdots & \vdots \\ x_{l,k_2-1} & x_{l,k_2} & \cdots & x_{l,n_2-1} \end{bmatrix}.$$

This enhanced form admits a low-rank decomposition

$$\boldsymbol{X}_{\mathrm{e}} = \boldsymbol{E}_{\mathrm{L}} \boldsymbol{D} \boldsymbol{E}_{\mathrm{R}},$$

for some matrices $\boldsymbol{E}_{\mathrm{L}}$ and $\boldsymbol{E}_{\mathrm{R}}$, where $\boldsymbol{D} := \mathrm{diag}(d_i)$. This reveals that $\mathrm{rank}(\boldsymbol{X}_{\mathrm{e}}) \le r$. We also let the SVD of the enhanced form $\boldsymbol{X}_{\mathrm{e}}$ be $\boldsymbol{X}_{\mathrm{e}} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{V}^*$.

The enhanced matrix completion (**EMaC**) algorithm can be presented as

$$\underset{\boldsymbol{M}_{\mathrm{e}}}{\text{minimize}} \ \|\boldsymbol{M}_{\mathrm{e}}\|_* \quad \text{subject to} \quad \mathcal{P}_\Omega(\boldsymbol{M}) = \mathcal{P}_\Omega(\boldsymbol{X}), \tag{2}$$

where $\boldsymbol{M}_{\mathrm{e}}$ denotes the enhanced form of $\boldsymbol{M}$.

II. Performance Guarantee

We define a measure of incoherence as follows. Let $\boldsymbol{G}_{\mathrm{L}}$ and $\boldsymbol{G}_{\mathrm{R}}$ be two $r \times r$ matrices such that

$$(\boldsymbol{G}_{\mathrm{L}})_{ij} := \frac{1}{k_1 k_2} \frac{1 - (y_i^* y_j)^{k_1}}{1 - y_i^* y_j} \frac{1 - (z_i^* z_j)^{k_2}}{1 - z_i^* z_j},$$

$$(\boldsymbol{G}_{\mathrm{R}})_{ij} := \frac{1 - (y_i^* y_j)^{n_1-k_1+1}}{(n_1-k_1+1)(1 - y_i^* y_j)} \frac{1 - (z_i^* z_j)^{n_2-k_2+1}}{(n_2-k_2+1)(1 - z_i^* z_j)},$$

with the convention that $(\boldsymbol{G}_{\mathrm{L}})_{ii} = (\boldsymbol{G}_{\mathrm{L}})_{ii} = 1$. We let $\Omega_{\mathrm{e}}(i,l)$ be the set of locations in the enhanced matrix $\boldsymbol{X}_{\mathrm{e}}$ containing copies of $x_{i,l}$, and denote $k_{(i,l)} = |\Omega_{\mathrm{e}}(i,l)|$. Besides, we define $\boldsymbol{A}_{(i,l)}$ such that $\left(\boldsymbol{A}_{(i,l)}\right)_{\alpha,\beta} = 1/\sqrt{k_{(i,l)}}$ if $(\alpha,\beta) \in \Omega_{\mathrm{e}}(i,l)$ and 0 otherwise. The incoherence can then be defined as follows.

Definition 1 (Incoherence). *$\boldsymbol{X}$ is said to have incoherence (μ_1, μ_2, μ_3) if they are respectively the smallest values obeying*

$$\sigma_{\min}(\boldsymbol{G}_{\mathrm{L}}) \ge \frac{1}{\mu_1}, \quad \sigma_{\min}(\boldsymbol{G}_{\mathrm{R}}) \ge \frac{1}{\mu_1}; \tag{3}$$

$$\max_{(i,l) \in [n_1] \times [n_2]} \frac{1}{|\Omega_e(i,l)|} \left|\left\langle \boldsymbol{U}\boldsymbol{V}^*, \boldsymbol{A}_{(i,l)} \right\rangle\right|^2 \le \frac{\mu_2 r}{n_1^2 n_2^2}; \tag{4}$$

$$\forall b \in [n_1] \times [n_2] : \sum_a \left|\left\langle \boldsymbol{U}\boldsymbol{U}^* \boldsymbol{A}_b \boldsymbol{V}\boldsymbol{V}^*, \sqrt{k_a}\boldsymbol{A}_a \right\rangle\right|^2 \le \frac{\mu_3 r}{n_1 n_2} k_b. \tag{5}$$

Theorem 1. *Define $c_s := \max\left(\frac{n_1 n_2}{k_1 k_2}, \frac{n_1 n_2}{(n_1-k_1+1)(n_2-k_2+1)}\right)$. Then there exist constants $c_0, \tilde{c}_0 > 0$ such that under either of the following conditions:*

1) *Condition (3), (4),and (5) hold and $m > c_0 \max(\mu_1 c_s, \mu_4, \mu_2)\, r \log^2(n_1 n_2)$;*
2) *Condition (3) holds and $m > \tilde{c}_0 \mu_1^2 c_s^2 r^2 \log^2(n_1 n_2)$;*

*$\boldsymbol{X}$ is the unique solution of **EMaC** with high probability.*

III. Numerical Results

Set $n := n_1 = n_2 = 11$. We run 200 Monte Carlo trials for each triple (n, m, r) when the frequencies are randomly generated. A trial is declared successful if the relative error $\|\boldsymbol{X}^\star - \boldsymbol{X}\|_{\mathrm{F}} / \|\boldsymbol{X}\|_{\mathrm{F}} \le 10^{-5}$. The phase transition diagram is plotted in Fig. 1, which illustrates the practicability of **EMaC**.

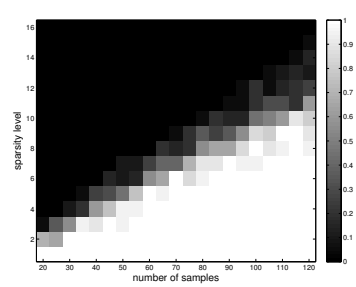

Figure 1. Phase transition of **EMaC** when $n = n_1 = n_2 = 11$.

References

[1] G. Tang, B. N. Bhaskar, P. Shah, and B. Recht, "Compressed sensing off the grid," July 2012. [Online]. Available: http://arxiv.org/abs/1207.6053
[2] Y. Hua, "Estimating two-dimensional frequencies by matrix enhancement and matrix pencil," *IEEE Transactions on Signal Processing*, vol. 40, no. 9, pp. 2267 –2280, Sep 1992.