Policy Mirror Descent for Regularized Reinforcement Learning: A Generalized Framework with Linear Convergence

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Abstract

Policy optimization, which learns the policy of interest by maximizing the value function via large-scale optimization techniques, lies at the heart of modern reinforcement learning (RL). In addition to value maximization, other practical considerations arise commonly as well, including the need of encouraging exploration, and that of ensuring certain structural properties of the learned policy due to safety, resource and operational constraints. These considerations can often be accounted for by resorting to regularized RL, which augments the target value function with a structure-promoting regularization term.

Focusing on an infinite-horizon discounted Markov decision process, this paper proposes a generalized policy mirror descent (GPMD) algorithm for solving regularized RL. As a generalization of policy mirror descent (Lan, 2021), the proposed algorithm accommodates a general class of convex regularizers as well as a broad family of Bregman divergence in cognizant of the regularizer in use. We demonstrate that our algorithm converges linearly over an entire range of learning rates, in a dimension-free fashion, to the global solution, even when the regularizer lacks strong convexity and smoothness. In addition, this linear convergence feature is provably stable in the face of inexact policy evaluation and imperfect policy updates. Numerical experiments are provided to corroborate the applicability and appealing performance of GPMD.

Keywords: policy mirror descent, Bregman divergence, regularization, nonsmooth, policy optimization

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1 Introduction
Policy optimization lies at the heart of recent successes of reinforcement learning (RL) (Mnih et al., 2015). In its basic form, the optimal policy of interest, or a suitably parameterized version, is learned by attempting to maximize the value function in a Markov decision processes (MDP). For the most part, the maximization step is carried out by means of first-order optimization algorithms amenable to large-scale applications, whose foundations were set forth in the early works of Williams (1992); Sutton et al. (2000). A partial list of widely adopted variants in modern practice includes policy gradient (PG) methods (Sutton et al., 2000), natural policy gradient (NPG) methods (Kakade, 2001), TRPO (Schulman et al., 2015), PPO (Schulman et al., 2017), soft actor critic methods (Haarnoja et al., 2018), to name just a few. In comparison with model-based and value-based approaches, this family of policy-based algorithms offers a remarkably flexible framework to work with both continuous and discrete action spaces, and lends itself well to the incorporation of powerful function approximation schemes like neural networks. Nevertheless, theoretical understanding of policy optimization remains severely limited compared to its practical success, owing to the ubiquitous nonconvexity issue underlying the objective function.

1.1 The role of regularization
In practice, there are often competing objectives and additional constraints that the agent has to deal with in conjunction to maximizing values, which motivate the studies of regularization techniques in RL. In what follows, we isolate a few representative examples.

• Promoting exploration. In the face of large problem dimensions and complex dynamics, it is often desirable to maintain a certain degree of randomness in the policy iterates, in order to encourage exploration and discourage premature convergence to sub-optimal policies. A popular strategy of this kind is to enforce entropy regularization (Williams and Peng, 1991), which penalizes policies that are not sufficiently stochastic. Along similar lines, the Tsallis entropy regularization (Chow et al., 2018b; Lee et al., 2018) further promotes sparsity of the learned policy while encouraging exploration, ensuring that the resulting policy does not assign non-negligible probabilities to too many sub-optimal actions.
• **Safe RL.** In a variety of application scenarios such as industrial robot arms and self-driving cars, the agent is required to operate safely both to itself and its surroundings (Amodei et al., 2016; Moldovan and Abbeel, 2012); for example, certain actions might be strictly forbidden in some states. One way to incorporate such prescribed operational constraints is through adding a regularizer (e.g., a certain log barrier or indicator function) to explicitly account for the constraints.

• **Cost-sensitive RL.** In reality, different actions of an agent might incur drastically different costs even for the same state. This motivates the design of new objective functions that trade off the cumulative rewards against the accumulated cost, which often take the form of regularized value functions.

Viewed in this light, it is of immediate value to develop a unified framework towards understanding the performance and limits of regularized policy optimization. While a recent line of works (Agarwal et al., 2020; Mei et al., 2020b; Cen et al., 2020) have looked into specific types of regularization techniques such as entropy regularization, existing convergence theory is still highly inadequate when it comes to more general regularizers.

### 1.2 Main contributions

The current paper focuses on policy optimization for regularized RL in a $\gamma$-discounted infinite horizon Markov decision process (MDP) with state space $S$, action space $A$, and reward function $r(\cdot, \cdot)$. The goal is to find an optimal policy that maximizes a regularized value function. Informally speaking, the regularized value function associated with a given policy $\pi$ takes the following form:

$$
V^\pi_\tau = V^\pi - \tau \mathbb{E}[h_s(\pi(\cdot|s))],
$$

where $V^\pi$ denotes the original (unregularized) value function, $\tau > 0$ is the regularization parameter, $h_s(\cdot)$ denotes a convex regularizer employed to regularize the policy in state $s$, and the expectation is taken over certain marginal distribution w.r.t. the MDP (to be made precise in Section 2.1). It is noteworthy that the regularizer $h_s(\cdot)$ does not need to be either strongly convex and or smooth.

Inspired by the recent work Lan (2021), we propose a generalized policy mirror descent (GPMD) algorithm. In each iteration, the policy is updated in parallel for each state via a mirror-descent style update rule. In stark contrast to Lan (2021) that freezes the Bregman divergence term to be the Kullback-Leibler (KL) divergence, our algorithm allows the Bregman divergence to be selected adaptively in cognizant of the regularizer. Several important features and theoretical appeal of GPMD are summarized as follows.

- GPMD significantly broadens the range of algorithmic choices for regularized RL, and subsumes several well-known algorithms as special cases. For example, it reduces to regularized policy iteration (Geist et al., 2019) when the learning rate tends to infinity, and reduces to entropy-regularized NPG methods when the Bregman divergence is taken to be the KL divergence (Cen et al., 2020).

- Assuming exact policy evaluation and perfect policy update in each iteration, GPMD converges linearly—in a dimension-free fashion—over the entire range of the learning rate $\eta > 0$. More precisely, it converges to an $\epsilon$-optimal regularized Q-function in no more than an order of

$$
\frac{1 + \eta\tau}{\eta\tau(1-\gamma)} \log \frac{1}{\epsilon}
$$

iterations (up to some logarithmic factor), as detailed in Section 3.1. Encouragingly, this appealing feature is valid for a broad family of convex and possibly nonsmooth regularizers.

- The intriguing convergence guarantees are robust in the face of inexact policy evaluation and imperfect policy updates, namely, the algorithm is guaranteed to converge linearly at the same rate until an error floor is hit. See Section 3.2 for details.

- Numerical experiments are provided in Section 5 to demonstrate the practical applicability and appealing performance of the proposed algorithm.
Finally, it is helpful to briefly compare the above findings with prior works. As soon as the learning rate exceeds \( \eta \geq 1/\tau \), the iteration complexity of our algorithm is on the order of \( \frac{1}{\tau} \log \frac{1}{\gamma} \), thus matching that of regularized policy iteration (Geist et al., 2019). In comparison with Lan (2021), the linear convergence of the algorithm studied therein critically requires the regularizer to be strongly convex and smooth, and is therefore much more restricted than our results.

1.3 Related works

**Global convergence of policy gradient methods.** Recent years have witnessed a surge of activities towards understanding the global convergence properties of policy gradient methods and their variants for both continuous and discrete RL problems, examples including Fazel et al. (2018); Bhandari and Russo (2019); Agarwal et al. (2020); Zhang et al. (2019); Wang et al. (2019); Mei et al. (2020a); Bhandari and Russo (2020); Khodadadian et al. (2021); Liu et al. (2020); Mei et al. (2020a); Agazzi and Lu (2020); Xu et al. (2019); Wang et al. (2019); Cen et al. (2020); Mei et al. (2021); Liu et al. (2019); Wang et al. (2020); Zhang et al. (2020a, 2021, 2020b); Shani et al. (2019), among other things. Neu et al. (2017) provided the first interpretation of NPG methods as mirror descent (Nemirovsky and Yudin, 1983), thereby enabling the adaptation of techniques for analyzing mirror descent to studying NPG-type algorithms such as TRPO (Shani et al., 2019; Tomar et al., 2020). Our work clearly belongs to the former type. Several recent results in prior literature merit particular attention: Agarwal et al. (2020) demonstrated sublinear convergence guarantees for PG methods in the presence of relative entropy regularization, Mei et al. (2020b) established linear convergence of entropy-regularized PG methods, whereas Cen et al. (2020) proved an almost dimension-free linear convergence for NPG methods with entropy regularization. Most of the existing literature focuses on the entropy or KL regularization, and the study of general regularizers is quite limited until the recent work of Lan (2021). The regularized MDP problems are also closely related to the studies of constrained MDPs, as both types of problems can be employed to model constraint satisfaction in RL, as recently investigated in, e.g., Chow et al. (2018a); Efroni et al. (2020); Ding et al. (2021); Yu et al. (2019); Xu et al. (2020). Note, however, that our algorithm cannot be directly compared with these algorithms, due to drastically different formulations and settings.

**Regularization in RL.** Regularization has been suggested to the RL literature either through the lens of optimization (Dai et al., 2018; Agarwal et al., 2020), or through the lens of dynamic programming (Geist et al., 2019). In comparison with Lan (2021), the linear convergence of the algorithm studied therein critically requires the regularizer to be strongly convex and smooth, and is therefore much more restricted than our results.

1.4 Notation

For any set \( \mathcal{A} \), we denote by \(|\mathcal{A}|\) the cardinality of a set \( \mathcal{A} \), and let \( \Delta(\mathcal{A}) \) indicate the probability simplex over the set \( \mathcal{A} \). For any convex and differentiable function \( h(\cdot) \), the Bregman divergence generated by \( h(\cdot) \) is defined as

\[
D_h(z, x) := h(z) - h(x) - \langle \nabla h(x), z - x \rangle.
\]

For any convex (but not necessarily differentiable) \( h(\cdot) \), we denote by \( \partial h \) the subdifferential of \( h \). Given two probability distributions \( \pi_1 \) and \( \pi_2 \) over \( \mathcal{A} \), the Kullback-Leibler (KL) divergence from \( \pi_2 \) to \( \pi_1 \) is defined by \( \text{KL}(\pi_1 \parallel \pi_2) := \sum_{a \in \mathcal{A}} \pi_1(a) \log \frac{\pi_1(a)}{\pi_2(a)} \). For any vectors \( a = [a_i]_{1 \leq i \leq n} \) and \( b = [b_i]_{1 \leq i \leq n} \), the notation \( a \leq b \) (resp. \( a \geq b \)) means that \( a_i \leq b_i \) (\( a_i \geq b_i \)) for all \( 1 \leq i \leq n \). We shall also use 1 (resp. 0) to denote the all-one (resp. all-zero) vector whenever it is clear from the context.
2 Model and algorithms

2.1 Problem settings

Markov decision process (MDP). The focus of this paper is a discounted infinite-horizon Markov decision process, as represented by $M = (\mathcal{S}, \mathcal{A}, P, r, \gamma)$ (Bertsekas, 2017). Here, $\mathcal{S} := \{1, \ldots, |\mathcal{S}|\}$ is the state space, $\mathcal{A} := \{1, \ldots, |\mathcal{A}|\}$ is the action space, $\gamma \in [0, 1]$ is the discount factor, $P : \mathcal{S} \times \mathcal{A} \to \Delta(\mathcal{S})$ is the probability transition matrix (so that $P(s, a)$ is the transition probability from state $s$ upon execution of action $a$), whereas $r : \mathcal{S} \times \mathcal{A} \to [0, 1]$ is the reward function (so that $r(s, a)$ indicates the immediate reward received in state $s$ after action $a$ is executed). Here, we focus on finite-state and finite-action scenarios, meaning that both $|\mathcal{S}|$ and $|\mathcal{A}|$ are assumed to be finite. A policy $\pi : \mathcal{S} \to \Delta(\mathcal{A})$ specifies a possibly randomized action selection rule, namely, $\pi(\cdot | s)$ represents the action selection probability in state $s$.

For any policy $\pi$, we define the associated value function $V^\pi : \mathcal{S} \to \mathbb{R}$ as follows

$$
\forall s \in \mathcal{S} : \quad V^\pi(s) := \mathbb{E}_{s_{t+1} \sim P(\cdot | s_t), \ a_t \sim \pi(\cdot | s_t), \ \forall t \geq 0} \left[ \sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \middle| s_0 = s \right],
$$

(2)

which can be viewed as the utility function we wish to maximize. Here, the expectation is taken over the randomness of the MDP trajectory $\{(s_t, a_t)\}_{t \geq 0}$ induced by policy $\pi$. Similarly, when the initial action $a$ is fixed, we can define the action-value function (or Q-function) as follows

$$
\forall (s, a) \in \mathcal{S} \times \mathcal{A} : \quad Q^\pi(s, a) := \mathbb{E}_{s_{t+1} \sim P(\cdot | s_t, a_t), \ a_{t+1} \sim \pi(\cdot | s_{t+1}), \ \forall t \geq 0} \left[ \sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \middle| s_0 = s, a_0 = a \right].
$$

(3)

As a well-known fact, the policy gradient of $V^\pi$ (w.r.t. the policy $\pi$) admits the following closed-form expression (see, e.g., Lan (2021, Lemma 1))

$$
\forall (s, a) \in \mathcal{S} \times \mathcal{A} : \quad \frac{\partial V^\pi(s_0)}{\partial \pi(a \mid s)} = \frac{1}{1 - \gamma} d^\pi_{s_0}(s) Q^\pi(s, a).
$$

(4)

Here, $d^\pi_{s_0} \in \Delta(\mathcal{S})$ is the so-called discounted state visitation distribution defined as follows

$$
d^\pi_{s_0}(s) := (1 - \gamma) \sum_{t=0}^{\infty} \gamma^t \mathbb{P}^\pi(s_t = s \mid s_0)
$$

(5)

where $\mathbb{P}^\pi(s_t = s \mid s_0)$ denotes the probability of $s_t = s$ when the MDP trajectory $\{s_t\}_{t \geq 0}$ is generated under policy $\pi$ given the initial state $s_0$.

Furthermore, the optimal value function and the optimal Q-function are defined and denoted by

$$
\forall (s, a) \in \mathcal{S} \times \mathcal{A} : \quad V^*(s) := \max_{\pi} V^\pi(s), \quad Q^*(s, a) := \max_{\pi} Q^\pi(s, a).
$$

(6)

It is well known that there exists at least one optimal policy, denoted by $\pi^*$, that simultaneously maximizes the value function and the Q-function for all state-action pairs (Agarwal et al., 2019).

Regularized MDP. In practice, the agent is often asked to design policies that possess certain structural properties in order to be cognizant of system constraints such as safety and cost, as well as encourage exploration during the optimization/learning stage. A natural strategy to achieve these is to resort to the following regularized value function w.r.t. a given policy $\pi$ (Neu et al., 2017; Mei et al., 2020b; Cen et al., 2020; Lan, 2021):

$$
\forall s \in \mathcal{S} : \quad V^\pi_r(s) := \mathbb{E}_{s_{t+1} \sim \pi(\cdot | s_t), \ a_t \sim \pi(\cdot | s_t), \ \forall t \geq 0} \left[ \sum_{t=0}^{\infty} \gamma^t \left\{ r(s_t, a_t) - \tau h_s(\pi(\cdot | s_t)) \right\} \middle| s_0 = s \right]
= V^\pi(s) - \frac{\tau}{1 - \gamma} \sum_{s' \in \mathcal{S}} d^\pi_{s}(s') h_{s'}(\pi(\cdot | s'))
$$

(7)
where \( h_s : \Delta_{\zeta}(A) \to \mathbb{R} \) stands for a convex and possibly nonsmooth regularizer for state \( s \), \( \tau > 0 \) denotes the regularization parameter, and \( d_f^\tau (\cdot) \) is defined in (5). Here, for technical convenience, we assume throughout that \( h_s(\cdot) (s \in S) \) is well-defined over an “\( \zeta \)-neighborhood” of the probability simplex \( \Delta(A) \) defined as follows

\[
\Delta_{\zeta}(A) = \left\{ x = [x_a]_{a \in A} \mid x_a \geq 0 \text{ for all } a \in A; \ 1 - \zeta \leq \sum_{a \in A} x_a \leq 1 + \zeta \right\},
\]

where \( \zeta > 0 \) can be an arbitrary constant. For instance, entropy regularization adopts the choice \( h_s(p) = \sum_{i \in A} p_i \log p_i \) for all \( s \in S \) and \( p \in \Delta(A) \), which coincides with the negative Shannon entropy of a probability distribution. Similar, a KL regularization adopts the choice \( h_s(p) = \text{KL}(p \parallel p_{\text{ref}}) \), which penalizes the distribution \( p \) that deviates from the reference \( p_{\text{ref}} \). As another example, a weighted \( \ell_1 \) regularization adopts the choice \( h_s(p) = \sum_{i \in A} w_{s,i} p_i \) for all \( s \in S \) and \( p \in \Delta(A) \), where \( w_{s,i} \geq 0 \) is the cost of taking action \( i \) at state \( s \), and the regularizer \( h_s(\pi(\cdot | s)) \) captures the expected cost of the policy at state \( s \). Throughout this paper, we impose the following assumption.

**Assumption 1.** Consider an arbitrarily small constant \( \zeta > 0 \). For any \( s \in S \), suppose that \( h_s(\cdot) \) is convex and

\[ h_s(p) = \infty \quad \text{for any } p \notin \Delta_{\zeta}(A). \quad (8) \]

Following the convention in prior literature (e.g., Mei et al. (2020b)), we also define the corresponding regularized Q-function as follows:

\[
\forall (s, a) \in S \times A : \quad Q^\tau_{\zeta}(s, a) := r(s, a) + \gamma \mathbb{E}_{s' \sim \pi(\cdot | s, a)} [V^\tau_{\zeta}(s')]. \quad (9a)
\]

As can be straightforwardly verified, one can also express \( V^\tau_{\zeta} \) in terms of \( Q^\tau_{\zeta} \) as

\[
\forall s \in S : \quad V^\tau_{\zeta}(s) := \mathbb{E}_{a \sim \pi(\cdot | s)} \left[ Q^\tau_{\zeta}(s, a) - \tau h_s(\pi(\cdot | s)) \right]. \quad (9b)
\]

The optimal regularized value function \( V^*_{\zeta} \) and the corresponding optimal policy \( \pi^*_{\zeta} \) are defined respectively as

\[
\forall s \in S : \quad V^*_{\zeta}(s) := V^{{\zeta}^*}_{\tau}(s) = \max_{\pi} V^\tau_{\zeta}(s), \quad \pi^*_{\zeta} := \arg \max_{\pi} V^\tau_{\zeta}. \quad (10)
\]

It is worth noting that Puterman (2014) asserts the existence of an optimal policy \( \pi^*_{\zeta} \) that achieves (10) simultaneously for all \( s \in S \). Correspondingly, we shall also define the resulting regularized Q-function as

\[
\forall (s, a) \in S \times A : \quad Q^\tau_{\zeta}(s, a) = Q^{{\zeta}^*}_{\tau}(s, a). \quad (11)
\]

### 2.2 Algorithm: generalized policy mirror descent

In order to maximize the regularized value function (9b), Lan (2021) exhibited an optimization-based algorithm called Policy Mirror Descent (PMD), which can be viewed as an adaptation of the mirror descent algorithm (Nemirovsky and Yudin, 1983; Beck and Teboulle, 2003; Zhou et al., 2020) for policy optimization. In particular, the natural policy gradient (NPG) method (Kakade, 2001) is a special case of PMD. Motivated by this approach, we put forward a more general variant of PMD that accommodates a broader family of Bregman divergence. A thorough comparison with Lan (2021) will be provided after introducing our generalized PMD algorithm.

**Review: mirror descent (MD) for the composite model.** To better elucidate our algorithmic idea, let us first briefly review the design of classical mirror descent—originally proposed by Nemirovsky and Yudin (1983)—in the optimization literature. Consider the following composite model:

\[
\text{minimize}_{x} \quad F(x) := f(x) + h(x),
\]

where the objective function consists of two components. The first component is assumed to be differentiable, while the second component \( h(\cdot) \) can be more general and is commonly employed to model some sort of
regularizers. To solve this composite problem, one variant of mirror descent adopts the following update rule (see also Beck (2017); Duchi et al. (2010)):

\[ x^{(k+1)} = \arg \min_x \left\{ f(x^{(k)}) + \langle \nabla f(x^{(k)}), x \rangle + h(x) + \frac{1}{\eta} D_h(x, x^{(k)}) \right\}, \]

(12)

where \( \eta > 0 \) is the learning rate or step size, and \( D_h(\cdot, \cdot) \) is defined in (1). Note that the first term within the curly brackets of (12) can be safely discarded as it is a constant given \( x^{(k)} \). In words, the above update rule approximates \( f(x) \) via its first-order Taylor expansion \( f(x^{(k)}) + \langle \nabla f(x^{(k)}), x \rangle \) at the point \( x^{(k)} \), employs the Bregman divergence \( D_h \) to monitor the difference between the new iterate and the current iterate \( x^{(k)} \), and attempts to optimize such (properly monitored) approximation instead. While one can further generalize the Bregman divergence to \( D_\omega \) for a different generator \( \omega \), we shall restrict attention to the case with \( h = \omega \) in the current paper.

The proposed algorithm. We are now ready to present the algorithm we come up with, which is an extension of the PMD algorithm (Lan, 2021). For notational simplicity, we shall write

\[ V^{(k)}_\tau := V^{(k)}_\tau(x), \quad Q^{(k)}_\tau(s, a) := Q^{(k)}_\tau(s, a) \quad \text{and} \quad d^{(k)}_s(s) := d^{(k)}_s(s) \]

(13)

throughout the paper, where \( \pi^{(k)} \) denotes our policy estimate in the \( k \)-th iteration.

To begin with, suppose for simplicity that \( h_s(\cdot) \) is differentiable everywhere. In the \( k \)-th iteration, a natural MD scheme that comes into mind for solving (7)—namely, maximize \( V^{(k)}_\tau(s_0) \) for a given initial state \( s_0 \)—is the following update rule:

\[
\pi^{(k+1)}(\cdot | s) = \arg \min_{p \in \Delta(A)} \left\{ -\langle \nabla \pi(\cdot | s), V^{(k)}(s_0) \rangle_{\pi=\pi(s)} + \frac{\tau}{1 - \gamma} d^{(k)}_s(s) h_s(p) + \frac{1}{\eta} D_h(s, \pi^{(k)}(\cdot | s)) \right\}
\]

\[
= \arg \min_{p \in \Delta(A)} \left\{ \frac{1}{1 - \gamma} d^{(k)}_s(s) \left\{ -\langle Q^{(k)}(s, \cdot), p \rangle + \tau h_s(p) \right\} + \frac{1}{\eta} D_h(s, \pi^{(k)}(\cdot | s)) \right\}
\]

\[
= \arg \min_{p \in \Delta(A)} \left\{ -\langle Q^{(k)}(s, \cdot), p \rangle + \tau h_s(p) + \frac{1}{\eta} D_h(s, \pi^{(k)}(\cdot | s)) \right\}
\]

(14)

for every state \( s \in S \), which is a direct application of (12) to our setting. Here, we start with a learning rate \( \eta' \), and simplify it later on by replacing \( \eta' \) with \( \eta(1 - \gamma)/d^{(k)}_s(s) \). Notably, the update strategy (14) is invariant to the initial state \( s_0 \), akin to natural policy gradient methods (Agarwal et al., 2020).

This update rule is well-defined for, say, the case when \( h_s(\cdot) \) is the negative entropy, since the algorithm guarantees \( \pi^{(k)} > 0 \) all the time and hence \( h_s \) is always differentiable w.r.t. the \( k \)-th iterate (see Cen et al. (2020)). In general, however, it is possible to encounter situations when the gradient of \( h_s \) does not exist on the boundary (e.g., when \( h_s \) represents a certain indicator function). To cope with such cases, we resort to a generalized version of Bregman divergence (e.g., Kiwiel (1997)). To be specific, we attempt to replace the usual Bregman divergence \( D_h(s, p, q) \) by the following metric

\[ D_h(s, p, q; g_s) := h_s(p) - h_s(q) - \langle q, p - q \rangle \geq 0, \]

(15)

where \( g_s \) can be any vector falling within the subdifferential \( \partial h_s(q) \). Here, the non-negativity condition in (15) follows directly from the definition of the subgradient for any convex function. The constraint on \( g_s \) can be further relaxed by exploiting the requirement \( p, q \in \Delta(A) \). In fact, for any vector \( \xi_s = g_s - c_s 1 \) (with \( c_s \in \mathbb{R} \) some constant and 1 the all-one vector), one can readily see that

\[
D_h(s, p, q; g_s) = h_s(p) - h_s(q) - \langle q, p - q \rangle = h_s(p) - h_s(q) - \langle \xi_s, p - q \rangle + c_s (1, p - q) = h_s(p) - h_s(q) - \langle \xi_s, p - q \rangle = D_h(s, p, q; \xi_s),
\]

(16)

where the last line is valid since \( 1^T p = 1^T q = 1 \). As a result, everything boils down to identifying a vector \( \xi_s \) that falls within \( \partial h_s(q) \) upon global shift.

Towards this, we propose the following iterative rule for designing such a sequence of vectors as surrogates for the subgradient of \( h_s \):

\[
\xi^{(0)}(s, \cdot) \in \partial h_s(\pi^{(0)}(\cdot | s));
\]

(17a)
Interestingly, several well-known algorithms can be recovered as special cases of GPMD: for all \( v \in \mathbf{R} \), the vectors we construct satisfy the desired property (i.e., lying within the subdifferential of \( h_s \) under suitable global shift).

**Lemma 1.** For all \( k \geq 0 \) and every \( s \in \mathcal{S} \), there exists a quantity \( c_s^{(k)} \in \mathbf{R} \) such that

\[
\xi^{(k)}(s, \cdot) - c_s^{(k)} 1 \in \partial h_s(\pi^{(k)}(\cdot | s)). \tag{18}
\]

In addition, for every \( s \in \mathcal{S} \), there exists a quantity \( c_s^* \in \mathbf{R} \) such that

\[
\tau^{-1} Q^*_\tau(s, \cdot) - c_s^* 1 \in \partial h_s(\pi^*_\tau(\cdot | s)). \tag{19}
\]

**Proof.** See Section A.1. \( \square \)

Thus far, we have presented all crucial ingredients of our algorithm. The whole procedure is summarized in Algorithm 1, and will be referred to as generalized policy mirror descent (GPMD) throughout the paper. Interestingly, several well-known algorithms can be recovered as special cases of GPMD:

- When the Bregman divergence \( D_h(\cdot, \cdot) \) is taken as the KL divergence, GPMD reduces to the established NPG algorithm (Kakade, 2001) when \( \tau = 0 \) (no regularization), and to the NPG algorithm with entropy regularization analyzed in (Cen et al., 2020) when \( h_s(\cdot) \) is taken as the negative Shannon entropy.
- When \( \eta = \infty \) (no divergence), GPMD reduces to regularized policy iteration in Geist et al. (2019); in particular, GPMD reduces to the standard policy iteration algorithm if in addition \( \tau = 0 \).

**Algorithm 1: PMD with generalized Bregman divergence (GPMD)**

1. **Input:** initial policy iterate \( \pi^{(0)} \), learning rate \( \eta > 0 \).
2. **Initialize** \( \xi^{(0)} \) so that \( \xi^{(0)}(s, \cdot) \in \partial h_s(\pi^{(0)}(\cdot | s)) \) for all \( s \in \mathcal{S} \).
3. **for** \( k = 0, 1, \ldots \), **do**
   4. **for** every \( s \in \mathcal{S} \), set
      \[
      \pi^{(k+1)}(\cdot | s) = \arg \min_{p \in \Delta(\mathcal{A})} \left\{ -(Q^*_\tau(s, \cdot) ; p) + \tau h_s(p) + \frac{1}{\eta} D_h_s(p, \pi^{(k)}(\cdot | s); \xi^{(k)}) \right\}, \tag{20a}
      \]
      where
      \[
      D_h_s(p, q; \xi) := h_s(p) - h_s(q) - \langle \xi(s, \cdot), p - q \rangle. \tag{20b}
      \]
   5. **for** every \( (s, a) \in \mathcal{S} \times \mathcal{A} \), compute
      \[
      \xi^{(k+1)}(s, a) = \frac{1}{1 + \eta \tau} \xi^{(k)}(s, a) + \frac{\eta}{1 + \eta \tau} Q^*_\tau(s, a). \tag{20c}
      \]

**Comparison with PMD (Lan, 2021).** Before continuing, let us take a moment to point out the key difference between our algorithm GPMD and the PMD algorithm proposed in Lan (2021). Specifically, the Bregman divergence term employed in the PMD algorithm is always taken to be the KL divergence (Lan, 2021) regardless of the choice of the regularizer \( h_s \), while our GPMD algorithm adaptively set this term to be the Bregman divergence generated by a general family of regularizer \( h_s \). Hence, the two algorithms become equivalent when the regularizer \( h_s \) is the negative Shannon entropy, but will be different in general. Moreover, in stark contrast to Lan (2021), our GPMD algorithm does not require differentiability of the regularizer, meaning that \( h_s \) can be nonsmooth, while Lan (2021) requires \( h_s \) to be smooth. Encouragingly, our approach preserves many advantages of PMD, while offering convergence guarantees for more general scenarios, which will be detailed later.
3 Main results

This section presents our convergence guarantees for the GPMD method presented in Algorithm 1. We shall start with the idealized case assuming that the update rule can be precisely implemented, and then discuss how to generalize it to the scenario with imperfect policy evaluation.

3.1 Convergence of exact GPMD

To start with, let us pin down the convergence behavior of GPMD, assuming that accurate evaluation of the policy \( Q^*(\tau) \) is available and the subproblem (20a) can be solved perfectly. Here and below, we shall refer to the algorithm in this case as exact GPMD. Encouragingly, exact GPMD provably achieves global linear convergence from an arbitrary initialization, as asserted by the following theorem.

**Theorem 1** (Exact GPMD). Suppose that Assumption 1 holds. Consider any learning rate \( \eta > 0 \), and set \( \alpha := \frac{1}{1+\eta \tau} \). Then the iterates of Algorithm 1 satisfy

\[
\| Q^* - Q^{(k+1)} \|_\infty \leq \gamma \left( 1 - (1 - \alpha)(1 - \gamma) \right)^k C_1, \quad k \geq 0,
\]

where \( C_1 := \| Q^*_\tau - Q^{(0)}_\tau \|_\infty + 2\alpha \| Q^*_\tau - \tau \xi^{(0)}_\tau \|_\infty \).

In addition, if \( h_s \) is 1-strongly convex w.r.t. the \( \ell_1 \) norm for some \( s \in S \), then one further has

\[
\| \pi^*_\tau(s) - \pi^{(k+1)}_\tau(s) \|_1 \leq \tau^{-1}(1 - (1 - \alpha)(1 - \gamma))^k C_1, \quad k \geq 0.
\]

Our theorem confirms the fast global convergence of the GPMD algorithm, in terms of both the resulting regularized Q-value (if \( h_s(\cdot) \) is convex) and the policy estimate (if \( h_s(\cdot) \) is strongly convex). In summary, it takes GPMD no more than

\[
\frac{1}{(1 - \alpha)(1 - \gamma)} \log \frac{C_1}{\varepsilon} = \frac{1 + \eta \tau}{\eta \tau (1 - \gamma)} \log \frac{C_1}{\varepsilon}
\]

iterations to converge to an \( \varepsilon \)-optimal regularized Q-function, or

\[
\frac{1 + \eta \tau}{\eta \tau (1 - \gamma)} \log \frac{C_1}{\varepsilon} \tau
\]

iterations to yield an \( \varepsilon \)-approximation (w.r.t. the \( \ell_1 \) norm error) of \( \pi^*_\tau \). The iteration complexity (23) is nearly dimension-free—namely, depending at most logarithmically on the dimension of the state/action space—making it scalable to large-dimensional problems.

**Comparison with Lan (2021, Theorems 1-2).** It is helpful to compare Theorem 1 with the state-of-the-art algorithm PMD in Lan (2021) to make clear our contributions.

- **Linear convergence for convex regularizers.** Our finding reveals that GPMD enjoys global linear convergence even—in terms of \( Q^*_\tau - Q^{(k+1)}_\tau \)—even when the regularizer \( h_s(\cdot) \) is only convex but not strongly convex. In contrast, Lan (2021, Theorem 2) only provided sub-linear convergence (with an iteration complexity proportional to \( 1/\varepsilon \)) for the case with convex regularizers.

- **A full range of learning rates.** Theorem 1 reveals linear convergence of GPMD for a full range of learning rates, namely, our result is applicable to any \( \eta > 0 \). In comparison, linear convergence was shown in Lan (2021) only when the learning rate is sufficiently large and when \( h_s(\cdot) \) is 1-strongly convex w.r.t. the KL divergence. Consequently, the linear convergence results in Lan (2021) do not extend to several widely used regularizers such as negative Tsallis entropy and log-barrier functions (even after scaling), which are covered by our theory. It is worth noting that the case with small-to-medium learning rates is often more challenging to cope with in theory, given that its behavior could be drastically different from regularized policy iteration.

- **From smooth to nonsmooth regularizers.** All results in Lan (2021) operate under the differentiability assumption on the regularizer \( h_s \), while our results permit the regularizer to be nonsmooth and hence cover a broader class of regularizers of interest.
3.2 Convergence of approximate GPMD

In reality, however, it is often the case that GPMD cannot be implemented in an exact manner, either because perfect policy evaluation is unavailable or because the subproblem (20a) cannot be solved exactly. To accommodate these practical considerations, this subsection generalizes our previous result by permitting inexact policy evaluation and non-zero optimization error in solving (20a). The following assumptions make precise this imperfect scenario.

**Assumption 2** (Policy evaluation error). Suppose for any \( k \geq 0 \), we have access to an estimate \( \hat{Q}_\tau^{(k)} \) obeying

\[
\|\hat{Q}_\tau^{(k)} - Q_\tau^{(k)}\|_\infty \leq \epsilon_{\text{eval}}. \tag{24}
\]

**Assumption 3** (Subproblem optimization error). Consider any policy \( \pi \) and any vector \( \xi \in \mathbb{R}^{|\mathcal{S}||\mathcal{A}|} \). Define

\[
f_s(p; \pi, \xi) := -(Q(s, \cdot), p) + \tau h_s(p) + \frac{1}{\eta} D_{h_s}(p, \pi(s, \cdot); \xi(s, \cdot)),
\]

where \( D_{h_s}(p, q; \xi) \) is defined in (15). Suppose there exists an oracle \( G_{s,\text{opt}}(Q, \pi, \xi) \), which is capable of returning \( \pi'(\cdot | s) \) such that

\[
f_s(\pi'(| s); \pi, \xi) \leq \min\limits_{p \in \Delta(\mathcal{A})} f_s(p; \pi, \xi) + \epsilon_{\text{opt}}. \tag{25}
\]

Note that the oracle in Assumption 3 can be implemented efficiently in practice via various first-order methods (Beck, 2017). Under Assumptions 2 and 3, we can modify Algorithm 1 by replacing \( \{Q_\tau^{(k)}\} \) with the estimate \( \{\hat{Q}_\tau^{(k)}\} \), and invoking the oracle \( G_{s,\text{opt}}(Q, \pi, \xi) \) to solve the subproblem (20a) approximately. The whole procedure, which we shall often refer to as approximate GPMD, is summarized in Algorithm 2.

**Algorithm 2:** Approximate PMD with generalized Bregman divergence

1. **Input:** initial policy \( \pi^{(0)} \), learning rate \( \eta > 0 \).
2. **Initialize** \( \xi^{(0)}(s) \in \partial h_s(\pi^{(0)}(\cdot | s)) \) for all \( s \in \mathcal{S} \).
3. **for** \( k = 0, 1, \ldots \), **do**
   4. **For every** \( s \in \mathcal{S} \), **invoke the oracle to obtain** (cf. (25))
      \[
      \pi^{(k+1)}(s) = G_{s,\text{opt}}(\hat{Q}_\tau^{(k)}, \pi^{(k)}, \xi^{(k)}). \tag{26}
      \]
   5. **For every** \( (s, a) \in \mathcal{S} \times \mathcal{A} \), **compute**
      \[
      \xi^{(k+1)}(s, a) = \frac{1}{1 + \eta \tau} \xi^{(k)}(s, a) + \frac{\eta}{1 + \eta \tau} \hat{Q}_\tau^{(k)}(s, a). \tag{27}
      \]

The following theorem uncovers that approximate GPMD converges linearly—at the same rate as exact GPMD—before an error floor is hit.

**Theorem 2** (Approximate GPMD). Suppose that Assumptions 1, 2 and 3 hold. Consider any learning rate \( \eta > 0 \). Then the iterates of Algorithm 2 satisfy

\[
\|Q_\tau^* - Q_\tau^{(k+1)}\|_\infty \leq \gamma \left( (1 - (1 - \alpha)(1 - \gamma))^k C_1 + C_2 \right), \tag{28}
\]

where \( \alpha := \frac{1}{1 + \eta \tau} \), \( C_1 \) is defined in Theorem 1, and

\[
C_2 := \frac{1}{1 - \gamma} \left[ 2 + \frac{2 \gamma}{(1 - \gamma)(1 - \alpha)} \right] \epsilon_{\text{eval}} + \left( 1 + \frac{2 \gamma}{(1 - \gamma)(1 - \alpha)} \right) \epsilon_{\text{opt}}.
\]

In addition, if \( h_s \) is 1-strongly convex w.r.t. the \( \ell_1 \) norm for any \( s \in \mathcal{S} \), then we can further obtain

\[
\|Q_\tau^* - Q_\tau^{(k+1)}\|_\infty \leq \gamma \left( (1 - (1 - \alpha)(1 - \gamma))^k C_1 + C_3 \right), \tag{29a}
\]

10
\[ \| \pi^*_r(\cdot | s) - \pi^*(\cdot | s) \|_1 \leq \tau^{-1} \left[ (1 - (1 - \alpha)(1 - \gamma))^k C_1 + C_3 \right] + \sqrt{\frac{2\gamma\epsilon_{\text{opt}}}{1 + \eta \tau}}, \quad (29b) \]

where
\[ C_3 := \frac{1}{1 - \gamma} \left[ \left( 2 + \frac{\epsilon_{\text{eval}}}{\tau(1 - \gamma)} \right) \epsilon_{\text{eval}} + \left( 1 + \frac{4\gamma}{(1 - \gamma)(1 - \alpha)} \right) \epsilon_{\text{opt}} \right]. \quad (30) \]

**Remark 1.** In the special case where \( \epsilon_{\text{opt}} = 0 \) and \( \eta = \infty \), Algorithm 2 reduces to regularized policy iteration, and the convergence result can be simplified as follows
\[ \| Q^*_r - Q^*_\tau \|_\infty \leq \gamma^k \| Q^*_r - Q^{(0)}_\tau \|_\infty + \frac{2\gamma\epsilon_{\text{eval}}}{(1 - \gamma)^2}. \]

**Remark 2.** When \( h_s \) is taken as the negative entropy, our result strengthens the prior result established in Cen et al. (2020) for approximate entropy-regularized NPG method with \( \epsilon_{\text{eval}} = 0 \). Specifically, (29) and (30) give a tighter bound on the final error in terms of \( \epsilon_{\text{eval}} \), since the factor does not depend on the learning rate \( \eta \). In comparison, the error bound in Cen et al. (2020) scales inversely as \( \eta \) approaches 0.

Roughly speaking, approximate GPMD is guaranteed to converge linearly to an error bound that scales linearly in both the policy evaluation error \( \epsilon_{\text{eval}} \) and the optimization error \( \epsilon_{\text{opt}} \), thus confirming the stability of our algorithm vis-à-vis imperfect implementation of the algorithm. As before, our theory improves upon prior works by demonstrating linear convergence for a full range of learning rates even in the absence of strong convexity and smoothness.

### 4 Analysis for exact GPMD (Theorem 1)

In this section, we present the analysis for our main result in Theorem 1. Here and throughout, we shall often employ the following shorthand notation when it is clear from the context:
\[ \pi^{(k)}(s) := \pi^{(k)}(\cdot | s) \in \Delta(A), \quad Q^\pi(s) := Q^\pi(s, \cdot) \in \mathbb{R}^{|A|}, \]
\[ \xi^{(k)}(s) := \xi^{(k)}(s, \cdot) \in \mathbb{R}^{|A|}, \quad Q^\pi_\tau(s) := Q^\pi_\tau(s, \cdot) \in \mathbb{R}^{|A|}. \quad (31) \]

in addition to those already defined in (13).

#### 4.1 Preparation: basic facts

In this subsection, we single out a few basic results that underlie the proof of our main theorems.

**Performance improvement.** To begin with, we demonstrate that GPMD enjoys a sort of monotonic improvements concerning the updates of both the value function and the Q-function, as stated in the following lemma. This lemma can be viewed as a generalization of the well-established policy improvement lemma in the analysis of NPG (Agarwal et al., 2020; Cen et al., 2020) as well as PMD (Lan, 2021).

**Lemma 2** (Pointwise monotonicity). For any \( (s, a) \in S \times A \) and any \( k \geq 0 \), Algorithm 1 achieves
\[ V^{(k+1)}_\tau(s) \geq V^{(k)}_\tau(s) \quad \text{and} \quad Q^{(k+1)}_\tau(s, a) \geq Q^{(k)}_\tau(s, a). \quad (32) \]

**Proof.** See Appendix A.2.

Interestingly, the above monotonicity holds simultaneously for all state-action pairs, and hence can be understood as a kind of pointwise monotonicity.
**Generalized Bellman operator.** Another key ingredient of our proof lies in the use of a generalized Bellman operator \( T_{\tau,h} : \mathbb{R}^{|S||A|} \to \mathbb{R}^{|S||A|} \) associated with the regularizer \( h = \{h_s\}_{s \in S} \). Specifically, for any state-action pair \((s,a)\) and any vector \(Q \in \mathbb{R}^{|S||A|}\), we define
\[
T_{\tau,h}(Q)(s,a) = r(s,a) + \gamma \mathbb{E}_{s' \sim P(\cdot |s,a)} \left[ \max_{p \in \Delta(A)} \left\{ \langle Q(s'), p \rangle - \tau h_{s'}(p) \right\} \right].
\]
It is worth noting that this definition shares similarity with the regularized Bellman operator proposed in Geist et al. (2019), where the operator defined there is targeted at \( V_\tau \), while ours is defined w.r.t. \( Q_\tau \).

The importance of this generalized Bellman operator is two-fold: it enjoys a desired contraction property, and its fixed point corresponds to the optimal regularized Q-function. These are generalization of the properties for the classical Bellman operator, and are formally stated in the following lemma. The proof is deferred to Appendix A.3.

**Lemma 3** (Properties of the generalized Bellman operator). For any \( \tau > 0 \), the operator \( T_{\tau,h} \) defined in (33) satisfies the following properties:

- \( T_{\tau,h} \) is a contraction operator w.r.t. the \( \ell_\infty \) norm, namely, for any \( Q_1, Q_2 \in \mathbb{R}^{|S||A|} \), one has
  \[
  \|T_{\tau,h}(Q_1) - T_{\tau,h}(Q_2)\|_\infty \leq \gamma \|Q_1 - Q_2\|_\infty.
  \]

- The optimal regularized Q-function \( Q_\tau^* \) is a fixed point of \( T_{\tau,h} \), that is,
  \[
  T_{\tau,h}(Q_\tau^*) = Q_\tau^*.
  \]

### 4.2 Proof of Theorem 1

Inspired by Cen et al. (2020), our proof consists of (1) characterizing the dynamics of \( \ell_\infty \) errors and establishing a connection to a useful linear system with two variables, and (2) analyzing the dynamics of this linear system directly. In what follows, we elaborate on each of these steps.

**Step 1: error contraction and its connection to a linear system.** With the assistance of the above preparations, we are ready to elucidate how to characterize the convergence behavior of \( \|Q_\tau^* - Q_\tau^{(k+1)}\|_\infty \). Recalling the update rule of \( \xi^{(k+1)} \) (cf. (20c)), we can deduce that
\[
Q_\tau^* - \tau \xi^{(k+1)} = \alpha (Q_\tau^* - \tau \xi^{(k)}) + (1 - \alpha) (Q_\tau^* - Q_\tau^{(k)})
\]
with \( \alpha = \frac{1}{1 + \eta \tau} \), thus indicating that
\[
\|Q_\tau^* - \tau \xi^{(k+1)}\|_\infty \leq \alpha \|Q_\tau^* - \tau \xi^{(k)}\|_\infty + (1 - \alpha) \|Q_\tau^* - Q_\tau^{(k)}\|_\infty.
\]

Interestingly, there exists an intimate connection between \( \|Q_\tau^* - Q_\tau^{(k+1)}\|_\infty \) and \( \|Q_\tau^* - \tau \xi^{(k+1)}\|_\infty \) that allows us to bound the former term by the latter. This is stated in the following lemma, with the proof postponed to Appendix A.4.

**Lemma 4.** Set \( \alpha = \frac{1}{1 + \eta \tau} \). The iterates of Algorithm 1 satisfy
\[
\|Q_\tau^* - Q_\tau^{(k+1)}\|_\infty \leq \gamma \|Q_\tau^* - \tau \xi^{(k+1)}\|_\infty + \gamma \alpha^{k+1} \|Q_\tau^0 - \tau \xi^{(0)}\|_\infty.
\]

The above inequalities (36) and (37) can be succinctly described via a useful linear system with two variables \( \|Q_\tau^* - Q_\tau^{(k)}\|_\infty \) and \( \|Q_\tau^* - \tau \xi^{(k)}\|_{\infty} \), that is,
\[
x_{k+1} \leq Ax_k + \gamma \alpha^{k+1} y,
\]
where
\[
A := \begin{bmatrix} \gamma(1 - \alpha) & \gamma \alpha \\ 1 - \alpha & \alpha \end{bmatrix}, \quad x_k := \begin{bmatrix} \|Q_\tau^* - Q_\tau^{(k)}\|_\infty \\ \|Q_\tau^* - \tau \xi^{(k)}\|_\infty \end{bmatrix} \quad \text{and} \quad y := \begin{bmatrix} \|Q_\tau^0 - \tau \xi^{(0)}\|_\infty \\ 0 \end{bmatrix}.
\]

This forms the basis for proving Theorem 1.
Step 2: analyzing the dynamics of the linear system (38). Before proceeding, we note that a linear system similar to (38) has been analyzed in Cen et al. (2020, Section 4.2.2). We intend to apply the following properties that have been derived therein:

\[
x_{k+1} \leq A^{k+1} \left[ x_0 + \gamma (\alpha^{-1} A - I)^{-1} y \right],
\]

\[
\gamma (\alpha^{-1} A - I)^{-1} y = \begin{bmatrix}
0 \\
\|Q^0 - \tau \xi^0\|_\infty
\end{bmatrix},
\]

\[
A^{k+1} = ((1 - \alpha) \gamma + \alpha)^k \begin{bmatrix}
\gamma \\ 1
\end{bmatrix} [1 - \alpha \quad \alpha].
\]

Substituting (40c) and (40b) into (40a) and rearranging terms, we reach

\[
x_{k+1} \leq ((1 - \alpha) \gamma + \alpha)^k \begin{bmatrix}
(1 - \alpha) \|Q^* - Q^0\|_\infty + \alpha \|Q^* - \tau \xi^0\|_\infty + \alpha \|Q^0 - \tau \xi^0\|_\infty
\end{bmatrix} \begin{bmatrix}
\gamma \\ 1
\end{bmatrix},
\]

which taken together with the definition of \(x_{k+1}\) gives

\[
\|Q^* - Q^{(k+1)}\|_\infty \leq \gamma ((1 - \alpha) \gamma + \alpha)^k \begin{bmatrix}
\|Q^* - Q^0\|_\infty + 2\alpha \|Q^* - \tau \xi^0\|_\infty
\end{bmatrix},
\]

\[
\|Q^* - \tau \xi^{(k+1)}\|_\infty \leq ((1 - \alpha) \gamma + \alpha)^k \begin{bmatrix}
\|Q^* - Q^0\|_\infty + 2\alpha \|Q^* - \tau \xi^0\|_\infty
\end{bmatrix}.
\]

Step 3: controlling \(\|\pi^*_s(s) - \pi^{(k+1)}(s)\|_1\). It remains to convert this result to an upper bound on \(\|\pi^*_s(s) - \pi^{(k+1)}(s)\|_1\). By virtue of Lemma 1, there exist two vectors \(g^* \in \partial h_s(\pi^*_s(s)), g' \in \partial h_s(\pi^{(k+1)}(s))\) and two scalars \(c^*_s, c^{(k+1)}_s \in \mathbb{R}\) that satisfy

\[
\tau^{-1} Q^*_s(s) - \xi^{(k+1)}(s) = g^* - c^*_s 1 - (g' - c^{(k+1)}_s 1).
\]

When \(h_s\) is 1-strongly convex w.r.t. the \(\ell_1\) norm, we can invoke the strong monotonicity property of a strongly convex function (Beck, 2017, Theorem 5.24) to obtain

\[
\|\pi^*_s(s) - \pi^{(k+1)}(s)\|_1 \leq \langle \pi^*_s(s) - \pi^{(k+1)}(s), g^* - g' \rangle
\]

\[
= \langle \pi^*_s(s) - \pi^{(k+1)}(s), g^* - c^*_s 1 - g' + c^{(k+1)}_s 1 \rangle
\]

\[
\leq \|\pi^*_s(s) - \pi^{(k+1)}(s)\|_1 \|g^* - c^*_s 1 - g' + c^{(k+1)}_s 1\|_\infty
\]

\[
= \tau^{-1} \|\pi^*_s(s) - \pi^{(k+1)}(s)\|_1 \|Q^*_s(s) - \tau \xi^{(k+1)}(s)\|_\infty,
\]

where the second line is valid since \(\langle \pi^*_s(s), 1 \rangle = \langle \pi^{(k+1)}(s), 1 \rangle = 1\). This taken together with (42) gives rise to the advertised bound

\[
\|\pi^*_s(s) - \pi^{(k+1)}(s)\|_1 \leq \tau^{-1} \|Q^*_s(s) - \tau \xi^{(k+1)}(s)\|_\infty
\]

\[
\leq \tau^{-1} ((1 - \alpha) \gamma + \alpha)^k \begin{bmatrix}
\|Q^*_s(s) - Q^0\|_\infty + 2\alpha \|Q^*_s - \tau \xi^0\|_\infty
\end{bmatrix}.
\]

5 Numerical experiments

In this section, we provide some simple numerical experiments to corroborate the effectiveness of the GPMD algorithm.

5.1 Tsallis entropy

While Shannon entropy is a popular choice of regularization, the discrepancy between the value function of the regularized MDP and the un-regularized counterpart scales as \(O\left(\frac{1}{1-\gamma} \log |A|\right)\). In addition, the optimal
5.2 Constrained RL

In reality, an agent with the sole aim of maximizing cumulative rewards might sometimes end up with unintended or even harmful behavior, due to, say, improper design of the reward function, or non-perfect simulation of physical laws. Therefore, it is sometimes necessary to enforce proper constraints on the policy in order to prevent it from taking certain actions too frequently.

To simulate this problem, we first solve a MDP with \(|S| = 200, |A| = 50\) that is generated in the same way as in the previous section and then pick 10 state-action pairs from the support of the optimal policy at random to form a set \(\Psi\). We can ensure that \(\pi^*_s(a\mid s) < \pi_{\text{max}} = 0.1\) for all \((s,a) \in \Psi\) by adding the following log-barrier regularization with \(\tau = 0.001\):

\[
h_s(p) = \begin{cases} 
\infty, & \text{if } (s,a) \in \Psi \text{ and } p(a) \geq \pi_{\text{max}}, \\
- \log \left( \pi_{\text{max}} - p(a) \right), & \text{if } (s,a) \in \Psi \text{ and } p(a) < \pi_{\text{max}}, \\
0, & \text{otherwise.}
\end{cases}
\]

Figure 1: \(|Q^*_\tau - Q^{(t)}_\tau|_\infty\) versus the iteration count for both PMD and GPMD, when the learning rate \(\eta\) varies. The left plot (a) is concerned with Tsallis entropy regularization, whereas the right plot (b) concerns with log-barrier regularization used in our constrained RL example. The error curves are averaged over 5 independent runs.

We now evaluate numerically the performance of PMD and GPMD when applied to a randomly generated MDP with \(|S| = 200, |A| = 50\). Here, the transition probability kernel and the reward function is generated as follows. For each state-action pair \((s,a)\), we randomly select 20 states to form a set \(S_{s,a}\), and set \(P(s'\mid s,a) = 1/20\) if \(s' \in S_{s,a}\), and 0 otherwise. The reward function is generated by \(r(s,a) \sim U_{s,a} \cdot U_s\), where \(U_{s,a}\) and \(U_s\) are independent uniform random variables over \([0,1]\). We shall set the regularizer as \(h_s(p) = -\nabla_{\text{Tsallis}}(p)\) for all \(s \in S\) with a regularization parameter \(\tau = 0.001\). As can been from the numerical results displayed in Figure 1 (a), GPMD enjoys a faster convergence rate compared to PMD.
Numerical comparisons of PMD and GPMD when applied to this problem are plotted in Figure 1 (b). It is observed that PMD methods stall after reaching an error floor on the order of $10^{-2}$, while GPMD methods are able to converge to the optimal policy efficiently.

6 Discussion

The present paper has introduced a generalized framework of policy optimization tailored to regularized RL problems. We have proposed a generalized policy mirror descent (GPMD) algorithm that achieves dimension-free linear convergence, which covers an entire range of learning rates and accommodates convex and possibly nonsmooth regularizers. Numerical experiments have been conducted to demonstrate the utility of the proposed GPMD algorithm. Our approach opens up a couple of future directions that are worthy of further exploration. For example, it is of interest to develop policy optimization algorithms when the regularizers are nonconvex and when sophisticated policy parameterization is adopted. Understanding the sample complexities of the proposed algorithm—when the policies are evaluated using samples collected over an online trajectory—is of interest to sample-constrained scenarios. Furthermore, it might be worthwhile to extend the proposed algorithm to accommodate multi-agent RL, with a representative example being regularized Markov games (Zhao et al., 2021).

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A Proof of key lemmas

In this section, we collect the proof of several key lemmas. Here and throughout, we use $\mathbb{E}_\pi[\cdot]$ to denote the expectation over the randomness of the MDP induced by policy $\pi$. We shall follow the notation convention in (31) throughout. In addition, to further simplify notation, we shall abuse the notation by letting

\[
D_{h_s}(\bar{\pi}, \pi; \xi) := D_{h_s}(\bar{\pi}(s), \pi(s); \xi(s)) \tag{44a}
\]

\[
D_{h_s}(p, \pi; \xi) := D_{h_s}(p, \pi(s); \xi(s)) \tag{44b}
\]

\[
D_{h_s}(\pi, p; \xi) := D_{h_s}(\pi(s), p; \xi(s)) \tag{44c}
\]

for any policy $\pi$ and $\bar{\pi}$ and any $p \in \Delta(A)$, whenever it is clear from the context.

A.1 Proof of Lemma 1

We start by relaxing the probability simplex constraint (i.e., $p \in \Delta(A)$) in (20a) with a simpler linear constraint $\sum_{a \in A} p(a) = 1$ as follows

\[
\begin{align*}
\text{minimize}_{p \in \mathbb{R}^{|A|}} & \quad -\eta \langle Q^{(k)}(s), p \rangle + \eta \tau h_s(p) + D_{h_s}(p, \pi^{(k)}; \xi^{(k)}) \\
\text{subject to} & \quad \sum_{a \in A} p(a) = 1.
\end{align*} \tag{45}
\]

To justify the validity of dropping the non-negative constraint, we note that for any $p$ obeying $p(a) < 0$ for some $a \in A$, our assumption on $h_s$ leads to $h_s(p) = \infty$, which cannot possibly be the optimal solution. This confirms the equivalence between (20a) and (45).

Observe that the Lagrangian w.r.t. (45) is given by

\[
\mathcal{L}_s(p, \lambda^{(k)}_s) = -\eta \langle Q^{(k)}(s), p \rangle + \eta \tau h_s(p) + h_s(p) - h_s(\pi^{(k)}(s)) - \langle p - \pi^{(k)}(s), \xi^{(k)}(s) \rangle + \lambda^{(k)}(\sum_{a \in A} p(a) - 1),
\]
where $\lambda^{(k)}_s \in \mathbb{R}$ denotes the Lagrange multiplier associated with the constraint $\sum_{a \in A} p(a) = 1$. Given that $\pi^{(k+1)}(s)$ is the solution to (20a) and hence (45), the optimality condition requires that

$$0 \in \partial_p L_s(p, \lambda^{(k)}_s) \big|_{p = \pi^{(k+1)}(s)} = -\eta Q^{(k)}_\tau(s) + (1 + \eta \tau)\partial h_s(\pi^{(k+1)}(s)) - \xi^{(k)}(s) + \lambda^{(k)}_s 1.$$  

Rearranging terms and making use of the construction (17), we are left with

$$\xi^{(k+1)}(s) - \frac{\lambda^{(k)}_s}{1 + \eta \tau} = \frac{1}{1 + \eta \tau} \left[ \eta Q^{(k)}_\tau(s) + \xi^{(k)}(s) - \lambda^{(k)}_s \right] \in \partial h_s(\pi^{(k+1)}(s)),$$

thus concluding the proof of the first claim (18).

We now turn to the second claim (19). In view of the property (35), we have

$$\pi^*_s(s) = \arg \min_{p \in \Delta(A)} -\langle Q^*_\tau(s), p \rangle + \tau h_s(p).$$

This optimization problem is equivalent to

$$\begin{align*}
\text{minimize}_{p \in \mathbb{R}^{|A|}} & \quad -\langle Q^*_\tau(s), p \rangle + \tau h_s(p), \\
\text{subject to} & \quad \sum_{a \in A} p(a) = 1,
\end{align*}$$

which can be verified by repeating a similar argument for (45). The Lagrangian associated with (46) is

$$L_s(p, \lambda^*_s) = -\langle Q^*_\tau(s), p \rangle + \tau h_s(p) + \lambda^*_s \left( \sum_{a \in A} p(a) - 1 \right),$$

where $\lambda^*_s \in \mathbb{R}$ denotes the Lagrange multiplier. Therefore, the first-order optimality condition requires that

$$0 \in \partial_p L_s(p, \lambda^*_s) \big|_{p = \pi^*_s(s)} = -Q^*_\tau(s) + \partial h_s(\pi^*_s(s)) + \lambda^*_s 1,$$

which finishes the proof.

### A.2 Proof of Lemma 2

We start by introducing the performance difference lemma that has previously been derived in Lan (2021, Lemma 2). For the sake of self-containedness, we include a proof of this lemma in Appendix A.2.1.

**Lemma 5** (Performance difference). For any two policies $\pi$ and $\pi'$, we have

$$V^\pi(s) - V^\pi'(s) = \frac{1}{1 - \gamma} \mathbb{E} \left[ \langle Q^\pi(s'), \pi'(s') - \pi(s') \rangle - \tau h_{\pi'}(\pi'(s')) + \tau h_{\pi'}(\pi(s')) \right],$$

where $d^\pi_s$ has been defined in (5).

Armed with Lemma 5, one can readily rewrite the difference $V^{(k+1)}_\tau(s) - V^{(k)}_\tau(s)$ between two consecutive iterates as follows

$$V^{(k+1)}_\tau(s) - V^{(k)}_\tau(s) = \frac{1}{1 - \gamma} \mathbb{E} \left[ \langle Q^{(k)}_\tau(s'), \pi^{(k+1)}(s') - \pi^{(k)}(s') \rangle - \tau h_{\pi'}(\pi^{(k+1)}(s')) + \tau h_{\pi'}(\pi^{(k)}(s')) \right].$$

It then comes down to studying the right-hand side of the relation (48), which can be accomplished via the following “3-point” lemma. The proof of this lemma can be found in Appendix A.2.2.

**Lemma 6.** For any $s \in S$ and any vector $p \in \Delta(A)$, we have

$$(1 + \eta \tau)D_{h_s}(p, \pi^{(k+1)}; \xi^{(k+1)}) + D_{h_s}(\pi^{(k+1)}, \pi^{(k)}; \xi^{(k)}) - D_{h_s}(p, \pi^{(k)}; \xi^{(k)}) = \eta \left[ \langle Q^*_\tau(s), \pi^{(k+1)}(s) - p \rangle + \tau h_s(p) - \tau h_s(\pi^{(k+1)}(s)) \right].$$
Taking \( p = \pi^{(k)}(s) \) in Lemma 6 and combining it with (48), we arrive at

\[
V^{(k+1)}_\tau(s) - V^{(k)}_\tau(s) = \frac{1}{(1 - \gamma)\eta} \sum_{a' \sim \pi} \mathbb{E}_{s' \sim \mathbb{D}_{s+1}} \left[ (1 + \eta\tau)D_{h_{s'}}(\pi^{(k)}, \pi^{(k+1)}; \xi^{(k+1)}) + D_{h_{s'}}(\pi^{(k+1)}, \pi^{(k)}; \xi^{(k)}) \right] \geq 0
\]

for any \( s \in S \), thus establishing the advertised pointwise monotonicity w.r.t. the regularized value function.

When it comes to the regularized Q-function, it is readily seen from the definition (9a) that

\[
Q^{(k+1)}_\tau(s, a) = r(s, a) + \gamma \sum_{s' \sim \mathbb{P}(\pi, s, a)} [V^{(k+1)}_\tau(s')] \geq r(s, a) + \gamma \sum_{s' \sim \mathbb{P}(\pi, s, a)} [V^{(k)}_\tau(s')] = Q^{(k)}_\tau(s, a)
\]

for any \((s, a) \in S \times A\), where the last line is valid since \( V^{(k+1)}_\tau \geq V^{(k)}_\tau \). This concludes the proof.

A.2.1 Proof of Lemma 5

For any two policies \( \pi' \) and \( \pi \), it follows from the definition (7) of \( V^{\pi}_\tau(s) \) that

\[
V^{\pi'}_\tau(s) - V^{\pi}_\tau(s) = \mathbb{E}_{\pi'} \left[ \sum_{t=0}^{\infty} \gamma^t \left[ r(s_t, a_t) - \tau h_{s_t}(\pi'(s_t)) \right] \middle| s_0 = s \right] - V^{\pi}_\tau(s) \]

\[
= \mathbb{E}_{\pi'} \left[ \sum_{t=0}^{\infty} \gamma^t \left[ r(s_t, a_t) - \tau h_{s_t}(\pi'(s_t)) + V^{\pi}_\tau(s_t) - V^{\pi}_\tau(s_t) \right] \middle| s_0 = s \right] - V^{\pi}_\tau(s) \]

\[
= \mathbb{E}_{\pi'} \left[ \sum_{t=0}^{\infty} \gamma^t \left[ r(s_t, a_t) - \tau h_{s_t}(\pi'(s_t)) + \gamma V^{\pi}_\tau(s_{t+1}) - V^{\pi}_\tau(s_t) \right] \middle| s_0 = s \right] + \mathbb{E}_{\pi'} \left[ V^{\pi}_\tau(s_0) \middle| s_0 = s \right] - V^{\pi}_\tau(s) \]

\[
= \mathbb{E}_{\pi'} \left[ \sum_{t=0}^{\infty} \gamma^t \left[ r(s_t, a_t) - \tau h_{s_t}(\pi'(s_t)) + \gamma V^{\pi}_\tau(s_{t+1}) - V^{\pi}_\tau(s_t) \right] \middle| s_0 = s \right] \]

\[
= \mathbb{E}_{\pi'} \left[ \sum_{t=0}^{\infty} \gamma^t \left[ Q^{\pi}_\tau(s_t, a_t) - \tau h_{s_t}(\pi(s_t)) - V^{\pi}_\tau(s_t) - \tau h_{s_t}(\pi'(s_t)) + \tau h_{s_t}(\pi(s_t)) \right] \middle| s_0 = s \right] \]

\[
= \frac{1}{1 - \gamma} \sum_{s' \sim d^{\pi}_\tau} \mathbb{E} \left[ (Q^{\pi'}_\tau(s'), \pi'(s') - \pi(s')) - \tau h_{s'}(\pi'(s')) + \tau h_{s'}(\pi(s')) \right], \tag{49}
\]

where the penultimate line comes from the definition (9a). To see why the last line of (49) is valid, we make note of the following identity

\[
\mathbb{E}_{a_t \sim \pi(s_t)} \left[ Q^{\pi'}_\tau(s_t, a_t) - \tau h_{s_t}(\pi(s_t)) - V^{\pi}_\tau(s_t) \right] \]

\[
= \mathbb{E}_{a_t \sim \pi(s_t)} \left[ Q^{\pi'}_\tau(s_t, a_t) - \tau h_{s_t}(\pi(s_t)) \right] - \mathbb{E}_{a_t \sim \pi(s_t)} \left[ Q^{\pi}_\tau(s_t, a_t) - \tau h_{s_t}(\pi(s_t)) \right] \]

\[
= \left( Q^{\pi'}_\tau(s_t) - \tau h_{s_t}(\pi(s_t)) \right) \cdot 1, \pi'(s_t) - \pi(s_t) \]

\[
= \left( Q^{\pi'}_\tau(s_t), \pi'(s_t) - \pi(s_t) \right), \tag{50}
\]

where the first identity results from the relation (9b), and the last relation holds since \( 1^T \pi'(s_t) = 1^T \pi(s_t) = 1 \). The last line of (49) then follows immediately from the relation (50) and the definition (5) of \( d^{\pi}_\tau \).
A.2.2 Proof of Lemma 6

For any state \( s \in S \), we make the observation that
\[
D_{h_s}(p, \pi^{(k)}; \xi^{(k)}) = h_s(p) - h_s(\pi^{(k)}(s)) - \langle p - \pi^{(k)}(s), \xi^{(k)}(s) \rangle
\]
\[
= h_s(p) - h_s(\pi^{(k+1)}(s)) - \langle p - \pi^{(k+1)}(s), \xi^{(k)}(s) \rangle
+ h_s(\pi^{(k+1)}(s)) - h_s(\pi^{(k)}(s)) - \langle \pi^{(k+1)}(s) - \pi^{(k)}(s), \xi^{(k)}(s) \rangle
\]
\[
= h_s(p) - h_s(\pi^{(k+1)}(s)) - \langle p - \pi^{(k+1)}(s), \xi^{(k+1)}(s) \rangle
+ h_s(\pi^{(k+1)}(s)) - h_s(\pi^{(k)}(s)) - \langle \pi^{(k+1)}(s) - \pi^{(k)}(s), \xi^{(k)}(s) \rangle
+ \langle p - \pi^{(k+1)}(s), \xi^{(k+1)}(s) - \xi^{(k)}(s) \rangle
\]
\[
= D_{h_s}(p, \pi^{(k+1)}; \xi^{(k+1)}) + D_{h_s}(\pi^{(k+1)}, \pi^{(k)}; \xi^{(k)}) + \langle p - \pi^{(k+1)}(s), \xi^{(k+1)}(s) - \xi^{(k)}(s) \rangle
\]
where the first and the fourth steps invoke the definition (15) of the generalized Bregman divergence and the last line results from the update rule (20c). Rearranging terms, we are left with
\[
\eta \langle Q^{(k)}_\tau(s), \pi^{(k+1)}(s) - p \rangle
= \left\{ D_{h_s}(p, \pi^{(k+1)}; \xi^{(k+1)}) + D_{h_s}(\pi^{(k+1)}, \pi^{(k)}; \xi^{(k)}) - D_{h_s}(p, \pi^{(k)}; \xi^{(k)}) \right\}
+ \eta \tau \langle \xi^{(k+1)}(s), \pi^{(k+1)}(s) - p \rangle.
\]

Adding the term \( \eta \tau \{ h_s(p) - h_s(\pi^{(k+1)}(s)) \} \) to both sides of this identity leads to
\[
\eta \left[ \langle Q^{(k)}_\tau(s), \pi^{(k+1)}(s) - p \rangle + \tau h_s(p) - \tau h_s(\pi^{(k+1)}(s)) \right]
= \left\{ D_{h_s}(p, \pi^{(k+1)}; \xi^{(k+1)}) + D_{h_s}(\pi^{(k+1)}, \pi^{(k)}; \xi^{(k)}) - D_{h_s}(p, \pi^{(k)}; \xi^{(k)}) \right\}
+ \eta \tau \left( h_s(p) - h_s(\pi^{(k+1)}(s)) - \langle \xi^{(k+1)}(s), p - \pi^{(k+1)}(s) \rangle \right)
= (1 + \eta \tau) D_{h_s}(p, \pi^{(k+1)}; \xi^{(k+1)}) + D_{h_s}(\pi^{(k+1)}, \pi^{(k)}; \xi^{(k)}) - D_{h_s}(p, \pi^{(k)}; \xi^{(k)})
\]
as claimed, where the last line makes use of the definition (15).

A.3 Proof of Lemma 3

In the sequel, we shall prove each claim in Lemma 3 separately.

**Proof of the contraction property** (34). For any \( Q_1, Q_2 \in \mathbb{R}^{||S||A} \), the definition (33) of the generalized Bellman operator obeys
\[
\mathcal{T}_{\tau,h}(Q_1) - \mathcal{T}_{\tau,h}(Q_2) = \gamma \mathbb{E}_{s' \sim P(|s,a)} \left[ \max_{p \in \Delta(A)} \left\{ \langle Q_1(s'), p \rangle - \tau h_s(p) \right\} - \max_{p \in \Delta(A)} \left\{ \langle Q_2(s'), p \rangle - \tau h_s(p) \right\} \right]
\leq \gamma \mathbb{E}_{s' \sim P(|s,a)} \left[ \max_{p \in \Delta(A)} \langle Q_1(s') - Q_2(s'), p \rangle \right]
\leq \gamma \mathbb{E}_{s' \sim P(|s,a)} \left[ \max_{||p||_1 = 1} \| Q_1 - Q_2 \|_{\infty} ||p||_1 \right]
= \gamma \| Q_1 - Q_2 \|_{\infty},
\]
where (a) arises from the elementary fact \( \max_x f(x) - \max_x g(x) \leq \max_x (f(x) - g(x)) \).
Proof of the fixed point property (35). Towards this, let us first define

$$\pi^\dagger(s) := \arg \max_{p, \in \Delta(A)} E \left[ Q^\star(s,a) - \tau h_s(p(s)) \right].$$

(51)

Then it can be easily verified that

$$Q^\star(s,a) = r(s,a) + \gamma \sum_{s_1 \sim P(s|s,a)} \left[ Q^\star(s_1,a_1) - \tau h_{s_1}(\pi^\star(s_1)) \right],$$

(52)

where the first identity results from (9), and the second line arises from the maximizing property of $\pi^\dagger$ (see (51)).

Note that the right-hand side of (52) involves the term $Q^\star(s_1,a_1)$, which can be further upper bounded via the same argument for (52). Successively repeating this upper bound argument (and the expansion) eventually allows one to obtain

$$Q^\star(s,a) \leq r(s,a) + \gamma \sum_{s_1 \sim P(s|s,a)} \left[ Q^\star(s_1,a_1) - \tau h_{s_1}(\pi^\star(s_1)) \right].$$

(53)

However, the fact that $\pi^\star$ is the optimal policy necessarily implies the following reverse inequality:

$$Q^\star(s,a) \geq Q^\pi^\dagger(s,a).$$

Therefore, one must have

$$Q^\star(s,a) = Q^\pi^\dagger(s,a).$$

(53)

To finish up, it suffices to show that $Q^\pi^\dagger = T_{\tau,h}(Q^\star)$. To this end, it is observed that

$$Q^\pi^\dagger(s,a) = r(s,a) + \gamma \sum_{s_1 \sim P(s|s,a)} \left[ Q^\pi^\dagger(s_1,a_1) - \tau h_{s_1}(\pi^\dagger(s_1)) \right].$$

(51)

(52)

where (b) utilizes the fact (53), (c) follows from the definition (51) of $\pi^\dagger$, and the last identity is a consequence of the definition (33) of $T_{\tau,h}$. The above results taken collectively demonstrate that $Q^\star = T_{\tau,h}(Q^\star)$ as claimed.

A.4 Proof of Lemma 4

Recall that $Q^{(k+1)} = Q^{(k+1)}$. In view of the relation (9), one obtains

$$Q^{(k+1)}(s,a) = r(s,a) + \gamma \sum_{s' \sim P(s|s,a)} \left[ V^{(k+1)}(s') \right].$$

(51)

$$= r(s,a) + \gamma \sum_{s' \sim P(s|s,a)} \left[ Q^{(k+1)}(s', a') - \tau h_{s'}(\pi^{(k+1)}(s')) \right]$$

(52)

$$= r(s,a) + \gamma \sum_{s' \sim P(s|s,a)} \left[ Q^{(k+1)}(s', \pi^{(k+1)}(s')) - \tau h_{s'}(\pi^{(k+1)}(s')) \right].$$
This combined with the fixed-point condition (35) allows us to derive
\[
Q_\tau^*(s,a) - Q_\tau^{(k+1)}(s,a) = T_{\tau,h}(Q_\tau^*)(s,a) - \left\{ r(s,a) + \gamma \mathbb{E}_{s' \sim P_{\tau}(s,a)} \left[ \langle Q_\tau^{(k+1)}(s'), \pi^{(k+1)}(s') \rangle - \tau h_{s'}(\pi^{(k+1)}(s')) \right] \right\}
\]
\[
= T_{\tau,h}(Q_\tau^*)(s,a) - \left\{ r(s,a) + \gamma \mathbb{E}_{s' \sim P_{\tau}(s,a)} \left[ \langle \tau \xi^{(k+1)}(s'), \pi^{(k+1)}(s') \rangle - \tau h_{s'}(\pi^{(k+1)}(s')) \right] \right\}
\]
\[
- \gamma \mathbb{E}_{s' \sim P_{\tau}(s,a)} \mathbb{E}_{a' \sim \pi^{(k+1)}(s')} [Q_\tau^{(k+1)}(s',a') - \tau \xi^{(k+1)}(s',a')] \, . \tag{54}
\]

In what follows, we control each term on the right-hand side of (61) separately.

**Step 1: bounding the 1st term on the right-hand side of (61).** Lemma 1 tells us that
\[
\xi^{(k+1)}(s) - c_s^{(k+1)} \in \partial h_s(\pi^{(k+1)}(s))
\]
f for some scalar \( c_s^{(k+1)} \in \mathbb{R} \). This important property allows one to derive
\[
0 \in -\xi^{(k+1)}(s) + c_s^{(k+1)} + \partial h_s(\pi^{(k+1)}(s)) = \partial \mathcal{L}_{k+1,s}(\pi^{(k+1)}(s); c_s^{(k+1)}), \tag{55}
\]
where
\[
\mathcal{L}_{k+1,s}(p; \lambda) := -\langle \xi^{(k+1)}(s), p \rangle + h_s(p) + \lambda 1^\top p.
\]

Recognizing that the function \( f_{k+1,s}(\cdot) \) is convex in \( p \), we can view \( \mathcal{L}_{k+1,s}(p; \lambda) \) as the Lagrangian of the following constrained convex problem with Lagrangian multiplier \( \lambda \in \mathbb{R} \):
\[
\text{minimize}_{p:1^\top p = 1} \quad f_{k+1,s}(p) = -\langle \xi^{(k+1)}(s), p \rangle + h_s(p). \tag{56}
\]
The condition (55) can then be interpreted as the optimality condition w.r.t. the program (56) and \( \pi^{(k+1)}(s) \), meaning that
\[
f_{k+1,s}(\pi^{(k+1)}(s)) = \min_{p:1^\top p = 1} f_{k+1,s}(p),
\]
or equivalently,
\[
\langle \xi^{(k+1)}(s), \pi^{(k+1)}(s) \rangle - h_s(\pi^{(k+1)}(s)) = \max_{p:1^\top p = 1} \langle \xi^{(k+1)}(s), p \rangle - h_s(p). \tag{57}
\]

In addition, for any vector \( p \) that does not obey \( p \geq 0 \), Assumption 1 implies that \( h_s(p) = \infty \), and hence \( p \) cannot possibly be the optimal solution to \( \max_{p \in \Delta(A)} \langle \xi^{(k+1)}(s), p \rangle - h_s(p) \). This together with (57) essentially implies that
\[
\langle \xi^{(k+1)}(s), \pi^{(k+1)}(s) \rangle - h_s(\pi^{(k+1)}(s)) = \max_{p \in \Delta(A)} \langle \xi^{(k+1)}(s), p \rangle - h_s(p). \tag{58}
\]

As a consequence, we arrive at
\[
T_{\tau,h}(Q_\tau^*)(s,a) - \left\{ r(s,a) + \gamma \mathbb{E}_{s' \sim P_{\tau}(s,a)} \left[ \langle \tau \xi^{(k+1)}(s'), \pi^{(k+1)}(s') \rangle - \tau h_{s'}(\pi^{(k+1)}(s')) \right] \right\}
\]
\[
= T_{\tau,h}(Q_\tau^*)(s,a) - \left\{ r(s,a) + \gamma \mathbb{E}_{s' \sim P_{\tau}(s,a)} \left[ \max_{p \in \Delta(A)} \left\{ \langle \tau \xi^{(k+1)}(s'), p \rangle - \tau h_{s'}(p) \right\} \right] \right\}
\]
\[
= T_{\tau,h}(Q_\tau^*)(s,a) - T_{\tau,h}(\tau \xi^{(k+1)})(s,a)
\]
\[
\leq \gamma \|Q_\tau^* - \tau \xi^{(k+1)}\|_\infty, \tag{59}
\]
where the last step results from the contraction property (34) in Lemma 3.
Step 2: bounding the 2nd term on the right-hand side of (61). Recall that \( \alpha = \frac{1}{1 + \eta} \). Invoking the monotonicity property in Lemma 2 and the update rule (20c), we obtain
\[
Q_{\tau}^{(k+1)}(s, a) - \tau \xi^{(k+1)}(s, a) = \alpha \left\{ Q_{\tau}^{(k+1)}(s, a) - \tau \xi^{(k)}(s, a) \right\} + (1 - \alpha) \left\{ Q_{\tau}^{(k+1)}(s, a) - Q_{\tau}^{(k)}(s, a) \right\} \\
\geq \alpha \left\{ Q_{\tau}^{(k)}(s, a) - \tau \xi^{(k)}(s, a) \right\}.
\]
Repeating this lower bound argument then yields
\[
Q_{\tau}^{(k+1)}(s, a) - \tau \xi^{(k+1)}(s, a) \geq \alpha^{k+1} \left\{ Q_{\tau}^{(0)}(s, a) - \tau \xi^{(0)}(s, a) \right\} \\
\geq -\alpha^{k+1} \| Q_{\tau}^{(0)} - \tau \xi^{(0)} \|_{\infty},
\]
thus revealing that
\[
- \mathbb{E}_{s' \sim P(s, a), a' \sim \pi^{k+1}(s')} \left[ Q_{\tau}^{(k+1)}(s', a') - \tau \xi^{(k+1)}(s', a') \right] \leq \alpha^{k+1} \| Q_{\tau}^{(0)} - \tau \xi^{(0)} \|_{\infty}. \tag{60}
\]

Step 3: putting all this together. Substituting (59) and (60) into (61) gives
\[
0 \leq Q_{\tau}^{*}(s, a) - Q_{\tau}^{(k+1)}(s, a) \leq \gamma \| Q_{\tau}^{*} - \tau \xi^{(k+1)} \|_{\infty} + \alpha^{k+1} \| Q_{\tau}^{(0)} - \tau \xi^{(0)} \|_{\infty} \tag{61}
\]
for all \((s, a) \in S \times A\), thus concluding the proof.

B Analysis for approximate GPMD (Theorem 2)

The proof consists of three steps: (i) evaluating the performance difference between \( \pi^{(k)} \) and \( \pi^{(k+1)} \), (ii) establishing a linear system to characterize the error dynamic, and (iii) analyzing this linear system to derive global convergence guarantees. We shall describe the details of each step in the sequel. As before, we adopt the notational convention (44) whenever it is clear from the context.

B.1 Step 1: bounding performance difference between consecutive iterates

When only approximate policy evaluation is available, we are no longer guaranteed to have pointwise monotonicity as in the case of Lemma 2. Fortunately, we are still able to establish an approximate version of Lemma 2, as stated below.

Lemma 7 (Performance improvement for approximate GPMD). For all \( s \in S \) and all \( k \geq 0 \), we have
\[
V_{\tau}^{(k+1)}(s) \geq V_{\tau}^{(k)}(s) - \frac{1 + \alpha}{1 - \gamma} \epsilon_{\text{opt}} - \frac{2}{1 - \gamma} \epsilon_{\text{eval}}.
\]
In addition, if \( h_{s} \) is \( 1 \)-strongly convex w.r.t. the \( \ell_{1} \) norm for all \( s \in S \), then one further has
\[
V_{\tau}^{(k+1)}(s) \geq V_{\tau}^{(k)}(s) - \frac{3 + \alpha}{1 - \gamma} \epsilon_{\text{opt}} - \frac{\eta}{(2 + \eta \tau)(1 - \gamma)^{2}} \epsilon_{\text{eval}}^{2}.
\]
In words, while monotonicity is not guaranteed, this lemma precludes the possibility of \( V_{\tau}^{(k+1)}(s) \) being much smaller than \( V_{\tau}^{(k)}(s) \), as long as both \( \epsilon_{\text{eval}} \) and \( \epsilon_{\text{opt}} \) are reasonably small.

B.1.1 Proof of Lemma 7

The case when \( h_{s} \) is convex. Let \( \tilde{\pi}^{(k+1)} \) be the exact solution of the following problem
\[
\tilde{\pi}^{(k+1)}(s) = \arg \min_{p \in \Delta(A)} \left\{ -\langle Q_{\tau}^{(k)}(s), p \rangle + \tau h_{s}(p) + \frac{1}{\eta} D_{h_{s}}(p, \pi^{(k)}; \tilde{\xi}^{(k)}) \right\}. \tag{62}
\]
With this auxiliary policy iterate \( \hat{\pi}^{(k+1)} \) in mind, we start by decomposing \( V^{(k+1)}_{\tau}(s) - V^{(k)}_{\tau}(s) \) into the following three parts:

\[
V^{(k+1)}_{\tau}(s) - V^{(k)}_{\tau}(s) = \frac{1}{1 - \gamma} \mathbb{E}_{s' \sim d^{(k+1)}_{\tau}} \left[ \langle Q^{(k)}_{\tau}(s'), \hat{\pi}^{(k+1)}(s') - \pi^{(k)}(s') \rangle - \tau h_{s'}(\hat{\pi}^{(k+1)}(s')) + \tau h_{s'}(\pi^{(k)}(s')) \right]
\]

\[
= \frac{1}{1 - \gamma} \mathbb{E}_{s' \sim d^{(k+1)}_{\tau}} \left[ \langle \hat{Q}^{(k)}_{\tau}(s'), \hat{\pi}^{(k+1)}(s') - \pi^{(k)}(s') \rangle - \tau h_{s'}(\hat{\pi}^{(k+1)}(s')) + \tau h_{s'}(\pi^{(k)}(s')) \right]
\]

\[
+ \frac{1}{1 - \gamma} \mathbb{E}_{s' \sim d^{(k+1)}_{\tau}} \left[ \langle \hat{Q}^{(k)}_{\tau}(s'), \pi^{(k+1)}(s') - \hat{\pi}^{(k+1)}(s') \rangle - \tau h_{s'}(\pi^{(k+1)}(s')) + \tau h_{s'}(\hat{\pi}^{(k+1)}(s')) \right]
\]

\[
+ \frac{1}{1 - \gamma} \mathbb{E}_{s' \sim d^{(k+1)}_{\tau}} \left[ \langle Q^{(k)}_{\tau}(s') - \hat{Q}^{(k)}_{\tau}(s'), \pi^{(k+1)}(s') - \pi^{(k)}(s') \rangle \right],
\]

where the first identity arises from the performance difference lemma (cf. Lemma 5). To continue, we seek to control each part of (63) separately.

- Regarding the first term of (63), replacing \( \xi \) (resp. \( Q_{\tau} \)) by \( \hat{\xi} \) (resp. \( \hat{Q}_{\tau} \)) in Lemma 6 indicates that

\[
\langle \hat{Q}^{(k)}_{\tau}(s'), \hat{\pi}^{(k+1)}(s') - \pi^{(k)}(s') \rangle - \tau h_{s'}(\hat{\pi}^{(k+1)}(s')) + \tau h_{s'}(\pi^{(k)}(s')) = \frac{1}{\eta} \left[ (1 + \eta \tau) D_{h_{s'}}(\pi^{(k+1)}, \hat{\pi}^{(k+1)}; \xi^{(k+1)}) + D_{h_{s'}}(\pi^{(k+1)}, \pi^{(k)}; \hat{\xi}^{(k)}) \right]
\]

for all \( s' \in S \).

- As for the second term of (63), the definition of the oracle \( G_{s, \epsilon_{opt}} \) (see Assumption 3) guarantees that

\[
- \langle \hat{Q}^{(k)}_{\tau}(s'), \pi^{(k+1)}(s') - \hat{\pi}^{(k+1)}(s') \rangle - \tau h_{s'}(\pi^{(k+1)}(s')) + \frac{1}{\eta} D_{h_{s'}}(\pi^{(k+1)}, \pi^{(k)}; \hat{\xi}^{(k)}) \leq - \langle \hat{Q}^{(k)}_{\tau}(s'), \hat{\pi}^{(k+1)}(s') \rangle + \tau h_{s'}(\hat{\pi}^{(k+1)}(s')) + \frac{1}{\eta} D_{h_{s'}}(\pi^{(k+1)}, \pi^{(k)}; \hat{\xi}^{(k)}(s')) + \epsilon_{opt}
\]

for any \( s' \in S \). Rearranging terms, we are left with

\[
\langle \hat{Q}^{(k)}_{\tau}(s'), \pi^{(k+1)}(s') - \hat{\pi}^{(k+1)}(s') \rangle - \tau h_{s'}(\pi^{(k+1)}(s')) + \tau h_{s'}(\hat{\pi}^{(k+1)}(s')) - \epsilon_{opt}
\]

\[
\geq \frac{1}{\eta} D_{h_{s'}}(\pi^{(k+1)}, \pi^{(k)}; \hat{\xi}^{(k)}) + \frac{1}{\eta} D_{h_{s'}}(\pi^{(k+1)}, \pi^{(k)}; \hat{\xi}^{(k)}) - \epsilon_{opt}
\]

\[
= \frac{1}{\eta} \left( D_{h_{s'}}(\pi^{(k+1)}, \pi^{(k)}; \hat{\xi}^{(k)}) + D_{h_{s'}}(\pi^{(k+1)}, \pi^{(k)}; \hat{\xi}^{(k)}) \right)
\]

\[
+ \frac{1}{\eta} \left( D_{h_{s'}}(\pi^{(k+1)}, \pi^{(k)}; \hat{\xi}^{(k)}) - D_{h_{s'}}(\pi^{(k+1)}, \pi^{(k)}; \hat{\xi}^{(k)}) \right) - \epsilon_{opt}.
\]

In addition, we note that the term

\[
D_{h_{s'}}(\pi^{(k+1)}, \pi^{(k)}; \hat{\xi}^{(k)})
\]

appears in both (64) and (65), which can be canceled out when summing these two equalities. Specifically, adding (64) and (65) gives

\[
\langle \hat{Q}^{(k)}_{\tau}(s'), \pi^{(k+1)}(s') - \pi^{(k)}(s') \rangle - \tau h_{s'}(\pi^{(k+1)}(s')) + \tau h_{s'}(\pi^{(k)}(s')) \]

\[
+ \langle \hat{Q}^{(k)}_{\tau}(s'), \pi^{(k+1)}(s') - \hat{\pi}^{(k+1)}(s') \rangle - \tau h_{s'}(\pi^{(k+1)}(s')) + \tau h_{s'}(\hat{\pi}^{(k+1)}(s')) \]

\[
\geq \frac{1}{\eta} \left[ (1 + \eta \tau) D_{h_{s'}}(\pi^{(k)}, \pi^{(k)}; \hat{\xi}^{(k+1)}) + D_{h_{s'}}(\pi^{(k+1)}, \pi^{(k)}; \hat{\xi}^{(k)}) \right]
\]

\[
+ \frac{1}{\eta} \left( D_{h_{s'}}(\pi^{(k+1)}, \pi^{(k)}; \hat{\xi}^{(k)}) - D_{h_{s'}}(\pi^{(k+1)}, \pi^{(k)}; \hat{\xi}^{(k)}) \right) - \epsilon_{opt}.
\]
Substituting this into (63) and invoking the elementary inequality $|\langle a, b \rangle| \leq \|a\|_1 \|b\|_\infty$ thus lead to

$$V^{(k+1)}_\tau(s) - V^{(k)}_\tau(s) \geq \frac{1}{1 - \gamma} \mathbb{E}_{s' \sim d_{\pi^{(k+1)}_{\tau}}}[\frac{1}{\eta} \left(1 + \eta \tau\right)D_{h_s}(\pi^{(k)}; \pi^{(k+1)}; \tilde{\xi}^{(k+1)}_{\tau}) + D_{h_s}(\pi^{(k+1)}; \pi^{(k)}; \tilde{\xi}^{(k)}_{\tau})]$$

$$+ \frac{1}{1 - \gamma} \mathbb{E}_{s' \sim d_{\pi^{(k+1)}_{\tau}}}[\frac{1}{\eta} \left(D_{h_{\pi^{(k+1)}}}(\pi^{(k+1)}; \pi^{(k)}; \tilde{\xi}^{(k)}_{\tau}) - D_{h_{\pi^{(k+1)}}}(\pi^{(k+1)}; \pi^{(k)}; \tilde{\xi}^{(k)}_{\tau})\right)]$$

$$- \frac{\epsilon_{\text{opt}}}{1 - \gamma} - \frac{1}{1 - \gamma} \mathbb{E}_{s' \sim d_{\pi^{(k+1)}_{\tau}}}[\|Q^{(k)}_{\tau}(s') - \tilde{Q}^{(k)}_{\tau}(s')\|_\infty \|\pi^{(k+1)}(s') - \pi^{(k)}(s')\|_1],$$  \hspace{1cm} (66)

where the last line makes use of Assumption 2 and the fact $\|\pi^{(k+1)}(s)\|_1 = \|\pi^{(k)}(s)\|_1 = 1$.

Following the discussion in Lemma 1, we can see that $\tilde{\xi}^{(k)} = \epsilon_s^{(k)} 1 \in \partial h_s(\pi^{(k)}(s))$ with some constant $\epsilon_s^{(k)}$ for all $k$. This together with the convexity of $h_s$ (see (15)) guarantees that

$$D_{h_s}(\pi^{(k)}; \pi^{(k+1)}; \tilde{\xi}^{(k+1)}_{\tau}) \geq 0 \quad \text{and} \quad D_{h_s}(\pi^{(k+1)}; \pi^{(k)}; \tilde{\xi}^{(k)}_{\tau}) \geq 0$$ \hspace{1cm} (67)

for any $s \in S$, thus implying that the first term of (66) is non-negative. It remains to control the second term in (66). Towards this, a little algebra gives

$$D_{h_s}(\pi^{(k+1)}; \pi^{(k)}; \tilde{\xi}^{(k+1)}_{\tau}) - D_{h_s}(\pi^{(k+1)}; \pi^{(k)}; \tilde{\xi}^{(k)}_{\tau})$$

$$= - \left\{ h_s(\pi^{(k)}(s)) - h_s(\pi^{(k+1)}(s)) - \tilde{\xi}^{(k)}_{\tau}(s), \pi^{(k)}(s) - \pi^{(k+1)}(s) \right\}$$

$$= - h_s(\pi^{(k)}(s)) + h_s(\pi^{(k+1)}(s)) + \left\langle \frac{1}{1 + \eta \tau} \tilde{Q}^{(k)}_{\tau}(s), \pi^{(k+1)}(s) - \pi^{(k)}(s) \right\rangle$$

$$= \frac{\eta}{1 + \eta \tau} \left\{ - \left\langle \tilde{Q}^{(k)}_{\tau}(s), \pi^{(k)}(s) \right\rangle + \tau h_s(\pi^{(k)}(s)) + \frac{1}{\eta} D_{h_s}(\pi^{(k+1)}; \pi^{(k)}; \tilde{\xi}^{(k+1)}_{\tau}) \right.$$ 

$$- \left[ - \left\langle \tilde{Q}^{(k)}_{\tau}(s), \pi^{(k)}(s) \right\rangle + \tau h_s(\pi^{(k)}(s)) + \frac{1}{\eta} D_{h_s}(\pi^{(k)}; \pi^{(k-1)}; \tilde{\xi}^{(k-1)}_{\tau}) \right]\}$$

$$\geq - \frac{\eta \epsilon_{\text{opt}}}{1 + \eta \tau}. \hspace{1cm} (68)$$

Here, the first and the third lines follow from the definition (15), the second inequality comes from the construction (27), whereas the last step invokes the definition of the oracle (25). Substitution of (67) and (68) into (66) gives

$$V^{(k+1)}_\tau(s) - V^{(k)}_\tau(s) \geq - \frac{1 + \alpha}{1 - \gamma} \epsilon_{\text{opt}} - \frac{1}{1 - \gamma} \mathbb{E}_{s' \sim d_{\pi^{(k+1)}_{\tau}}}[\|Q^{(k)}_{\tau}(s') - \tilde{Q}^{(k)}_{\tau}(s')\|_\infty \|\pi^{(k+1)}(s') - \pi^{(k)}(s')\|_1] \hspace{1cm} (69)$$

$$\geq - \frac{1 + \alpha}{1 - \gamma} \epsilon_{\text{opt}} - 2 \frac{\epsilon_{\text{eval}}}{1 - \gamma}.$$ 

The case when $h_s$ is strongly convex. When $h_{\pi^{(k+1)}}$ is 1-strongly convex w.r.t. the $\ell_1$ norm, the objective function of sub-problem (62) is $\frac{1 + \eta \tau}{\eta}$ strongly convex w.r.t. the $\ell_1$ norm. Taking this together with the $\epsilon_{\text{opt}}$-approximation guarantee in Assumption 3, we can demonstrate that

$$\frac{1 + \eta \tau}{2 \eta} \|\pi^{(k+1)}(s') - \pi^{(k)}(s')\|_1^2 \leq \epsilon_{\text{opt}} \quad \text{for all } k \geq 0 \text{ and } s' \in S.$$ \hspace{1cm} (70)

Additionally, the strong convexity assumption also implies that

$$D_{h_{\pi^{(k+1)}}}(\pi^{(k+1)}(s'); \pi^{(k)}(s'); \xi^{(k)}(s')) \geq \frac{1}{2} \|\pi^{(k)}(s') - \pi^{(k+1)}(s')\|_1^2$$

$$= \frac{1}{2} \left(\|\pi^{(k)}(s') - \pi^{(k+1)}(s')\|_1^2 + \|\pi^{(k)}(s') - \pi^{(k)}(s')\|_1^2\right) - \frac{1}{2} \|\pi^{(k)}(s') - \pi^{(k)}(s')\|_1^2.$$

\[ \geq \frac{1}{4} \left( \| \tilde{\pi}^{(k)}(s') - \pi^{(k+1)}(s') \|_1 + \| \tilde{\pi}^{(k)}(s') - \pi^{(k)}(s') \|_1 \right)^2 - \frac{1}{4} \| \tilde{\pi}^{(k)}(s') - \pi^{(k)}(s') \|_1^2 \]
\[ \geq \frac{1}{4} \| \pi^{(k)}(s') - \pi^{(k+1)}(s') \|_1^2 - \frac{\eta \epsilon_{\text{opt}}}{1 + \eta \tau}, \]

where the third line results from the elementary inequality \( a^2 + b^2 \geq (a+b)^2/2 \) for any \( a, b \in \mathbb{R} \), and the final step follows from (70). We can develop a similar lower bound on \( D_{h,\tau}(\pi^{(k)}, \pi^{(k+1)}; \xi^{(k+1)}) \) as well. Taken together, these lower bounds give

\[ \frac{1}{\eta} \left[ (1 + \eta \tau) D_{h,\tau}(\pi^{(k)}(s'), \pi^{(k+1)}(s'); \xi^{(k+1)}(s')) + D_{h,\tau}(\pi^{(k+1)}(s'), \pi^{(k)}(s'); \xi^{(k)}(s')) \right] \]
\[ \geq \frac{2 + \eta \tau}{4 \eta} \| \pi^{(k)}(s') - \pi^{(k+1)}(s') \|_1 - 2 \epsilon_{\text{opt}}. \]

In addition, it is easily seen that

\[ - \| Q_s^{(k)}(s') - \hat{Q}_s^{(k)}(s') \|_1 \| \pi^{(k+1)}(s') - \pi^{(k)}(s') \|_1 \]
\[ \geq \frac{1}{2} \left( \frac{2 \eta}{2 + \eta \tau} \| Q_s^{(k)}(s') - \hat{Q}_s^{(k)}(s') \|_1^2 + \frac{2 + \eta \tau}{2 \eta} \| \pi^{(k+1)}(s') - \pi^{(k)}(s') \|_1^2 \right) \]
\[ \geq \frac{\eta}{2 + \eta \tau} \| \pi^{(k+1)}(s') - \pi^{(k)}(s') \|_1^2. \]

Combining the above two inequalities with (69), we arrive at the advertised bound

\[ V^{(k+1)}(s) - V^{(k)}(s) \geq - \frac{3 + \alpha}{1 - \gamma} \frac{\eta}{2 + \eta \tau} \epsilon_{\text{eval}}. \]

### B.2 Step 2: connecting with the algorithm dynamic with a linear system

Now we are ready to discuss how to control \( \| Q_s^{(k)} - Q_{s,a}^{(k)} \|_\infty \). In short, we intend to establish the connection among several intertwined quantities, and identify a simple linear system that captures the algorithm dynamic.

**Bounding** \( \| Q_s^{(k)} - \tau \tilde{\xi}^{(k+1)} \|_\infty \). From the definition of \( \tilde{\xi}^{(k+1)} \) in (27), we have

\[ \| Q_s^{(k)} - \tau \tilde{\xi}^{(k+1)} \|_\infty = \| \alpha (Q_s^* - \tau \tilde{\xi}^{(k)}) + (1 - \alpha) (Q_s^{(k)} - \tau \tilde{\xi}^{(k)}) \|_\infty \]
\[ \leq \alpha \| Q_s^* - \tau \tilde{\xi}^{(k)} \|_\infty + (1 - \alpha) \| Q_s^{(k)} - \tau \tilde{\xi}^{(k)} \|_\infty \]
\[ \leq \alpha \| Q_s^* - \tau \tilde{\xi}^{(k)} \|_\infty + (1 - \alpha) \| Q_s^{(k)} - \tau \tilde{\xi}^{(k)} \|_\infty + (1 - \alpha) \epsilon_{\text{eval}}, \]

where the last inequality is a consequence of Assumption 2.

**Bounding** \( \min_{s,a} (Q^{(k+1)}(s,a) - \tau \tilde{\xi}^{(k+1)}(s,a)) \). Applying the definition in (27) once again, we obtain

\[ - (Q^{(k+1)}(s,a) - \tau \tilde{\xi}^{(k+1)}(s,a)) = - \alpha (Q^{(k)}(s,a) - \tau \tilde{\xi}^{(k)}(s,a)) + (1 - \alpha) (\tilde{Q}_s^{(k)}(s,a) - Q^{(k)}(s,a)) \]
\[ + (Q^{(k)}(s,a) - Q^{(k+1)}(s,a)) \]
\[ \leq - \alpha (Q^{(k)}(s,a) - \tau \tilde{\xi}^{(k)}(s,a)) + (1 - \alpha + c_1) \epsilon_{\text{eval}} + c_2 \epsilon_{\text{opt}}, \]

where

\[ c_1 = \begin{cases} \frac{2 \gamma}{1 - \gamma} \frac{\eta \epsilon_{\text{opt}}}{(2 + \eta \tau)(1 - \gamma)}, & \text{if } h_s \text{ is convex but not strongly convex}, \\ \frac{(a+1) \gamma}{1 - \gamma}, & \text{if } h_s \text{ is 1-strongly convex w.r.t. the } \ell_1 \text{ norm}, \end{cases} \]
\[ c_2 = \begin{cases} \frac{(a+1) \gamma}{1 - \gamma}, & \text{if } h_s \text{ is convex but not strongly convex}, \\ \frac{(a+3) \gamma}{1 - \gamma}, & \text{if } h_s \text{ is 1-strongly convex w.r.t. the } \ell_1 \text{ norm}. \end{cases} \]
In the sequel, we control the three terms in (75) separately.

Invoking the relation (35) in Lemma 3 as well as the property (9), we reach

Here, the last step of (72) follows from Assumption 2 as well as the following relation:

To begin with, we repeat a similar argument as for (59) to show that

The second term of (75) can be bounded by applying (68) with \( k \) replaced by \( k + 1 \):

As for the third term of (75), taking the maximum over all \( (s, a) \in \mathcal{S} \times \mathcal{A} \) gives

Taken together, the above bounds and the decomposition (75) lead to

\[
\|Q^*_\tau - Q^{(k+1)}\|_\infty \leq \gamma \|Q^*_\tau - \tau \xi^{(k+1)}\|_\infty - \gamma \min_{s,a} \left( Q^{(k+1)}(s,a) - \tau \xi^{(k+1)}(s,a) \right) + \gamma(1 - \alpha)\epsilonopt. \tag{76}
\]
A linear system of interest. Combining (71), (74) and (76), we reach the following linear system

\[ z_{k+1} \leq Bz_k + b, \]  

(77)

where

\[
B := \begin{bmatrix}
\gamma(1-\alpha) & \gamma\alpha & \gamma\alpha \\
1 - \alpha & \alpha & 0 \\
0 & 0 & \alpha
\end{bmatrix}, \quad z_k := \begin{bmatrix}
\|Q^*_\tau - Q^{(k)}\|_\infty \\
\|Q^*_\tau - \tau\xi^{(k)}\|_\infty \\
-\min_{s,a} \left( Q^{(k)}_s(s,a) - \tau\xi^{(k)}(s,a) \right)
\end{bmatrix},
\]

\[
b := \begin{bmatrix}
\gamma(2-2\alpha + c_1)\epsilon_{\text{eval}} + \gamma(1-\alpha + c_2)\epsilon_{\text{opt}} \\
(1-\alpha)\epsilon_{\text{eval}} \\
(1-\alpha + c_1)\epsilon_{\text{eval}} + c_2\epsilon_{\text{opt}}
\end{bmatrix}.
\]

(78)

This linear system of three variables captures how the estimation error progresses as the iteration count increases.

B.3 Step 3: linear system analysis

In this step, we analyze the behavior of the linear system (77) derived above. Observe that the eigenvalues and respective eigenvectors of the matrix B are given by

\[
\lambda_1 = \alpha + (1-\alpha)\gamma, \quad \lambda_2 = \alpha, \quad \lambda_3 = 0,
\]

(79)

\[
v_1 = \begin{bmatrix}
\gamma \\
1 \\
0
\end{bmatrix}, \quad v_2 = \begin{bmatrix}
0 \\
-1 \\
1
\end{bmatrix}, \quad v_3 = \begin{bmatrix}
\alpha \\
\alpha - 1 \\
0
\end{bmatrix}.
\]

(80)

Armed with these, we can decompose \( z_0 \) in terms of the eigenvectors of \( B \) as follows

\[
z_0 \leq \begin{bmatrix}
\|Q^*_\tau - Q^{(0)}_\tau\|_\infty \\
\|Q^*_\tau - \tau\xi^{(0)}\|_\infty \\
\|Q^{(0)}_\tau - \tau\xi^{(0)}\|_\infty
\end{bmatrix}
\]

\[
= \frac{1}{\alpha + (1-\alpha)\gamma} \left[ (1-\alpha)\|Q^*_\tau - Q^{(0)}_\tau\|_\infty + \alpha\|Q^*_\tau - \tau\xi^{(0)}\|_\infty + \alpha\|Q^{(0)}_\tau - \tau\xi^{(0)}\|_\infty \right] v_1
\]

\[
+ \|Q^*_\tau - \tau\xi^{(0)}\|_\infty v_2 + \epsilon_z v_3
\]

\[
\leq \frac{1}{\alpha + (1-\alpha)\gamma} \left[ \|Q^*_\tau - Q^{(0)}_\tau\|_\infty + 2\alpha\|Q^*_\tau - \tau\xi^{(0)}\|_\infty \right] v_1 + \|Q^*_\tau - \tau\xi^{(0)}\|_\infty v_2 + \epsilon_z v_3,
\]

(81)

where \( \epsilon_z \in \mathbb{R} \) is some constant that does not affect our final result. Also, the vector \( b \) defined in (78) satisfies

\[
b \leq \begin{bmatrix}
\gamma(2-2\alpha + c_1)\epsilon_{\text{eval}} + \gamma(1-\alpha + c_2)\epsilon_{\text{opt}} \\
(1-\alpha)\epsilon_{\text{eval}} + (1-\alpha)\epsilon_{\text{opt}} \\
(1-\alpha + c_1)\epsilon_{\text{eval}} + c_2\epsilon_{\text{opt}}
\end{bmatrix}
\]

(82)

Using the decomposition in (81) and (82) and applying the system relation (77) recursively, we can derive

\[
z_{k+1} \leq B^{k+1}z_0 + \sum_{t=0}^{k} B^{k-t}b
\]

\[
\leq B^{k+1} \left[ \frac{1}{\alpha + (1-\alpha)\gamma} \left[ \|Q^*_\tau - Q^{(0)}_\tau\|_\infty + 2\alpha\|Q^*_\tau - \tau\xi^{(0)}\|_\infty \right] v_1 + \|Q^*_\tau - \tau\xi^{(0)}\|_\infty v_2 + \epsilon_z v_3 \right]
\]

\[
+ \sum_{t=0}^{k} B^{k-t} \left[ (2-2\alpha + c_1)\epsilon_{\text{eval}} + (1-\alpha + c_2)\epsilon_{\text{opt}} \right] v_1 + \left[ (1-\alpha + c_1)\epsilon_{\text{eval}} + c_2\epsilon_{\text{opt}} \right] v_2
\]
Recognizing that the first two entries of $v_2$ are non-positive, we can discard the term involving $v_2$ and obtain

$$\begin{aligned}
\left[\|Q^*_\tau - Q^{(k+1)}_\tau\|_\infty \bigg/ \|Q^*_\tau - \tau \xi^{(k+1)}\|_\infty \right]
&\leq \left[\lambda_1^k \left(\|Q^*_\tau - Q^{(0)}_\tau\|_\infty + 2\|Q^*_\tau - \tau \xi^{(0)}\|_\infty\right) + \frac{1 - \lambda_1^k}{1 - \lambda_1} \left(2 - 2\alpha + c_1\right)\epsilon_{eval} + (1 - \alpha + c_2)\epsilon_{opt}\right] \left[\gamma \right] \left[1\right]
&\leq \left[\lambda_1^k \left(\|Q^*_\tau - Q^{(0)}_\tau\|_\infty + 2\|Q^*_\tau - \tau \xi^{(0)}\|_\infty\right) + \frac{1}{1 - \lambda_1} \left(2 - 2\alpha + c_1\right)\epsilon_{eval} + (1 - \alpha + c_2)\epsilon_{opt}\right] \left[\gamma \right] \left[1\right].
\end{aligned}$$

Making use of the fact that $1 - \lambda_1 = (1 - \alpha)(1 - \gamma)$, we can conclude

$$C = \frac{1}{1 - \gamma} \left(2 + \frac{c_1}{1 - \alpha}\right)\epsilon_{eval} + \left(1 + \frac{c_2}{1 - \alpha}\right)\epsilon_{opt}.$$ 

The above bound essentially says that

$$\|Q^*_\tau - Q^{(k+1)}_\tau\|_\infty \leq \gamma \left[\lambda_1^k \left(\|Q^*_\tau - Q^{(0)}_\tau\|_\infty + 2\|Q^*_\tau - \tau \xi^{(0)}\|_\infty\right) + C\right]$$

and

$$\|Q^*_\tau - \tau \xi^{(k+1)}\|_\infty \leq \lambda_1^k \left(\|Q^*_\tau - Q^{(0)}_\tau\|_\infty + 2\|Q^*_\tau - \tau \xi^{(0)}\|_\infty\right) + C.$$

Plugging in the choices of $c_1$ and $c_2$ (cf. (73)), we have $C \leq C_2$ when $\{h_s\}$ is convex, and $C \leq C_3$ when $\{h_s\}$ is 1-strongly convex w.r.t. the $\ell_1$ norm. In addition, for the latter case, we can follow a similar argument as for (43) to demonstrate that

$$\|\pi_\tau^*(s) - \tilde{\pi}_\tau^{(k)}(s)\|_1 \leq \tau^{-1}((1 - \alpha)\gamma + \alpha)^k \left(\|Q^*_\tau - Q^{(0)}_\tau\|_\infty + 2\alpha\|Q^*_\tau - \tau \xi^{(0)}\|_\infty\right) + \tau^{-1}C_3,$$

which taken together with (70) gives

$$\|\pi_\tau^*(s) - \pi_\tau^{(k)}(s)\|_1 \leq \|\pi_\tau^*(s) - \tilde{\pi}_\tau^{(k)}(s)\|_1 + \|\pi_\tau^{(k)}(s) - \tilde{\pi}_\tau^{(k)}(s)\|_1 \leq \tau^{-1}((1 - \alpha)\gamma + \alpha)^k \left(\|Q^*_\tau - Q^{(0)}_\tau\|_\infty + 2\alpha\|Q^*_\tau - \tau \xi^{(0)}\|_\infty\right)$$

$$+ \tau^{-1}C_3 + \sqrt{\frac{2\gamma\epsilon_{opt}}{1 + \eta \tau}}.$$ 

This concludes the proof.

References


