Foundations of Reinforcement Learning

Markov decision processes: basics

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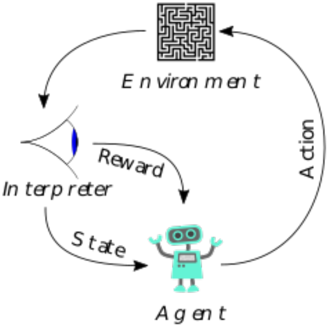
Outline

Reinforcement learning

In reinforcement learning (RL), an agent learns through interaction with the (unknown) environment.



Reinforcement learning

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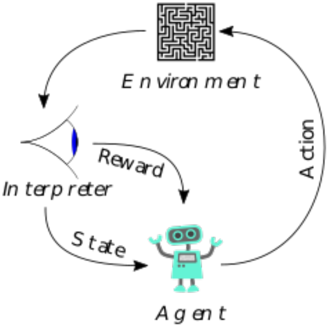

RL has deep connection with control theory, and is also sometimes called approximate dynamic programming. It can be viewed as a type of optimal control theory with no pre-defined model of the environment.

Applications of RL

RL can be applied to many different areas.

- Robotics: in which direction and how fast should a robot arm move?
- Mobility: where should taxis go to pick up passengers?
- Transportation: when should traffic lights turn green?
- Recommendations: which news stories will users click on?
- Network configuration: which parameter settings lead to the best allocation of resources?

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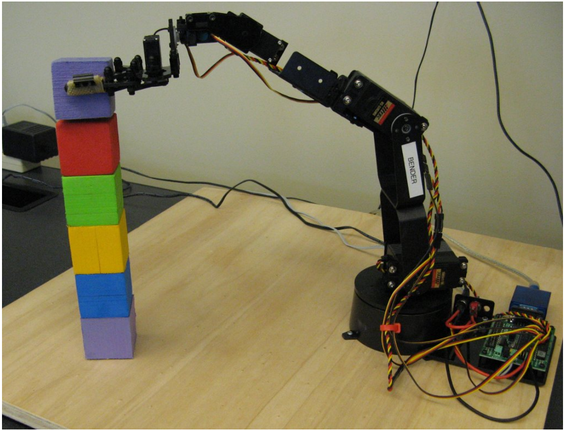
Similar to multi-armed bandits, but with a notion of state or context.

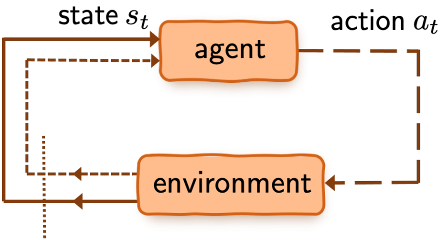
Example: grasping an object



RL reinforces the agents' decisions over time by observing the reward and state that result from taking different actions.

Markov decision processes

Infinite-horizon Markov decision process (MDP)



- $\mathcal{S}$: state space
- $\mathcal{A}$: action space

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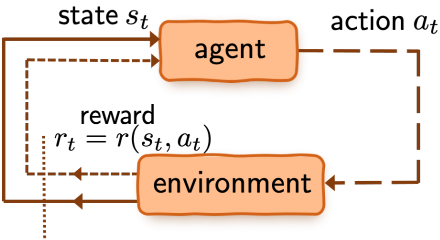

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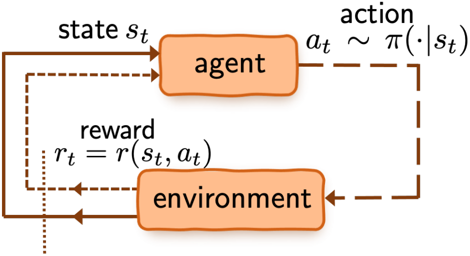

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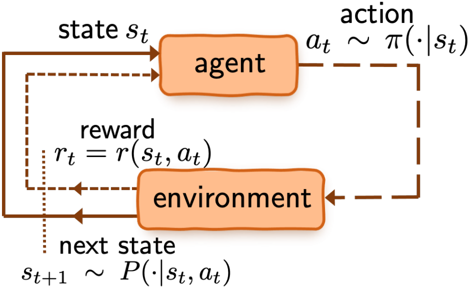

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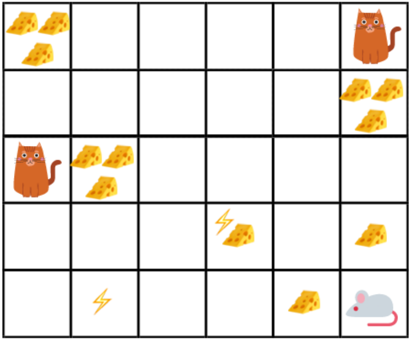
- $\pi(\cdot|s)$: policy (or action selection rule), deterministic or random
- $P(\cdot|s,a)$: transition probabilities

Help the mouse!



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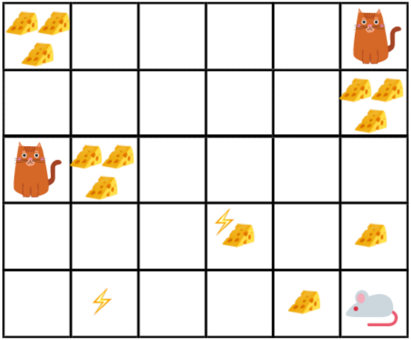

- state space $\mathcal{S}$: positions in the maze

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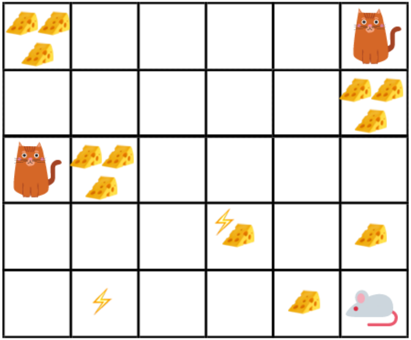

- state space $\mathcal{S}$: positions in the maze
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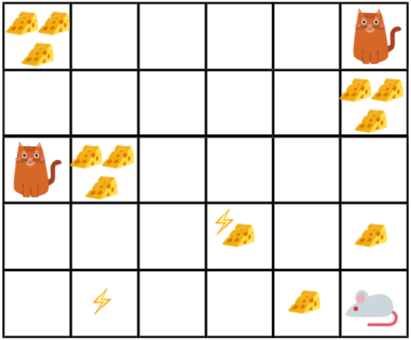

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- immediate reward r: cheese (+1), electricity shocks (-1), cats (-10000)

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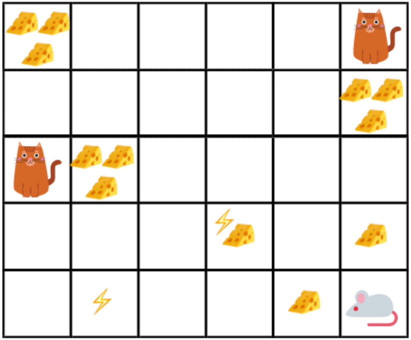

- state space $\mathcal{S}$: positions in the maze
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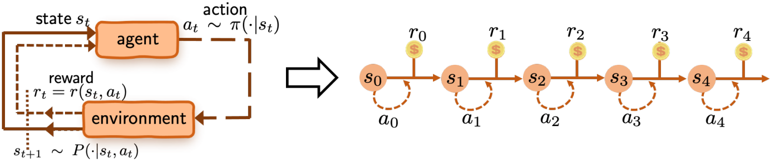
- immediate reward r: cheese ($+1$), electricity shocks (-1), cats (-10000)
- policy $\pi(\cdot|s)$: the way to find cheese

Value function

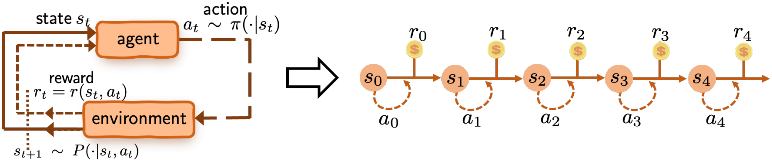


Value of policy π: cumulative discounted reward

$$\forall s \in \mathcal{S}: \quad V^\pi(s) := \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \,\middle|\, s_0 = s\right]$$

- $(a_0, s_1, a_1, s_2, a_2, \cdots)$: induced by policy π

Value function



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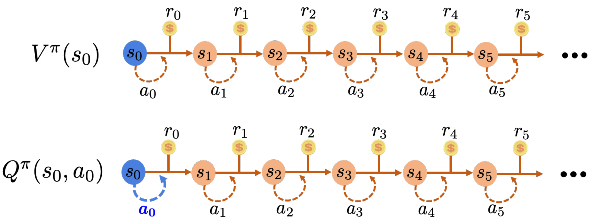
- $(a_0, s_1, a_1, s_2, a_2, \cdots)$: induced by policy π
- $\gamma \in [0, 1)$: discount factor,
  - γ close to 0 leads to "myopic" evaluation
  - γ close to 1 leads to "far-sighted" evaluation

Q-function (action-value function)



Q-function of policy π:

$$\forall (s,a) \in \mathcal{S} \times \mathcal{A}: \quad Q^\pi(s,a) := \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t r_t \,\middle|\, s_0 = s, a_0 = a\right]$$

- $(\cancel{a_0,}\, s_1, a_1, s_2, a_2, \cdots)$: induced by policy π

Effective horizon

Since $r(s, a) \in [0, 1]$,

$$0 \leq V^\pi(s), Q^\pi(s, a) \leq \frac{1}{1 - \gamma}.$$

Often think of $\frac{1}{1-\gamma}$ as the **effective horizon** of the problem.

Why Markov transitions?

- By the Markovian property,

$$
\begin{aligned}
P(s_{t+1}, s_t, \ldots, s_0) &= P(s_0)P(s_1|s_0)P(s_2|s_1, s_0)\ldots P(s_{t+1}|s_t, \ldots, s_0) \\
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&= P(s_0)\prod_{i=0}^{t} P(s_{i+1}|s_i).
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$$

Low computation and memory complexity!

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  Low computation and memory complexity!

- The world is Markovian when the state space is large enough. For example, if $s_{t+1} \sim P(\cdot|s_t, s_{t-1})$ depends on the previous two steps, by working with $\widetilde{s}_t = (s_t, s_{t-1})$ (and $s_{-1} = s_0$), we have

$$\widetilde{s}_{t+1} \sim P(\cdot|\widetilde{s}_t)$$

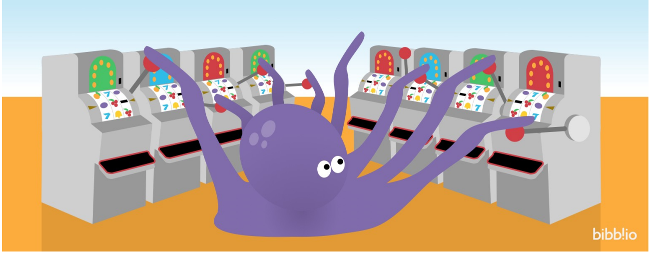
  is Markovian.

All models are wrong, but some are useful

Why discounting?

- Mathematically convenient: the limit always exists
- Immediate rewards earn more interest than future rewards
- Account for variability and uncertainty in the future which may not be fully captured
- Undiscounted MDP is possible, e.g. if all sequences terminate (like in a maze or game).
- Alternatives: average reward and finite-horizon episodic settings.

Reduction to multi-arm bandits



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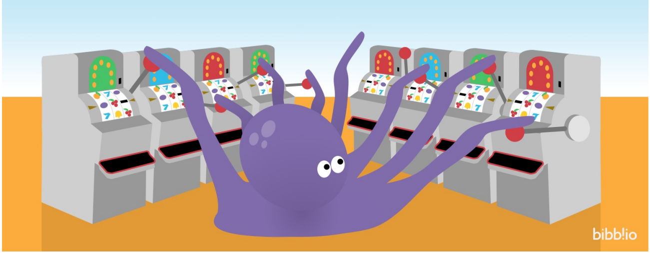

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- The reward function is action-dependent (action = arm): $r(a)$
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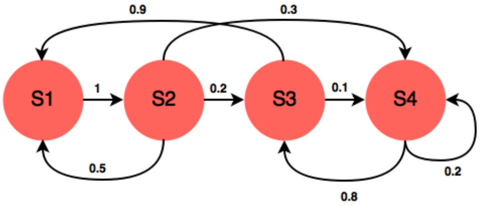
- The value of a policy π becomes $V^\pi := \mathbb{E}_\pi[r(a)]$.

Reduction to Markov reward process



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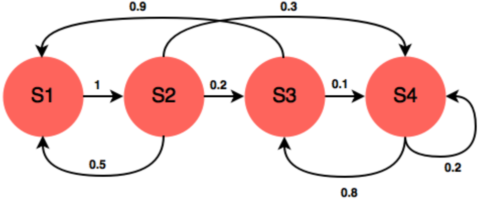

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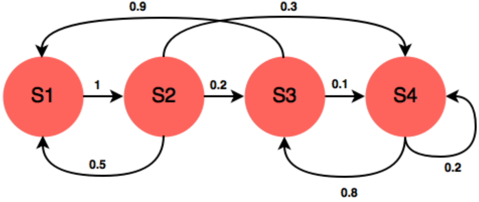

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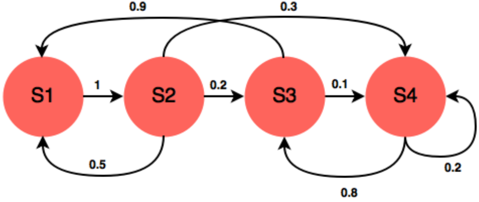

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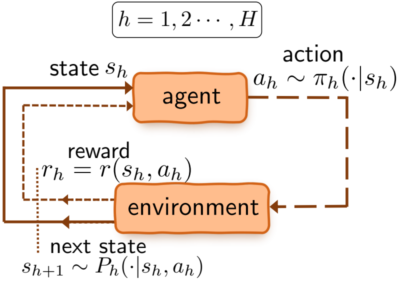
- The reward function is state-dependent: $r(s)$
- The value becomes $V(s) := \mathbb{E}_{\pi}[\sum_{t=0}^{\infty} \gamma^t r(s_t) \mid s_0 = s]$.

Finite-horizon episodic MDP



- H: horizon length

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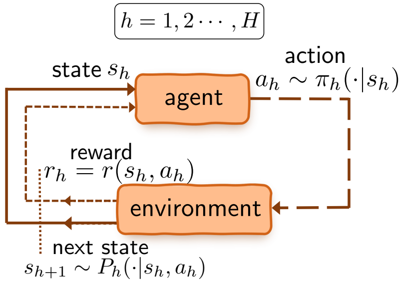

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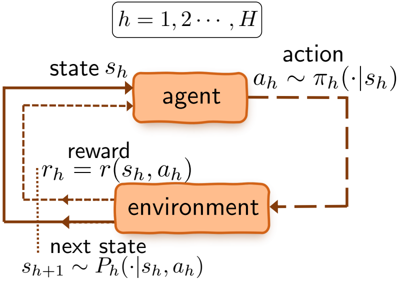

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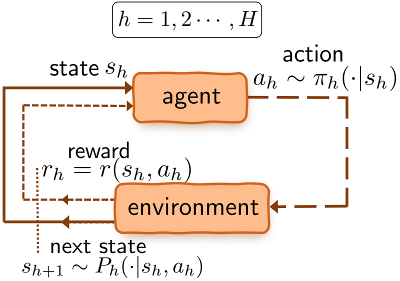

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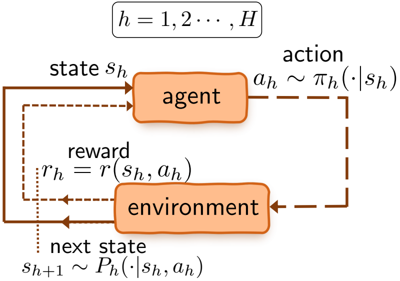

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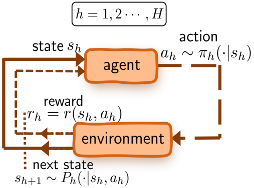
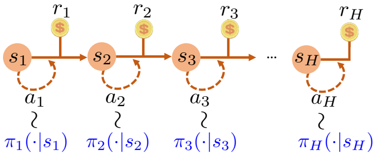
- $P_h(\cdot \mid s, a)$: transition probabilities in step h

Value function and Q-function





$$V_h^\pi(s) := \mathbb{E}\left[\sum_{t=h}^{H} r_t(s_t, a_t) \,\Big|\, s_h = s\right]$$

$$Q_h^\pi(s, a) := \mathbb{E}\left[\sum_{t=h}^{H} r_t(s_t, a_t) \,\Big|\, s_h = s, a_h = a\right]$$

- execute policy π to generate sample trajectory

Basic tasks

Policy evaluation:

- given a policy π, how good is it?

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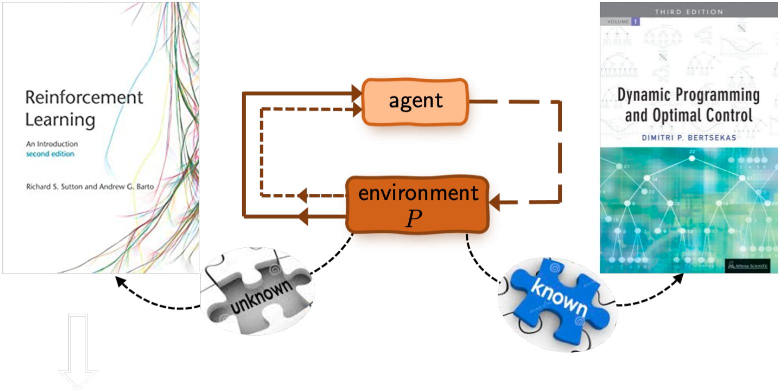
Policy optimization:

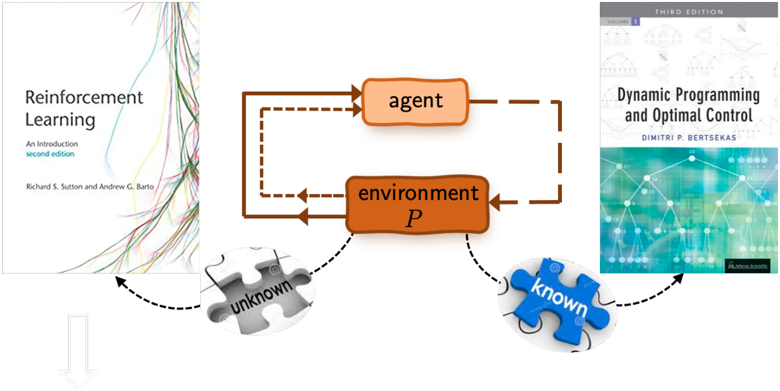
- can we find the best policy for the given MDP?

Planning versus learning



- **Planning:** solve for a desired policy given model specification

Planning versus learning



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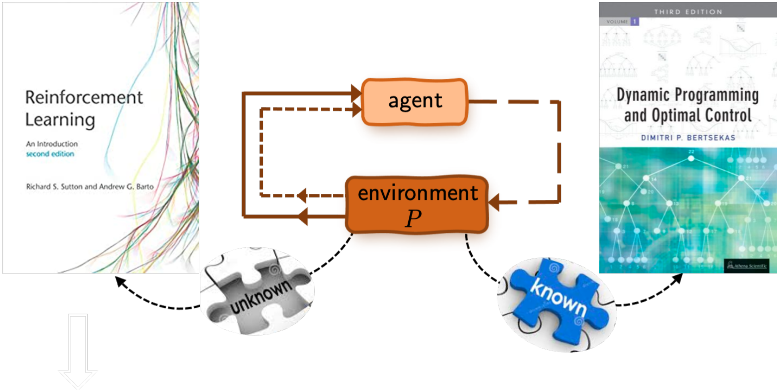

- **Planning:** solve for a desired policy given model specification
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  We'll focus on planning first.

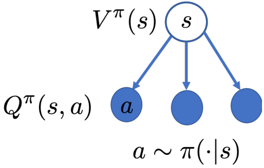
Policy evaluation

Policy evaluation: evaluating V via Q

$$V^\pi(s) = \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \,\middle|\, s_0 = s\right]$$

$$= \sum_{a \in \mathcal{A}} \pi(a_0 = a | s = s_0) \underbrace{\mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) \,\middle|\, s_0 = s, a_0 = a\right]}_{=: Q^\pi(s,a)}$$

$$= \mathbb{E}_{a \sim \pi(\cdot|s)}[Q^\pi(s, a)]$$



Policy evaluation: evaluating Q via V

$$Q^\pi(s,a) = \mathbb{E}\left[r(s,a) + \sum_{t=1}^{\infty}\gamma^t r(s_t,a_t) \,\middle|\, s_0 = s, a_0 = a\right]$$

$$= \mathbb{E}\left[r(s,a)\right] + \mathbb{E}_{s'\sim P(\cdot|s,a)}\left[\mathbb{E}\left[\sum_{t=1}^{\infty}\gamma^t r(s_t,a_t) \,\middle|\, s_1 = s', s_0 = s, a_0 = a\right]\right]$$

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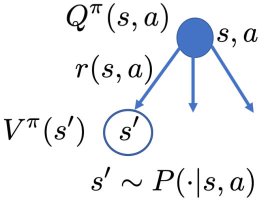

Bellman's consistency equation

- V^π / Q^π: value / action-value function under policy π

Bellman's consistency equation

$$V^\pi(s) = \mathbb{E}_{a\sim\pi(\cdot|s)}\left[Q^\pi(s,a)\right]$$

$$Q^\pi(s,a) = \underbrace{\mathbb{E}[r(s,a)]}_{\text{immediate reward}} + \gamma \mathop{\mathbb{E}}_{s'\sim P(\cdot|s,a)}\left[\underbrace{V^\pi(s')}_{\text{next state's value}}\right]$$

The value/Q function can be decomposed into two parts:

- immediate reward $\mathbb{E}\left[r(s,a)\right]$
- discounted value of at the successor state $\gamma\mathbb{E}_{s'\sim P(\cdot|s,a)}V(s')$

Richard Bellman

Matrix-vector representation

- Plugging Q^π into V^π, we have

$$V^\pi(s) = \mathbb{E}_{a\sim\pi(\cdot|s)}[r(s,a)] + \gamma \mathop{\mathbb{E}}_{a\sim\pi(\cdot|s), s'\sim P(\cdot|s,a)}[V^\pi(s')].$$

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- Let P^π be the state-state transition matrix induced by π, namely,

$$P^\pi(s'|s) = \sum_{a\in\mathcal{A}} \pi(a|s)P(s'|s,a).$$

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- We can write the above in a matrix-vector form as

$$V^\pi = r^\pi + \gamma P^\pi V^\pi,$$

where $V^\pi = [V^\pi(s)]_{s\in\mathcal{S}}$, and $r^\pi = \left[\mathbb{E}_{a\sim\pi(\cdot|s)}[r(s,a)]\right]_{s\in\mathcal{S}}$.

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a similar treatment applies to Q^π, too.

Solving the Bellman's consistency equation

$$V^\pi = r^\pi + \gamma P^\pi V^\pi \quad \Longrightarrow \quad V^\pi = (I - \gamma P^\pi)^{-1} r^\pi$$

Solving the Bellman's consistency equation

$$V^\pi = r^\pi + \gamma P^\pi V^\pi \quad \Longrightarrow \quad V^\pi = (I - \gamma P^\pi)^{-1} r^\pi$$

Invertibility of $I - \gamma P^\pi$: Gershgorin's circle theorem, or for any $x \in \mathbb{R}^{|\mathcal{S}|}$, verify

$$\begin{aligned}
\|(I - \gamma P^\pi)x\|_\infty &\geq \|x\|_\infty - \gamma \|P^\pi x\|_\infty \\
&\geq \|x\|_\infty - \gamma \|x\|_\infty \qquad (\|P^\pi x\|_\infty \leq \|P^\pi\|_1 \|x\|_\infty = \|x\|_\infty) \\
&\geq (1 - \gamma)\|x\|_\infty \\
&> 0.
\end{aligned}$$

Thus, $I - \gamma P^\pi$ is full rank and invertible.

Solving the Bellman's consistency equation

$$V^\pi = r^\pi + \gamma P^\pi V^\pi \quad \Longrightarrow \quad V^\pi = (I - \gamma P^\pi)^{-1} r^\pi$$

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Thus, $I - \gamma P^\pi$ is full rank and invertible.

Computationally expensive for problems with large state space!

Bellman's policy operator

Bellman's policy operator: denote the operator $\mathcal{T}^\pi : \mathbb{R}^{|\mathcal{S}|} \mapsto \mathbb{R}^{|\mathcal{S}|}$ as

$$\forall V \in \mathbb{R}^{|\mathcal{S}|} : \qquad \mathcal{T}^\pi(V) = r^\pi + \gamma P^\pi V.$$

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Fixed-point equation:

$$V = \mathcal{T}^\pi(V)$$

- V^π is the unique fixed point of this fixed-point equation.

Contraction property of the Bellman's operator

Lemma 1

The operator $\mathcal{T}^\pi$ *is a* γ*-contraction on* $\mathbb{R}^{|\mathcal{S}|}$*, i.e. for any* V *and* V' *in* $\mathbb{R}^{|\mathcal{S}|}$*, it follows*

$$\|\mathcal{T}^\pi(V) - \mathcal{T}^\pi(V')\|_\infty \leq \gamma \|V - V'\|_\infty.$$

Proof: For any V and V',

$$\begin{aligned}\|\mathcal{T}^\pi(V) - \mathcal{T}^\pi(V')\|_\infty &= \|\gamma P^\pi V - \gamma P^\pi V'\|_\infty \\ &\leq \gamma \|P^\pi\|_1 \|V - V'\|_\infty = \gamma \|V - V'\|_\infty,\end{aligned}$$

using $\|P^\pi\|_1 = 1$.

Fast computation without inversion

Value iteration for policy evaluation

For $t = 0, 1, 2, \ldots$

$$V^{(t+1)} = \mathcal{T}^{\pi}(V^{(t)}).$$

Linear convergence:

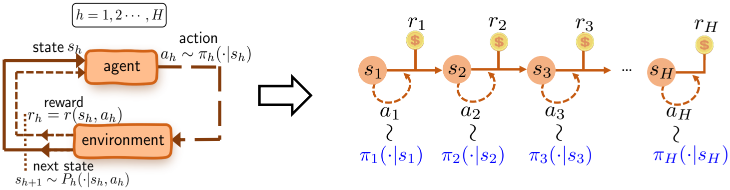
$$\begin{aligned} \|V^{(t+1)} - V^{\pi}\|_{\infty} &= \|\mathcal{T}^{\pi}(V^{(t)}) - \mathcal{T}^{\pi}(V^{\pi})\|_{\infty} \\ &\leq \gamma \|V^{(t)} - V^{\pi}\|_{\infty} \\ &\leq \gamma^{t} \|V^{(0)} - V^{\pi}\|_{\infty}. \end{aligned}$$

Implication: to achieve $\|V^{(t+1)} - V^{\pi}\|_{\infty} \leq \epsilon$, it takes no more than

$$\frac{1}{1-\gamma} \log\left(\frac{\|V^{(0)} - V^{\pi}\|_{\infty}}{\epsilon}\right)$$

iterations.

Policy evaluation for finite-horizon MDPs



1. Begin with the terminal step $h = H + 1$:

$$V^\pi_{H+1} = 0, \quad Q^\pi_{H+1} = 0.$$

2. Backtrack $h = H, H-1, \ldots, 1$:

$$Q^\pi_h(s,a) := \underbrace{\mathbb{E}\left[r_h(s_h, a_h)\right]}_{\text{immediate reward}} + \underbrace{\mathbb{E}_{s' \sim P_h(\cdot|s,a)} V^\pi_{h+1}(s')}_{\text{next step's value}}$$

$$V^\pi_h(s) := \mathbb{E}_{a \sim \pi_h(\cdot|s)} Q^\pi_h(s,a)$$