An integrative perspective to LQ and ℓ_∞ control for delayed and quantized systems

Yorie Nakahira and Lijun Chen

Abstract—**Deterministic and stochastic approaches to handle uncertainties may incur very different complexities in computation time and memory usage, in addition to different uncertainty models. For linear systems with delay and rate constrained communications between the observer and the controller, previous work shows that a deterministic approach, the ℓ_∞ control has low complexity but can only handle bounded disturbances. In this paper, we first take a stochastic approach and propose an LQ controller that can handle arbitrarily large disturbance but has large complexity in time and space. The differences in robustness and complexity of the ℓ_∞ and LQ controllers motivate the design of a hybrid controller that interpolates between the ℓ_∞ and LQ controllers. Using both theoretical bounds and numerical examples, we show that the hybrid controller can achieve a sweet spot in the robustness-complexity tradeoff, *i.e.*, reject occasional large disturbance while operating with low complexity most of the time.**

Index Terms—**Robustness-complexity tradeoff, the LQ control, the ℓ_∞ control, communication constraints, robust control.**

I. INTRODUCTION

In the design of cyber-physical systems, it is essential to account for a broad range of uncertainties such as disturbances due to environmental changes and control errors due to delays and quantizations in feedback loops. Two approaches are typically used to handle uncertainties: deterministic or stochastic. In the deterministic approach, uncertain input or parameters are assumed to be in an uncertainty set, and the design goal is to optimize the worst-case performance over the uncertainty set. In the stochastic approach, uncertain input or parameter is assumed to have a certain distribution, and the design goal is to optimize the average performance. It is obvious that the applicability of each approach depends on the characterization of uncertainty. However, it is not clear which approach incurs less complexity in time and space (i.e., memory). In the paper, we investigate some of the related issues in controller design for linear systems with delay and quantization.

Specifically, we consider a linear dynamical system with delay and rate constrained communications between the observer and the controller; see Fig. 1 for a schematic. Previous work [2], [3] takes the deterministic approach of ℓ_∞ control, i.e., to design an optimal controller that minimizes the worst-case infinity-norm of the system output under infinity-norm bounded disturbances. The resulting controller uses static memoryless quantizers and therefore has low time and space complexity. However, the efficacy of this approach partly depends on how "tight" the uncertainty set is in covering all possible disturbances, and the assumption of bounded uncertainty set will necessarily leave out large disturbance that may occasionally occur.

In contrast, in this paper, we take a stochastic approach that can better handle (occasional) large disturbances and study the linear-quadratic (LQ) control problem with costs (i.e., performance) in both the state and the control action. Building upon controller design methods for the quantized system [4], [5], we design a controller for the delayed and quantized system. We further derive a lower bound on the optimal performance and compare the performance of the proposed LQ controller against it. The comparison shows that the LQ controller can reject large disturbance while achieving near-optimal performance. However, the LQ controller needs to store the whole distribution of the system state, which incurs a much higher time and space complexity than the optimal ℓ_∞ controller.

The above optimal/near-optimal controllers based on the two approaches have different advantages and limitations regarding robustness to uncertainty and complexity in time and space. An interesting question that arises from these differences is if it is possible to design a controller that has the advantages of both the above controllers. In this paper, we take a hybrid approach to create such a controller. Specifically, we assume that the *typical* disturbance is relatively small and covered by a bounded set, while the large disturbance (outside of the bounded set) is *rare* event that has a (tail) Gaussian distribution. Under this assumption, we construct a hybrid controller that interpolates between the ℓ_∞ controller and the LQ controller. Using both theoretical bounds and numerical examples, we show that the hybrid controller can achieve a sweet spot in the robustness-complexity tradeoff, *i.e.*, reject occasional large disturbance while operating with low complexity most of the time.

Related work: There is a large literature on the topics studied in this paper. Here we briefly review only those that are directly relevant. Applications of the model studied in this paper range from cyber-physical systems [2], [6]–[10] to neuroscience [3] and cell biology [11]–[13]. Motivated by these applications, there exists a large literature on control under communication constraints, based on either the deterministic approach or the stochastic approach. For the former, stability conditions are known for a broad class of linear systems with quantization or data rate constraints [14], [15], and optimal controllers for systems with delay and quantization are given

Y. Nakahira is with Computing and Mathematical Science, California Institute of Technology, Pasadena, CA 91120, USA (email: ynakahir@caltech.edu).

L. Chen is with Computer Science, University of Colorado, Boulder, CO 80309, USA (email: lijun.chen@colorado.edu).

Preliminary result of this paper was presented at IEEE Conference on Decision and Control (CDC), Las Vegas, NV, December 2016 [1].

in [2], [3]. For the latter, stability conditions are known for linear systems with quantization or data rate constraints [16]–[18], and performance bounds are given in [19]–[29]. The relation between the optimal cost and the causal rate-distortion function is studied in [29]–[34]. The information-theoretic quantities used to model communication constraints include mutual information [22], [35], anytime capacity [36], and directed information [27], [37], among others. The optimal controllers are studied for quantized systems [4], [5]. In contrast, in this paper, we study optimal controller design for delayed and quantized systems, and further, we take a hybrid deterministic-stochastic approach.

Notation and preliminaries: We use lower case letter to denote a sequence, e.g., $x = \{x_0, x_1, x_2, \cdots\}$, x_τ^t to denote a truncated sequence $\{x_\tau, x_{\tau+1}, \cdots, x_t\}$ from τ to t, and for simplicity let $x^t = x_0^t$. We use $'$ to denote matrix transpose. The ℓ_∞ norm of a sequence x is defined as $\|x\|_\infty := \sup_t |x_t|$. We use $f(x)$ to denote the probability density function of a random variable x, and $f(x|y)$ to denote the conditional probability density function of a random variable x given y.

The rest of this paper is organized as follows. Section II describes the system model, as well as summarizes the existing result on the ℓ_∞ control. Section III present the LQ control design and its analysis. Section IV presents the hybrid controller and its analysis. Section V concludes the paper.

II. System Model

Consider a feedback dynamical system with delay and rate constrained communications between the observer and the controller as shown in Fig. 1. The plant follows the discrete-time dynamics:

$$x_{t+1} = Ax_t + u_t + w_t, \tag{1}$$

where $x_t \in \mathbb{R}$ is the state, $w_t \in \mathbb{R}$ is the disturbance, and $u_t \in \mathbb{R}$ is the control action at time t. Without loss of generality, assume the initial condition $x_0 = 0$ and $w_t = 0$ for $t < 0$.

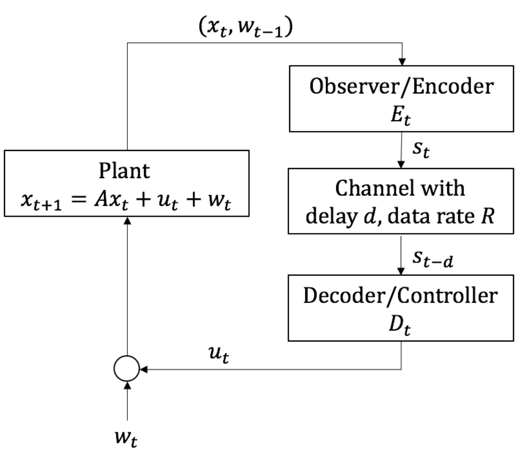

Fig. 1: The system model.

The communication channel between the observer and the controller is characterized by delay d and bandwidth R, with $R > \log_2 |A|$ to ensure stability [17]. Associated with the observer is an encoder that at time t is defined by a mapping E_t from the available information $\mathcal{I}_t = \{\{x_\tau\}_{\tau=0,\dots,t}, \{w_\tau\}_{\tau=0,\dots,t-1}\}$ to a proper codeword s_t, i.e.,

$$s_t = E_t(\mathcal{I}_t) \in S, \tag{2}$$

where the set S of codewords has cardinality of at most 2^R. Associated with the controller is an decoder that at time t recovers the information on state and disturbance upon the received (delayed) information $\mathcal{J}_{t-d} = \{s_\tau\}_{\tau=0,\dots,t-d}$, based on which the controller will decide the control action u_t. The encoder and controller can be jointly defined by a mapping D_t:

$$u_t = D_t(\mathcal{J}_{t-d}) \in \mathbb{R}. \tag{3}$$

We may loosely refer to D_t as decoder, controller or decoder-encoder, whichever is more convenient in the relevant context.

Let $K := \{(E_0, D_0), (E_1, D_1), \cdots, (E_t, D_t), \cdots\}$, which we also broadly call the controller, and denote by $\mathcal{K}(R, d)$ the space of such controllers with delay d and bandwidth R. The design goal for the controller is to achieve a good performance (small state deviation under disturbance) with small control effort (small actuation, small computation time, and low memory usage), which can be quantified in terms of $\|x\|$, $\|u\|$ for certain norm $\|\cdot\|$ and by the functional form of (E_t, D_t).

A. The ℓ_∞ System

In this subsection, we summarize the existing robust control theory for the ℓ_∞ system with delay and quantization [2] [3], where the design objective is to minimize $\max_w \|x\|_\infty$. For disturbance with bounded support $\|w\|_\infty \le L$ and stabilizing bandwidth $R > \log_2 |A|$, the optimal performance is given by:

$$\max_{\|w\|_\infty \le L} \|x\|_\infty = \left\{ \sum_{i=0}^{d} |A^i| + \frac{|A^{d+1}|}{(2^R - |A|)^{-1}} \right\} L. \tag{4}$$

Let $\Psi(L) := \left\{ |A^{d+2}|(2^R - |A|)^{-1} + |A^{d+1}| \right\} L$. The optimal performance is achieved by the ℓ_∞ controller as shown in Algorithm 1. In Algorithm 1, $\mathcal{Q}_\ell : \mathbb{R} \to S_R$ denotes a uniform quantizer of rate R (i.e., with 2^R levels) over the interval $[-\ell, \ell]$, and $|S_R| = 2^R$.

Algorithm 1: The ℓ_∞ controller.

$$\begin{aligned}
\text{Encoder:} \quad & q_t = \mathcal{Q}^{-1}_{\Psi(L)}(s_{t-d-1}) - u^*_{t-1} \\
& z_t = A^d w_{t-d-1} + q_t \\
& u^*_t = -Az_t \\
& s_{t-d} = \mathcal{Q}_{\Psi(L)}(u^*_t) \\
\text{Decoder:} \quad & u_t = \mathcal{Q}^{-1}_{\Psi(L)}(s_{t-d})
\end{aligned}$$

The advantage of this controller is that it requires little computation and storage: the encoder only needs to store the last codeword and perform minimum computation, and the decoder is static and memoryless. In addition, this controller requires minimum actuation effort when $|A| \ge 1$: the stabilizing control law that minimizes $\max_{\|w\|_\infty \le 1} \|u\|_\infty$ is identical to the above control law, which minimizes $\max_{\|w\|_\infty \le 1} \|x\|_\infty$. However, the low complexity of the ℓ_∞ controller does not come for free. For a disturbance with unbounded support, the fixed quantizer in Algorithm 1 is not stabilizing because there is always a nonzero probability that the quantizer saturates. In next section, we will consider the LQ controller that can better handle large disturbance.

III. The Linear Quadratic System

In this section, we study the robust control problem for the linear quadratic (LQ) system with delay and quantization. The disturbance $w_t,\ t \ge 0$ is assumed to be *i.i.d.* Gaussian with zero mean and variance σ^2, i.e., $w_t \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2)$ for $t \ge 0$. The control objective is to minimize an average cost subject to the plant dynamics (1):[1]

$$\underset{K \in \mathcal{K}(R,d)}{\text{minimize}} \quad \lim_{t \to \infty} \mathbb{E}[x_t' P x_t + u_t' Q u_t] \tag{5}$$

where $P \ge 0$ and $Q \ge 0$ balance the cost of state deviation and control action.

A. Performance Bound and Optimal Controller Structure

The following result gives a lower bound on the theoretically opimal LQ cost.

Theorem 1: The optimal performance of the robust control problem (5) is bounded below as follows:

$$\begin{aligned} &\lim_{t \to \infty} \mathbb{E}[x_t' P x_t + u_t' Q u_t] \\ &\ge P \sum_{i=0}^{d-1} A^{2i} \sigma^2 + P^\star A^{2d} \sigma^2 + G^\star A^{2d} \frac{\sigma^2}{2^{2R} - A^2}, \end{aligned} \tag{6}$$

where $P^\star$ and $G^\star$ are the unique solution to the equations:

$$\begin{aligned} P^\star &= A' \left[P^\star + P - P^\star (Q + P^\star)^{-1} P^\star \right] A, \\ G^\star &= A' P^\star A + P - P^\star. \end{aligned} \tag{7}$$

Proof: See Appendix. ■

The first and second terms in the lower bound (6), $P \sum_{i=0}^{d-1} A^{2i}\sigma^2 + P^\star A^{2d} \sigma^2$, are due to delay in control action, while the third term $G^\star A^{2d} \frac{\sigma^2}{2^{2R} - A^2}$ is mainly due to limited data rate. The lower bound is derived using the following lemma, which characterizes the structure of the optimal controller to (5) and holds generally for multiple-input-multiple-output (MIMO) systems.

Lemma 1: Consider a MIMO system

$$x_{t+1} = A x_t + B u_t + w_t \tag{8}$$

with $x_t \in \mathbb{R}^m$, $u_t \in \mathbb{R}^n$, $w_t \in \mathbb{R}^m$ and $w_t \overset{i.i.d}{\sim} \mathcal{N}(0, \Sigma)$ with covariance matrix $\Sigma \succeq 0$, and the corresponding robust control problem

$$\min_{K \in \mathcal{K}(R,d)} \lim_{N \to \infty} \mathbb{E} \left[x_N' P x_N + \sum_{t=0}^{N-1} (x_t' P x_t + u_t' Q u_t) \right] \tag{9}$$

with $P \succeq 0$, $Q \succeq 0$. Given any encoding scheme $\{E_t\}$, the optimal decoder-controller D_t has the following structure:

$$u_t = L_t \, \mathbb{E}[z_t | s^{t-d}], \tag{10}$$

where z_t is defined by the recursion

$$z_{t+1} = A z_t + A^d w_{t-d} + B_t u_t, \quad z_0 = 0, \tag{11}$$

and

$$L_t = -(Q + B' P_{t+1} B)^{-1} B' P_{t+1} A, \tag{12}$$

with P_t defined by the recursion

$$\begin{aligned} P_N &= P, \\ P_t &= A' \left[P_{t+1} + P - P_{t+1} B (Q + B' P_{t+1} B)^{-1} B' P_{t+1} \right] A. \end{aligned} \tag{13}$$

Proof: See Appendix. ■

Notice that Lemma 1 does not specify what an optimal encoder is. Also, seen from the proof of Theorem 1, the first two terms in the lower bound (6) of the optimal performance are tight for any delay d if the decoder-controller has the structure (10).

Remark 1 (Certainty equivalence): The definition of certainty equivalence and its extension to quantized systems are given in [21], [38]. The optimal controller structure in Lemma 1 is an extension of certainty equivalence to systems with delay and quantization. The auxiliary sequence $\{z_t\}$ and Lemma 1 together allow us to bound the objective value by studying an estimation problem of a Gauss-Markov source and an LQ control problem of a fully observed system. The sequence $\{z_t\}$ also plays an important role in Section IV.

B. The LQ Controller

Based on the optimal decoder-controller structure characterized in Lemma 1, we propose a controller, referred to as *the LQ controller*, in Algorithm 2. The encoder and decoder in Algorithm 2 use an adaptive quantizer generated by the Lloyd algorithm [4], [39], [40] and estimate z_t using recursive Bayesian estimation. The encoder computes the prior density function[2]

$$\begin{aligned} f(z_t | s^{t-d-1}) &= \int_{-\infty}^{\infty} f(z_t, z_{t-1} | s^{t-d-1}) dz_{t-1} \\ &= \int_{-\infty}^{\infty} f(z_t | z_{t-1}, s^{t-d-1}) f(z_{t-1} | s^{t-d-1}) dz_{t-1}, \end{aligned} \tag{14}$$

where $f(z_t | z_{t-1}, s^{t-d-1})$ can be computed by

$$\begin{aligned} f(z_t | z_{t-1}, s^{t-d-1}) &= f(z_t | z_{t-1}) \\ &= f(A z_{t-1} + A^d w_{t-d-1} + u_{t-1} | z_{t-1}). \end{aligned}$$

Then, $f(z_t | s^{t-d-1})$ is used to run the Lloyd algorithm [4], [39], [40] to find a quantizer $\mathcal{Q}_t$ that maps z_t to s_t. Given

[1] In this paper we consider the scalar system (1), except for Lemma 1 that is for the vector system (8). But notice that we treat a scalar as a vector or matrix (of dimension one) in many equations.

[2] With a slight abuse of notation, we use $f(x|y)$ to denote both the probability density function of a random variable x conditioned on another random variable y and the function that is computed by the controller to approximate the actual density function.

the received codeword s_{t-d} at the decoder, the update process computes the posterior density function

$$\begin{aligned} f(z_t|s^{t-d}) &= \frac{f(z_t, s_{t-d}|s^{t-d-1})}{f(s_{t-d}|s^{t-d-1})} \\ &= \frac{f(z_t|s^{t-d-1})f(s_{t-d}|z_t, s^{t-d-1})}{f(s_{t-d}|s^{t-d-1})} \\ &\propto f(z_t|s^{t-d-1})f(s_{t-d}|z_t, s^{t-d-1}), \end{aligned} \tag{15}$$

where $f(z_t|s^{t-d-1})$ is the prior density function computed in (14), and $f(s_{t-d}|z_t, s^{t-d-1})$ is determined by the quantizer $\mathcal{Q}_t$. Finally, $f(z_t|s^{t-d})$ is used to generate an estimate of z_t as follows:

$$\hat{z}_t \;=\; \mathbb{E}[z_t|s^{t-d}] = \int_{-\infty}^{\infty} z_t f(z_t|s^{t-d}) dz_t. \tag{16}$$

Algorithm 2: The LQ controller

Initialize:
1) Compute $f(z_d|s^0) = \mathcal{N}(0, \sigma^2)$.
2) Set $z_d = 0, u_0 = 0$.

Encoder: At time t, the encoder performs the following procedures:
1) Update the auxiliary variable (11).
2) Generate the prior density function by (14).
3) Run the Lloyd algorithm to obtain $\mathcal{Q}_t$.
4) Send the codeword $s_t = \mathcal{Q}_t(z_t)$ to the decoder.
5) Generate the posterior density function by (15).

Decoder: At time t, the decoder receives the codeword s_{t-d} that was generated d sampling intervals before, and performs the following procedures:
1) Compute the prior density function by (14).
2) Run the Lloyd algorithm to recover $\mathcal{Q}_t$.
3) Use the delayed codeword s_{t-d} to generate the posterior density function by (15).
4) Calculate the estimate $\hat{z}_t$ of z_t by (16).
5) Compute the control action:

$$u_t = -(Q + P^*)^{-1} P^* A \, \hat{z}_t. \tag{17}$$

The proposed LQ controller may not be optimal, but can be shown to achieve near optimal performance by comparing with the lower bound (6) of the optimal performance. As mentioned in the above, the first two terms of the lower bound are tight for any delay d if the decoder-controller has the structure (10), which is the case for the LQ controller. Thus, the performance gap to the lower bound reduces mostly to the difference between the achievable $(z_t - \hat{z}_t)G^*(z_t - \hat{z}_t)$ and the lower bound of $\mathbb{E}[(z_t - \hat{z}_t)G^*(z_t - \hat{z}_t)]$. Fig. 2 shows a comparison between the LQ controller and the lower bound. We see that the LQ controller achieves near optimal performance when the bandwidth R is large enough.

The Gaussian distribution has infinite support, i.e., the LQ controller can handle large disturbance, as opposed to the ℓ_∞ controller that can only handle bounded disturbance. However, the LQ controller is demanding in both computation and memory, due to the use of an adaptive quantizer that is

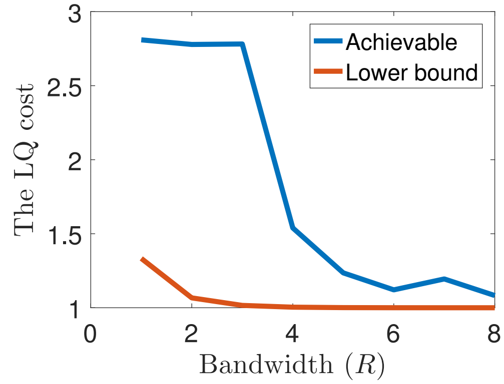

Fig. 2: The achievable performance of the LQ controller versus the lower bound (6) on the optimal performance for the system with $A = 1$, $d = 0$, and $\sigma^2 = 1$.

necessary for stabilizing an unstable system if the disturbance has an infinite support [15].

IV. A Hybrid Robust Controller

We have seen from the previous sections that the ℓ_∞ controller has low time and space complexity but can only handle bounded disturbance, while the LQ controller can reject arbitrarily large disturbance but incurs much higher time and space complexity. An interesting question that arises from these differences is if it is possible to design a controller that has the advantages of both controllers. In this section, we take a hybrid approach to design such a controller.

Specifically, we assume that the *typical* disturbance is relatively small and covered by a bounded set, while the large disturbance (outside of the bounded set) is *rare* event that has a (tail) Gaussian distribution. Under this assumption, we construct a hybrid controller that interpolates between the ℓ_∞ controller and the LQ controller. Using both theoretical bounds and numerical simulations, we show that the hybrid controller can achieve a sweet spot in the robustnes-complexity tradeoff, *i.e.*, reject occasional large disturbance while operate with low complexity most of the time.

A. The Hybrid Controller

We now assume that the LQ cost function has no control cost, *i.e.*, $Q = 0$ in (5), yielding the optimal LQ controller

$$u_t = -A\hat{z}_t \tag{18}$$

to replace (17) in Algorithm 2. This simplification allows the ℓ_∞ and LQ controllers to be considered in an unified framework.

The proposed hybrid controller has two modes: *normal mode* that runs the ℓ_∞ controller (Algorithm 1) and *acute mode* that runs the LQ controller (Algorithm 2). We now explain the switching policy between the ℓ_∞ and LQ controllers using a bridging variable z_t and a design parameter L. Notice that the sequences $\{z_t\}$ in the ℓ_∞ and LQ controllers have identical role (storing the sum of the quantization error from past control action and the scaled disturbance $A^d w_{t-d-1}$), and thus can

Algorithm 3: The hybrid controller

Initialize: $mode \leftarrow$ '$normal$'
  $\Psi(L) \leftarrow \{|A^{d+2}|(2^R - |A|)^{-1} + |A^{d+1}|\}L$
for $t \in \mathbb{N}$ **do**
  if *mode* = *'normal'* **then**
    Perform the ℓ_∞ controller (Algorithm 1)
    if $|z_t| > \Psi(L)/A$ **then**
      $mode \leftarrow$ '$acute$'
    end
  else
    Perform the LQ controller (Algorithm 2)
    if $|z_t| \leq \Psi(L)/A$ **then**
      $mode \leftarrow$ '$normal$'
    end
  end
end

serve as a bridging variable to connect the two controllers. Re-define the sequence $\{q_t\}$ as

$$q_{t+1} = Aq_t + u_t + A^{d+1}w_{t-d-1} \tag{19}$$

with $w_t = 0$ for $t < 0$. The definition (19) does not rely on the particular realization of the controller, so q_t is well-defined in both Algorithms 1 and 2. Using q_t, z_t can be written as

$$z_t = A^d w_{t-d-1} + q_t \tag{20}$$

with the $z_t = 0$ for $t \leq d$. Thus, z_t in Algorithm 2 satisfies

$$\begin{aligned} z_{t+1} &= Az_t + A^d w_{t-d} + u_t \\ &= A^d w_{t-d} + Aq_t + u_t + A^{d+1}w_{t-d-1} \\ &= A^d w_{t-d} + q_{t+1}, \end{aligned} \tag{21}$$

where the first equality follows form (11), the second equality from (20), and the third equality from (19). Therefore, z_t takes the same value in both Algorithms 1 and 2. The proposed controller sets a threshold on the absolute value of z_t to determine whether the ℓ_∞ controller or the LQ controller should be used.

Let the design parameter $L \in \mathbb{R}$ be the size of the disturbance up to which the controller stays in normal mode, *i.e.*, normal mode when $\|w\|_\infty \leq L$. Since $\|w_0^{t-d-1}\|_\infty \leq L$ implies $|z_t| \leq \Psi(L)/A$, equivalently $|z_t| > \Psi(L)/A$ implies $|w_\tau| \geq L$ for some $\tau \leq t-d-1$. Thus, the condition

$$|z_t| > \Psi(L)/A \tag{22}$$

is a sufficient condition for $\|w_0^{t-d-1}\|_\infty > L$. We use this sufficient condition to define the switching policy as follows:

$$mode = \begin{cases} \text{'}normal\text{'} & |z_t| \leq \Psi(L)/A, \\ \text{'}acute\text{'} & |z_t| > \Psi(L)/A. \end{cases} \tag{23}$$

The proposed hybrid controller is described in Algorithm 3.

The design parameter L impacts the system performance and controller complexity, and there exists a tradeoff between the two. We will next discuss its choice and the resulting performance and complexity tradeoff.

B. Switching Behavior

In this subsection, we analyze the behavior of the hybrid controller using the switching time from normal to acute mode and the recovery time from acute to normal mode. We denote the set of times at which the controller switches from normal to acute mode as

$$\mathcal{T}_s = \{t \in \mathbb{N} : |z_t| > \Psi(L)/A \;\; \& \;\; |z_{t-1}| \leq \Psi(L)/A\},$$

and the set of time at which the controller switches from acute to normal mode as

$$\mathcal{T}_r = \{t \in \mathbb{N} : |z_t| \leq \Psi(L)/A \;\; \& \;\; |z_{t-1}| > \Psi(L)/A\}.$$

Let $t_r \in \{0\} \cup \mathcal{T}_r$ be the beginning of a normal mode, the switching time T_s is defined as

$$T_s(t_r) = \min\{t > t_r : |z_t| > \Psi(L)/A\} - t_r. \tag{24}$$

Let $t_s \in \mathcal{T}_s$ be the beginning of an acute mode, the recovery time T_r is similarly defined as

$$T_r(t_s) = \min\{t > t_s : |z_t| \leq \Psi(L)/A\} - t_s. \tag{25}$$

Long switching time and short recovery time imply that the controller stays in normal mode most of the time, and thus requires less computation and memory. Therefore, the controller complexity can be roughly characterized by the time of operating in acute mode.

Let a random variable w be drawn from the same distribution with the disturbance w_t, *i.e.*, $w, w_t \overset{i.i.d.}{\sim} \mathcal{N}(0,\sigma)$. The following result characterizes the relation between the design parameter L and the expected switching time $\mathbb{E}[T_s(t_r)]$.

Theorem 2: Define a mappng $\hat{T}_s : \mathbb{R} \to \mathbb{R}_+$

$$\hat{T}_s(t_r) = \begin{cases} d + \mathbb{P}(|w| > L)^{-1} & t_r = 0, \\ \mathbb{P}(|w| > L)^{-1} & t_r \in \mathcal{T}_r. \end{cases}$$

The expected switching time $T_s(t_r)$ is lower bounded by

$$\mathbb{E}[T_s(t_r)] \geq \hat{T}_s(t_r), \tag{26}$$

and the lower bound becomes tight as the bandwidth $R \to \infty$.

Proof: See appendix. ∎

The proof of Theorem 2 uses the concept of *majorization* to approximate the switching time by *geometric distribution*. Theorem 2 suggests that the expected switching time can be approximated by $\mathbb{E}[T_s(t_r)] \approx \hat{T}_s(t_r)$.

Similarly, the expected recovery time $T_r(\cdot)$ can be approximated by

$$\mathbb{E}[T_r(\cdot)] \approx \hat{T}_r = \mathbb{P}(|w| \leq L)^{-1}. \tag{27}$$

Recall from (21) that the evolution of z_t follows $z_{t+1} = A^d w_{t-d} + q_{t+1}$ where q_{t+1} is a function of z_t. Assuming the quantizer (defined from the encoder and decoder) is near-optimal, a large z_{t_s} at the beginning of the an acute mode is approximately reduced by rate $|A|2^{-R}$ per unit time and by $|A^\tau|2^{-\tau R}$ after τ times. Thus, for sufficiently large $|A|2^{-R}$, the term $A^d w_{t-d}$ in (21) dominates. In this situation, observing a small disturbance, *i.e.*, $|w_{t-d}| \leq L$, is enough to lessen the value of z_t below $\Psi(L)/A$. This explains why the recovery time can be approximated by a geometric distribution with success probability $\mathbb{P}(|w_t| \leq L)$.

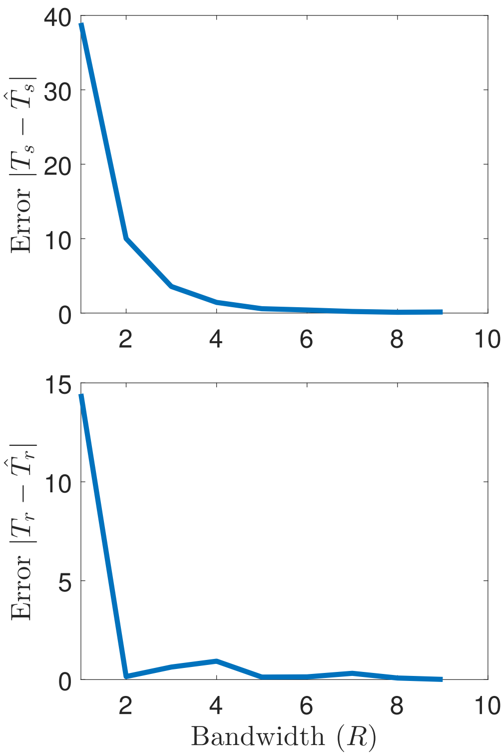

Fig. 3: The accuracy of the theoretical approximations (26) of the switching time and (27) of the recovery time for a system with $A = 1$ and $d = 1$. The empirical values of T_s and T_r are first generated by averaging 100 trials for different values of $L \in [0.1, 2]$ and $R \in \{1, 2, \cdots, 9\}$. Then, the approximation errors $|T_s - \hat{T}_s|$ and $|T_r - \hat{T}_r|$ are averaged over all L, and their mean values are plotted for different values of R.

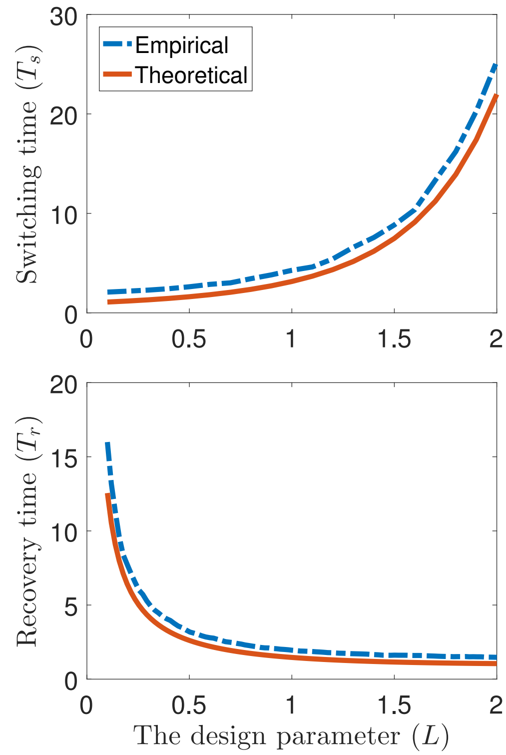

Fig. 4: The switching and recovery times as a function of L for a system with $A = 1$, $d = 1$, and $R = 6$. The averages over 100 trials are plotted for the empirical values.

Fig. 3 shows a comparison between the empirical value of the expected switching time $T_s(0)$ and the theoretical approximation $\hat{T}_s(0)$ and between the empirical value of the expected recovery time $T_r(\cdot)$ and the theoretical approximation $\hat{T}_r$. We see that the approximation becomes tight when the bandwidth R is large enough.

C. The Performance versus Complexity Tradeoff

The above theoretical approximations suggest that, for sufficiently large bandwidth ($|A|2^{-R} \ll 1$), a greater L implies larger switching time (from $T_s(t_r) \approx \hat{T}_s(t_r) = \mathbb{P}(|w_t| > L)^{-1}$) and smaller recovery time (from $T_r(t_s) \approx \hat{T}_r(t_s) = \mathbb{P}(|w_t| \leq L)^{-1}$). This can be empirically verified; see, e.g., Fig. 4. Since the switching (recovery) time is an increasing (decreasing) function of L, the complexity of the hybrid controller decreases as L increases.

One the other hand, the decrease in controller complexity comes with cost of degraded performance because a larger L also implies a coarser quantizer in Algorithm 1 (and thus larger quantization error). Specifically, in normal mode,

$$|x_t| \leq \left(\sum_{i=0}^{d} |A^i| + |A^{d+1}|(2^R - |A|)^{-1} \right) L. \tag{28}$$

So, the worst-case ℓ_∞ cost in normal model is an increasing function of L, and a smaller L leads to better performance.

Therefore, there is a tradeoff between performance and complexity, as shown in, e.g., Fig. 5. Fig. 5 (and other numerical experiments) also shows that significant increase (decrease) in switching (recovery) time can be achieved with small performance degradation (notice that the vertical axes are in log-scale).

D. Performance under Mixed Disturbance

We now take a look at the performance of the proposed controllers (Algorithms 1-3) under the mixed disturbance:

$$w_t = v_t + r_t \tag{29}$$

with $v_t \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma_v^2)$ and $\|r\|_\infty \leq 1$. We use this type of structured disturbance to model the common situation where the system experiences bounded disturbance most of the time and large disturbance occasionally (*i.e.*, with small probability).

For a feedback system with perfect communications, the optimal ℓ_∞ controller and LQ controller for the scalar system (1) are identical when the control cost is not considered. However, with communication constraints, the optimal ℓ_∞ controller and LQ controller are radically different, and the mixed disturbance poses significant challenge in encoding/decoding strategies as the system state can be defined neither in a worst-case framework nor in a stochastic framework.

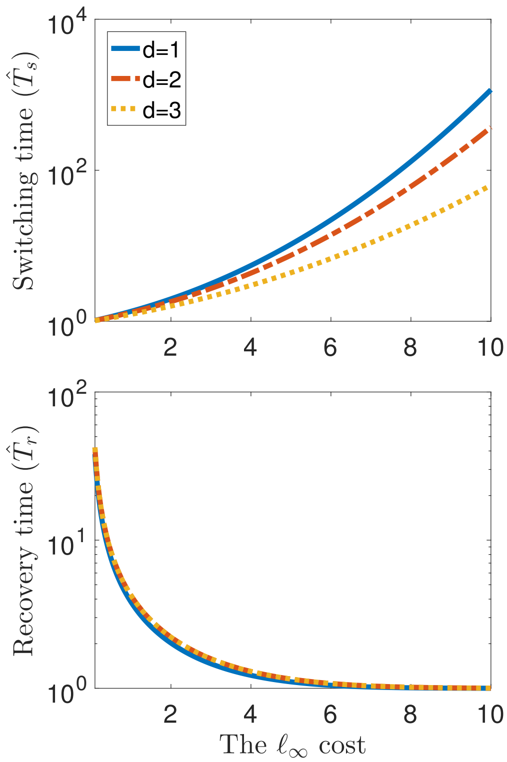

Fig. 5: The tradeoff between complexity (as implied by the switching time and the recovery time) and performance (as represented by the worst-case ℓ_∞ cost in normal mode) for a system with $A = 1$ and $(d, R) = \{(1, 1), (2, 2), (3, 3)\}$.

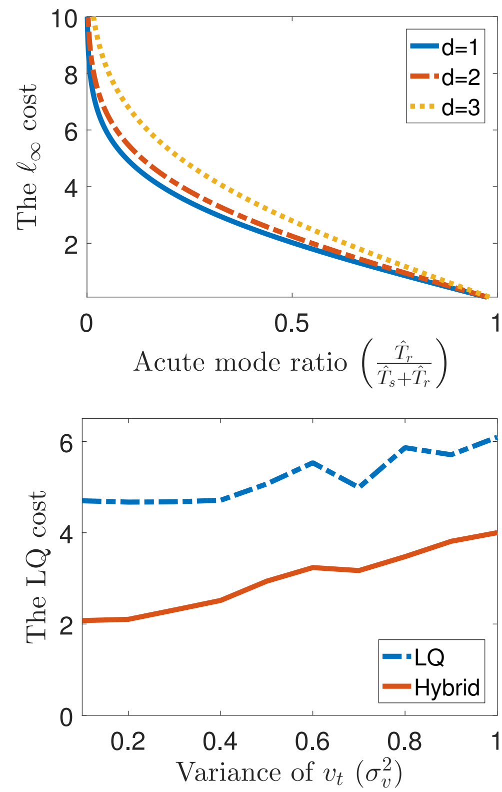

Fig. 6: Performance of the hybrid controller. The figure on the top shows the tradeoff between the normal mode performance (in the ℓ_∞ cost) and the approximated acute mode ratio $\hat{T}_s/(\hat{T}_r + \hat{T}_s)$ for a system with $A = 1$ and $(d, R) = \{(1, 1), (2, 2), (3, 3)\}$. The figure on the bottom shows the performance (in the LQ cost) for a system with $A = 1$, $d = 1$, $R = 3$ and under the mixed disturbance with different variances σ_v^2. The averaged LQ costs for 100 trials are plotted.

The ℓ_∞ controller cannot stabilize such systems because there is a non-zero probability for the fixed quantizer to saturate. The performance of the LQ controller and the proposed hybrid controller is compared in Fig. 6. The LQ controller has degraded performance when there exists an additional disturbance r that cannot be well-defined using probability density function. However, the proposed hybrid controller consistently achieves robust performance under such disturbance. By exploiting the additional dimension in the controller design space, the right inegration of stochastic (LQ) and worst-case (ℓ_∞) enables a robust controller under communication constraints.

V. Conclussion

We have considered robust control design for linear systems with delayed and rate constrained communications between the observer and the controller. We first take a stochastic approach and propose an LQ controller that can handle arbitrarily large disturbance but has large complexity in time and space. This is different from the ℓ_∞ control (a deterministic approach) that previous work have shown to have low time/space complexity but can only handle bounded disturbance. The differences in robustness and complexity of the LQ and ℓ_∞ controllers motivate the design of a hybrid controller that interpolates between the ℓ_∞ and LQ controllers. Using both theoretical bounds and numerical examples, we show that the hybrid controller can achieve a sweet spot in the robustness-complexity tradeoff, *i.e.*, reject occasional large disturbance while operate with low complexity most of the time.

VI. Acknowledgments

We thank John Doyle, Victoria Kostina, Ivan Papusha, and David Hui for insightful discussion.

VII. Appendix

In this section, we provide the proofs for the main results in the paper.

A. *Proof of Lemma 1*

Define

$$e_t = w_{t-1} + Aw_{t-2} + \cdots + A^{d-1}w_{t-d}, \tag{30}$$

$$z_t = x_t - e_t, \tag{31}$$

where e_t captures the component in the state x_t that results from the disturbance w_{t-d}^{t-1} and cannot be mitigated due to the delay in control, while z_t depends on the information of

w_0^{t-d-1} and the control action in response to it. Obviously, z_t and e_t are *independent*. Moreover, $\mathbb{E}[e_t]=0$, and z_t satisfies equation (11), restated below??

$$z_{t+1}=Az_t+A^d w_{t-d}+Bu_t, \quad z_0=0.$$

In order to decompose the effects of control action and disturbance, we define $\bar{z}_t$ to be the state z_t that would be generated at time t when the system (8) has zero control $u_t\equiv 0$. Setting $u_t=0$ in the above equation, we obtain

$$\bar{z}_{t+1}=A\bar{z}_t+A^d w_{t-d}, \quad \bar{z}_0=0. \tag{32}$$

Recall that $\{s_t\}$ is the codewords generated by $\{z_t\}$. We introduce an auxiliary encoder

$$f(\bar{s}_{t-d}|\bar{z}^t,\bar{s}^{t-d-1})=f(s_{t-d}|\bar{z}^t,s^{t-d-1}) \tag{33}$$

to generate another sequence of codewords $\{\bar{s}_t\}$.

Lemma 2: The following relation holds:

$$z_t-\mathbb{E}[z_t|s^{t-d}]=\bar{z}_t-\mathbb{E}[\bar{z}_t|\bar{s}^{t-d}].$$

Proof: (Lemma 2) We first use mathematical induction to show

$$f(s^{t-d},\bar{z}^t)=f(\bar{s}^{t-d},\bar{z}^t). \tag{34}$$

Obviously, (34) holds at $t=0$. If (34) holds until t, then (34) also holds for $t+1$ because

$$\begin{aligned}
&f(\bar{s}^{t-d+1},\bar{z}^{t+1})\\
=&f(\bar{s}^{t-d},\bar{z}^t)f(\bar{z}_{t+1}|\bar{s}^{t-d},\bar{z}^t)f(\bar{s}_{t-d+1}|\bar{s}^{t-d},\bar{z}^{t+1})\\
=&f(s^{t-d},\bar{z}^t)f(\bar{z}_{t+1}|s^{t-d},\bar{z}^t)f(s_{t-d+1}|s^{t-d},\bar{z}^{t+1})\\
=&f(s^{t-d+1},\bar{z}^{t+1}),
\end{aligned}$$

where the second equality is due to construction (33), the induction hypothesis (34), and the fact that $f(\bar{z}_{t+1}|\bar{s}^{t-d},\bar{z}^t)=f(\bar{z}_{t+1}|\bar{z}^t)=f(\bar{z}_{t+1}|s^{t-d},\bar{z}^t)$. By (34), we obtain

$$\begin{aligned}
\mathbb{E}[z_t|s^{t-d}]&=\mathbb{E}[\bar{z}_t+\sum_{k=1}^{t}A^{k-1}Bu_{t-k}|s^{t-d}]\\
&=\mathbb{E}[\bar{z}_t|s^{t-d}]+\sum_{k=1}^{t}A^{k-1}Bu_{t-k}\\
&=\mathbb{E}[\bar{z}_t|\bar{s}^{t-d}]+\sum_{k=1}^{t}A^{k-1}Bu_{t-k},
\end{aligned}$$

and thus

$$\begin{aligned}
&z_t-\mathbb{E}[z_t|s^{t-d}]\\
=&\bar{z}_t+\sum_{k=1}^{t}A^{k-1}Bu_{t-k}-\left(\mathbb{E}[\bar{z}_t|\bar{s}^{t-d}]+\sum_{k=1}^{t}A^{k-1}Bu_{t-k}\right)\\
=&\bar{z}_t-\mathbb{E}[\bar{z}_t|\bar{s}^{t-d}].
\end{aligned}$$

■

Lemma 2 implies that we can negate all the effect of the control action to obtain $\bar{z}_t$. Intuitively, this is because u_0^t is generated from s^{t-d}. This separation allows us to prove Lemma 1.

Proof: (Lemma 1) Consider the cost-to-go:

$$J_t(s^{t-d})=\mathbb{E}\left[x_N'Px_N+\sum_{\tau=t}^{N-1}x_\tau'Px_\tau+u_\tau'Qu_\tau\middle|s^{t-d}\right]$$

for any $k<N$ and $J_N=\mathbb{E}[x_N'Px_N]$. We use mathematical induction to show the following properties:

(i) The optimal cost-to-go satisfies

$$J_t(s^{t-d})=\mathbb{E}\left[\hat{z}_t'P_t\hat{z}_t|s^{t-d}\right]+\alpha_t(s^{t-d}), \tag{35}$$

where $\hat{z}_t=\mathbb{E}[z_t|s^{t-d}]$ and $\alpha_t(s^{t-d})$ is a function of s^{t-d} whose expected value does not depend on the choice of control action, *i.e.*,

$$\mathbb{E}\left[\alpha_t(s^{t-d})\right]=\mathbb{E}\left[\alpha_t(\bar{s}^{t-d})\right]. \tag{36}$$

(ii) The optimal controller admits the form (10).

At $t=N$, the cost-to-go satisfies

$$\begin{aligned}
J_N&=\mathbb{E}[x_N'Px_N|s^{N-d}]\\
&=\mathbb{E}[(\hat{z}_N+\tilde{z}_N+e_N)'P(\hat{z}_N+\tilde{z}_N+e_N)|s^{N-d}]\\
&=\mathbb{E}[\hat{z}_N'P\hat{z}_N|s^{N-d}]+\mathbb{E}[\tilde{z}_N'P\tilde{z}_N|s^{N-d}]+\mathbb{E}[e_N'Pe_N],
\end{aligned}$$

where $\tilde{z}_t:=z_t-\hat{z}_t$, and the last equality holds because e_N, $\hat{z}_N$ and $\tilde{z}_N$ are uncorrelated and e_N is independent of s^{N-d}. By Lemma 2, $\mathbb{E}[\tilde{z}_N'P\tilde{z}_N|s^{N-d}]$ does not depend on the choice of control action. Letting $\alpha_N=\mathbb{E}[\tilde{z}_N'P\tilde{z}_N|s^{N-d}]+\mathbb{E}[e_N'P_Ne_N]$ yields (35) for $t=N$.

Assume now that (35) holds for $t=k+1$. The optimal cost-to-go at time $t=k$ can be derived as follows:

$$\begin{aligned}
&J_k(s^{k-d})\\
&=\min_{u_k}\mathbb{E}[x_k'Px_k+u_k'Qu_k+J_{k+1}|s^{k-d}] \qquad (37)\\
&=\min_{u_k}\mathbb{E}[x_k'Px_k+u_k'Qu_k\\
&\qquad+\mathbb{E}\left[\hat{z}_{k+1}'P_{k+1}\hat{z}_{k+1}|s^{k-d+1}\right]+\alpha_{k+1}|s^{k-d}]\\
&=\min_{u_k}\mathbb{E}[\hat{z}_k'(P+A'P_{k+1}A)\hat{z}_k+u_k'(Q+B'P_{k+1}B)u_k\\
&\qquad+u_k'B'P_{k+1}A\hat{z}_k+\hat{z}_k'A'P_{k+1}Bu_k|s^{k-d}] \qquad (38)\\
&\qquad+\mathbb{E}[e_k'Pe_k+\hat{w}_k'P_{k+1}\hat{w}_k+\tilde{z}_k'P\tilde{z}_k|s^{k-d}]\\
&\qquad+\mathbb{E}[\alpha_{k+1}(s^{k-d+1})|s^{k-d}],
\end{aligned}$$

where $\hat{w}_k=\mathbb{E}[A^dw_{k-d}+A\tilde{z}_k|s^{k-d+1}]$, and by induction hypothesis the second equality holds. By Lemma 2 and induction hypothesis, $e_k'Pe_k+\hat{w}_k'P_{k+1}\hat{w}_k+\tilde{z}_k'P\tilde{z}_k$ does not depend on the control action u_t. Therefore, we can just consider minimizing the first term (38). The control action that minimizes this term is given by (10), i.e.,

$$u_k=-(Q+B'P_{k+1}B)^{-1}B'P_{k+1}A\ \hat{z}_k,$$

where

$$P_k=A'\left[P_{k+1}+P-P_{k+1}B(Q+B'P_{k+1}B)^{-1}B'P_{k+1}\right]A.$$

Substituting this control action u_k into J_k, we obtain the optimal cost-to-go

$$J_k(s^{k-d})=\mathbb{E}\left[\hat{z}_k'P_k\hat{z}_k|s^{k-d}\right]+\alpha_k(s^{k-d})$$

with

$$\alpha_k(s^{k-d}) = \mathbb{E}[e_k' P e_k + \hat{w}_k' P_{k+1} \hat{w}_k + \tilde{z}_k' P \tilde{z}_k + \alpha_{k+1} | s^{k-d}]$$
$$= \mathbb{E}[e_k' P e_k + \tilde{z}_t' A' P_{k+1} A \tilde{z}_t + w_{t-d}' (A^d)' P_{k+1} A^d w_{t-d}$$
$$- \tilde{z}_{t+1}' P_{k+1} \tilde{z}_{t+1} + \tilde{z}_k' P \tilde{z}_k + \alpha_{k+1}(s^{k-d+1}) | s^{k-d}], \quad (39)$$

where the second equality is obtained as follows. Given s^{k-d}, the random variable $\hat{w}_k$ is the estimate of $A^d w_{k-d} + A\tilde{z}_k$ given s_{k-d+1}, and the random variable $\tilde{z}_{k+1}$ is the resulting estimation error, *i.e.*,

$$\hat{w}_k + \tilde{z}_{k+1}' = A^d w_{k-d} + A\tilde{z}_k. \quad (40)$$

Therefore, the weighted covariance of the estimation target equals the sum of the weighted estimation error covariance and the weighted estimate covariance

$$\mathbb{E}[(A^d w_{k-d} + A\tilde{z}_k)' P_{k+1} (A^d w_{k-d} + A\tilde{z}_k) | s^{k-d}]$$
$$= \mathbb{E}[\tilde{z}_{k+1}' \tilde{z}_{k+1} | s^{k-d}] + \mathbb{E}[\hat{w}_k' P_{k+1} \hat{w}_k | s^{k-d}]$$

Combining above with

$$\mathbb{E}[(A^d w_{k-d} + A\tilde{z}_k)' P_{k+1} (A^d w_{k-d} + A\tilde{z}_k) | s^{k-d}]$$
$$= \mathbb{E}[\tilde{z}_t' A' P_{k+1} A \tilde{z}_t + w_{k-d}' (A^d)' P_{k+1} A^d w_{k-d} | s^{k-d}]$$

yields (39). By Lemma 2 and the induction hypothesis $\mathbb{E}[\alpha_{k+1} | s^{k-d}] = \mathbb{E}[\alpha_{k+1} | \bar{s}^{k-d}]$, α_k does not depend on the choice of control action. So, equation (35) holds for $t = k$. ■

From the proof of lemma, we can observe that, given any encoder, the optimal decoder are essentially the optimal LQ controller for the sequence $\hat{z}_t$, which evolves according to the dynamics

$$\hat{z}_{t+1} = A\hat{z}_t + Bu_t + \hat{w}_t. \quad (41)$$

In other words, the optimal decoder are the certainty equivalent controller for the sequence z_t, the estimation target of $\hat{z}_t$. When there is no delay in the control action, *i.e.*, $d = 0$, then this optimal decoder reduces to the certainty equivalent controller for x_t, as is given by [22].

B. Proof of Theorem 1

We first describe a result that will be used later.

Lemma 3 ([22], [29]): Consider a scalar Gauss-Markov sequence $\{y_t\}$ satisfying

$$y_{t+1} = Ay_t + v_t, \quad y_0 = 0, \quad (42)$$

where $A \in \mathbb{R}$, $y_t \in \mathbb{R}$, and $v_t \overset{i.i.d.}{\sim} \mathcal{N}(0, \sigma^2)$. Assume that at each time t, only $R(> \log_2 |A|)$ bits of information about y^t can be transmitted to $s_t \in S$, where $|S| = 2^R$ and s_t is a function of (y^t, s^{t-1}). Let $\hat{y}_t$ be an estimate of y_t using only the information of s^t. Then, the following inequality holds:

$$\lim_{t\to\infty} \frac{1}{N} \mathbb{E}\left[\sum_{t=1}^{N} (y_t - \hat{y}_t)^2\right] \geq \frac{\sigma^2}{2^{2R} - A^2}.$$

With Lemmas 1 and 3, we are ready to proveTheorem 1.

Proof: (Theorem 1) By equation (35),[3]

$$\begin{aligned} &\lim_{t\to\infty} \mathbb{E}[x_t' P x_t + u_t' Q u_t] \\ =\; &\lim_{N\to\infty} \frac{1}{N} \mathbb{E}\left[x_N' P x_N + \sum_{t=0}^{N-1} x_t' P x_t + u_t' Q u_t\right] \\ =\; &\lim_{N\to\infty} \frac{1}{N} \mathbb{E}[J_1] \\ =\; &\lim_{N\to\infty} \frac{1}{N} \mathbb{E}[J_d(s^0)] \\ =\; &\lim_{N\to\infty} \frac{1}{N} \mathbb{E}\left[\mathbb{E}\left[\hat{z}_d P_d \hat{z}_d | s^0\right] + \alpha_d(s^0)\right] \\ =\; &\lim_{N\to\infty} \frac{1}{N} \mathbb{E}\left[\alpha_d(s^0)\right] \end{aligned}$$

Next we observe that $\mathbb{E}[\alpha_t]$ satisfies the relation

$$\mathbb{E}[\alpha_k(s^{k-d})] \quad (43)$$
$$= \mathbb{E}[\mathbb{E}[\alpha_{k+1}(s^{k-d+1}) + e_k' P e_k + w_{k-d}' (A^d)' P_{k+1} A^d w_{k-d} \quad (44)$$
$$+ \tilde{z}_k' (A' P_{k+1} A + P) \tilde{z}_k' - \tilde{z}_{k+1}' P_{k+1} \tilde{z}_{k+1} | s^{k-d}]] \quad (45)$$
$$= \mathbb{E}[\alpha_{k+1}(s^{k-d+1})] + \mathbb{E}[e_k' P e_k + w_{k-d}' (A^d)' P_{k+1} A^d w_{k-d} \quad (46)$$
$$+ \tilde{z}_k' (A' P_{k+1} A + P) \tilde{z}_k' - \tilde{z}_{k+1}' P_{k+1} \tilde{z}_{k+1}] \quad (47)$$
$$= \mathbb{E}[\alpha_N(s^{N-d})] + \sum_{\tau=k}^{N-1} \mathbb{E}[e_\tau' P e_\tau + w_{\tau-d}' (A^d)' P_{\tau+1} A^d w_{\tau-d} \quad (48)$$
$$+ \tilde{z}_\tau' (A' P_{\tau+1} A + P) \tilde{z}_\tau' - \tilde{z}_{\tau+1}' P_{\tau+1} \tilde{z}_{\tau+1}] \quad (49)$$
$$(50)$$

Because the system is controllable, the Riccati difference equation (7) has a unique solution $P^\star$, and $\lim_{N\to\infty} P_k = P^\star$. Therefore, we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\tau=d}^{N-1} \mathbb{E}[w_{\tau-d}' (A^d)' P_{\tau+1} A^d w_{\tau-d}] \quad = P^\star A^{2d} \sigma^2 \quad (51)$$

and

$$\lim_{N\to\infty} \frac{1}{N} \sum_{\tau=d}^{N-1} \mathbb{E}[\tilde{z}_t' (A' P_{k+1} A + P - P_{k+1}) \tilde{z}_t] \quad (52)$$
$$= \lim_{N\to\infty} \frac{1}{N} \sum_{t=1}^{N-1} \mathbb{E}[\tilde{z}_t' (A' P^\star A + P - P^\star) \tilde{z}_t]. \quad (53)$$

[3]With a slight abuse of notation, we use J_1 without the conditioning of the sequence s_t because it is purely determined from the initial condition.

Combining (43)–(49) and (51)–(53), we obtain that

$$\begin{aligned}
&\lim_{N\to\infty}\frac{1}{N}\mathbb{E}[\alpha_d(s^0)]\\
&= P(1+A^2+A^4+\cdots+A^{2(d-1)})\sigma^2 + P^\star A^{2d}\sigma^2\\
&\quad+\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N-1}\mathbb{E}[\tilde{z}_t'(A'P_{k+1}A+P-P_{k+1})\tilde{z}_t]\\
&= P\sum_{i=0}^{d-1}A^{2i}\sigma^2 + P^\star A^{2d}\sigma^2\\
&\quad+\lim_{N\to\infty}\frac{1}{N}\sum_{t=1}^{N-1}\mathbb{E}[\tilde{z}_t'(A'P^\star A+P-P^\star)\tilde{z}_t].
\end{aligned}$$

When $x_t \in \mathbb{R}$, from Lemma 3, the second term is lower bounded by

$$\begin{aligned}
&\mathbb{E}[(\bar{z}_t-\mathbb{E}[\bar{z}_t|\bar{s}^{t-d}])'G^\star(\bar{z}_t-\mathbb{E}[\bar{z}_t|\bar{s}^{t-d}])]\\
&\geq G^\star A^{2d}\frac{\sigma^2}{2^{2R}-A^2}
\end{aligned}$$

Therefore, we have obtained (9). ∎

C. Proof of Theorem 2

Proof: (Theorem 2) We first prove the lower bound for $\tau=0$. Let $\{E_k\}$ be the event that the controller switches at time k, *i.e.*,

$$E_k = \{|z_t|\leq \Psi(L)/A \text{ for all } t<k \text{ and } |z_k|>\Psi(L)/A\}.$$

Notice that $\{E_k\}$ a sequence of a mutually exclusive set of events, and that $\mathbb{P}(E_k)=0$ for $k\leq d$ (since $z_t=0$ for $t\leq d$ by definition). Let $\{F_k\}$ be the event that the disturbance first exceeds L in amplitude at time k, *i.e.*,

$$F_k \quad = \{|w_t|\leq L \text{ for all } t<k \text{ and } |w_k|>L\}.$$

The sequence $\{E_k\}$ is a mutually exclusive set of events, and $\lim_{\tau\to\infty}\sum_{i=0}^{\tau}\mathbb{P}(E_i)=1$. Same holds for $\{F_k\}$, *i.e.*, $\lim_{\tau\to\infty}\sum_{i=0}^{\tau}\mathbb{P}(F_i)=1$. From $\cup_{i\geq k}E_i\subset\cup_{i\geq k}F_i$, we obtain

$$\sum_{i=k-d-1}^{\infty}\mathbb{P}(F_i)\leq\sum_{i=k}^{\infty}\mathbb{P}(E_i) \tag{54}$$

for any $k\in\mathbb{N}$. Using (54), the expected switching time can be bounded below by

$$\begin{aligned}
\mathbb{E}[T_s(\tau)] &= \sum_{k=0}^{\infty}k\,\mathbb{P}(E_k)\\
&= \sum_{k=0}^{\infty}k\,\mathbb{P}(E_k)-\sum_{k=0}^{\infty}k\,\mathbb{P}(E_{k+d})+\sum_{k=0}^{\infty}k\,\mathbb{P}(E_{k+d})\\
&= d+\sum_{k=0}^{\infty}k\,\mathbb{P}(E_{k+d})\\
&= d+\sum_{k=1}^{\infty}\sum_{i=k}^{\infty}\mathbb{P}(E_{i+d})\\
&\geq d+\sum_{k=1}^{\infty}\sum_{i=k}^{\infty}\mathbb{P}(F_{i-1})\\
&= d+\sum_{k=1}^{\infty}k\,\mathbb{P}(F_{k-1})\\
&= d+\sum_{k=1}^{\infty}k\,\big(1-\mathbb{P}(|w|>L)\big)^{k-1}\mathbb{P}(|w|>L)\\
&= d+\mathbb{P}(|w|>L)^{-1},
\end{aligned}$$

where the last equality can be interpreted as computing the mean of a geometric distribution with failure probability $\mathbb{P}(|w|>L)$.

Next, notice that $|z_t|\leq\Psi(L)/A$ and $|w_{t-d}|\leq L$ implies $|z_{t+1}|\leq\Psi(L)/A$. Thus, we can apply the argument in $\tau=0$ to obtain the lower bound for $\tau\in\mathcal{T}$:

$$\mathbb{E}[T_s(\tau)]\leq\mathbb{P}(|w|>L)^{-1}.$$

Next, we prove the convergence for $\tau=0$, *i.e.*, $\mathbb{E}[T_s(0)]\overset{R\to\infty}{\to} d+\mathbb{P}(|w|>L)^{-1}$. Since $d+\sum_{k=1}^{\infty}\sum_{i=k}^{\infty}\mathbb{P}(E_{i+d})\geq d+\sum_{k=1}^{\infty}\sum_{i=k}^{\infty}\mathbb{P}(F_{i-1})$ is the only inequality from the above analysis, it is suffice to show that $|\sum_{k=1}^{\infty}\sum_{i=k}^{\infty}\mathbb{P}(E_{i+d})-\sum_{k=1}^{\infty}\sum_{i=k}^{\infty}\mathbb{P}(F_{i-1})|\to 0$. By $\|q\|_\infty\overset{R\to\infty}{\to}0$ and $z_t\to A^d w_{t-d-1}$, $\mathbb{P}(F_{t-d-1})\to\mathbb{P}(E_t)$. This implies that

$$\begin{aligned}
&\left|\sum_{i=k-1}^{\infty}\mathbb{P}(F_{i-1})-\sum_{i=k}^{\infty}\mathbb{P}(E_{i+d})\right|\\
&=\left|\left(1-\sum_{i=0}^{k-2}\mathbb{P}(F_i)\right)-\left(1-\sum_{i=0}^{k-1}\mathbb{P}(E_{i+d})\right)\right|\\
&\to 0 \text{ as } R\to\infty
\end{aligned}$$

holds for any $k\in\mathbb{N}$. Since both $\sum_{k=1}^{\infty}\sum_{i=k}^{\infty}\mathbb{P}(E_{i+d})$ and $\sum_{k=1}^{\infty}\sum_{i=k}^{\infty}\mathbb{P}(F_{i-1})$ are bounded, for any $\epsilon>0$ there exits a sufficiently large T such that $\tau>T$ implies

$$\sum_{k=\tau}^{\infty}\sum_{i=k}^{\infty}\mathbb{P}(E_{i+d})\leq\epsilon/4 \quad\text{and}\quad \sum_{k=\tau}^{\infty}\sum_{i=k}^{\infty}\mathbb{P}(F_{i-1})\leq\epsilon/4,$$

and sufficiently large $\bar{R}$ such that $R>\bar{R}$ implies

$$\sum_{k=1}^{\tau}\sum_{i=k}^{\infty}\mathbb{P}(E_{i+d})\leq\epsilon/4 \quad\text{and}\quad \sum_{k=1}^{\tau}\sum_{i=k}^{\infty}\mathbb{P}(F_{i-1})\leq\epsilon/4,$$

which jointly yields

$$\left|\sum_{k=1}^{\infty}\sum_{i=k}^{\infty}\,\mathbb{P}(E_{i+d})-\sum_{k=1}^{\infty}\sum_{i=k}^{\infty}\,\mathbb{P}(F_{i-1})\right|\leq\epsilon. \tag{55}$$

The case for $\tau\in\mathcal{T}$ follows the same argument and is omitted here. ■

References

[1] Y. Nakahira, "Lq vs. ℓ_∞ in controller design for systems with delay and quantization," in *IEEE 55th Conference on Decision and Control (CDC)*, Dec. 2016, pp. 2382–2389.

[2] G. N. Nair, F. Fagnani, S. Zampieri, and R. J. Evans, "Feedback control under data rate constraints: An overview," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 108–137, 2007.

[3] Y. Nakahira, N. Matni, and J. C. Doyle, "Hard limits on robust control over delayed and quantized communication channels with applications to sensorimotor control," in *IEEE 54th Annual Conference on Decision and Control (CDC)*, 2015, pp. 7522–7529.

[4] L. Bao, M. Skoglund, and K. H. Johansson, "Iterative encoder-controller design for feedback control over noisy channels," *IEEE Transactions on Automatic Control*, vol. 56, no. 2, pp. 265–278, 2011.

[5] A. Khina, Y. Nakahira, Y. Su, and B. Hassibi, "Algorithms for optimal control with fixed-rate feedback," in *IEEE 56th Conference on Decision and Control (CDC)*, 2017.

[6] J. Baillieul and P. J. Antsaklis, "Control and communication challenges in networked real-time systems," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 9–28, 2007.

[7] B. R. Andrievsky, A. S. Matveev, and A. L. Fradkov, "Control and estimation under information constraints: Toward a unified theory of control, computation and communications," *Automation and Remote Control*, vol. 71, no. 4, pp. 572–633, 2010.

[8] S. Fang, J. Chen, and I. Hideaki, *Towards integrating control and information theories*. Springer, 2017.

[9] L. Schenato, B. Sinopoli, M. Franceschetti, K. Poolla, and S. S. Sastry, "Foundations of control and estimation over lossy networks," *Proceedings of the IEEE*, vol. 95, no. 1, pp. 163–187, 2007.

[10] S. Yüksel and T. Basar, "Stochastic networked control systems," *AMC*, vol. 10, p. 12, 2013.

[11] D. D. Vecchio and R. M. Murray, *Biomolecular Feedback Systems*. Princeton University Press, 2014.

[12] I. Lestas, G. Vinnicombe, and J. Paulsson, "Fundamental limits on the suppression of molecular fluctuations," *Nature*, vol. 467, no. 7312, pp. 174–178, 2010.

[13] Y. Nakahira, F. Xiao, V. Kostina, and J. C. Doyle, "Fundamental limits and achievable performance in biomolecular control," in *The American Control Conference*, 2018.

[14] W. S. Wong and R. W. Brockett, "Systems with finite communication bandwidth constraints. ii. stabilization with limited information feedback," *IEEE Transactions on Automatic Control*, vol. 44, no. 5, pp. 1049–1053, 1999.

[15] G. N. Nair and R. J. Evans, "Stabilization with data-rate-limited feedback: Tightest attainable bounds," *Systems & Control Letters*, vol. 41, no. 1, pp. 49–56, 2000.

[16] S. Yuksel, "Stochastic stabilization of noisy linear systems with fixed-rate limited feedback," *IEEE Transactions on Automatic Control*, vol. 55, no. 12, pp. 2847–2853, 2010.

[17] G. N. Nair and R. J. Evans, "Stabilizability of stochastic linear systems with finite feedback data rates," *SIAM Journal on Control and Optimization*, vol. 43, no. 2, pp. 413–436, 2004.

[18] S. Tatikonda and S. Mitter, "Control over noisy channels," *IEEE Transactions on Automatic Control*, vol. 49, no. 7, pp. 1196–1201, 2004.

[19] T. Fischer, "Optimal quantized control," *IEEE Transactions on Automatic Control*, vol. 27, no. 4, pp. 996–998, 1982.

[20] M. Fu, "Lack of separation principle for quantized linear quadratic Gaussian control," *IEEE Transactions on Automatic Control*, vol. 57, no. 9, pp. 2385–2390, 2012.

[21] Y. Bar-Shalom and E. Tse, "Dual effect, certainty equivalence, and separation in stochastic control," *IEEE Transactions on Automatic Control*, vol. 19, no. 5, pp. 494–500, 1974.

[22] S. Tatikonda, A. Sahai, and S. Mitter, "Stochastic linear control over a communication channel," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1549–1561, 2004.

[23] C. D. Charalambous and A. Farhadi, "LQG optimality and separation principle for general discrete time partially observed stochastic systems over finite capacity communication channels," *Automatica*, vol. 44, no. 12, pp. 3181–3188, 2008.

[24] J. S. Freudenberg, R. H. Middleton, and V. Solo, "Stabilization and disturbance attenuation over a Gaussian communication channel," *IEEE Transactions on Automatic Control*, vol. 55, no. 3, pp. 795–799, 2010.

[25] E. I. Silva, M. S. Derpich, and J. Ostergaard, "A framework for control system design subject to average data-rate constraints," *IEEE Transactions on Automatic Control*, vol. 56, no. 8, pp. 1886–1899, 2011.

[26] S. Yuksel, "Jointly optimal LQG quantization and control policies for multi-dimensional systems," *IEEE Transactions on Automatic Control*, vol. 59, no. 6, pp. 1612–1617, 2014.

[27] T. Tanaka, P. M. Esfahani, and S. K. Mitter, "LQG control with minimum directed information: Semidefinite programming approach," *arXiv preprint arXiv:1510.04214*, 2015.

[28] E. I. Silva, M. S. Derpich, J. Østergaard, and M. A. Encina, "A characterization of the minimal average data rate that guarantees a given closed-loop performance level," *IEEE Transactions on Automatic Control*, vol. 61, no. 8, pp. 2171–2186, 2016.

[29] V. Kostina and B. Hassibi, "Rate-cost tradeoffs in control," in *arXiv:1612.02126*, 2016.

[30] A. Gorbunov and M. S. Pinsker, "Nonanticipatory and prognostic epsilon entropies and message generation rates," *Problemy Peredachi Informatsii*, vol. 9, no. 3, pp. 12–21, 1973.

[31] R. Bucy, "Distortion-rate theory and filtering," in *Advances in Communications*. Springer, 1980, pp. 11–15.

[32] M. S. Derpich and J. Ostergaard, "Improved upper bounds to the causal quadratic rate-distortion function for Gaussian stationary sources," *IEEE Transactions on Information Theory*, vol. 58, no. 5, pp. 3131–3152, 2012.

[33] F. Rezaei, N. Ahmed, and C. D. Charalambous, "Rate distortion theory for general sources with potential application to image compression," *International Journal of Applied Mathematical Sciences*, vol. 3, no. 2, pp. 141–165, 2006.

[34] C. D. Charalambous, P. A. Stavrou, and N. U. Ahmed, "Nonanticipative rate distortion function and relations to filtering theory," *IEEE Transactions on Automatic Control*, vol. 59, no. 4, pp. 937–952, 2014.

[35] T. M. Cover and J. A. Thomas, *Elements of information theory*. John Wiley & Sons, 2012.

[36] A. Sahai and S. Mitter, "The necessity and sufficiency of anytime capacity for stabilization of a linear system over a noisy communication link - part i: Scalar systems," *IEEE Transactions on Information Theory*, vol. 52, no. 8, pp. 3369–3395, 2006.

[37] J. Massey, "Causality, feedback and directed information," in *Proceedings of the International Symposium on Information Theory and Applications (ISITA-90)*, 1990, pp. 303–305.

[38] S. Tatikonda, A. Sahai, and S. Mitter, "Stochastic linear control over a communication channel," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1549–1561, 2004.

[39] S. Lloyd, "Least squares quantization in pcm," *IEEE transactions on information theory*, vol. 28, no. 2, pp. 129–137, 1982.

[40] Q. Du, V. Faber, and M. Gunzburger, "Centroidal voronoi tessellations: Applications and algorithms," *SIAM review*, vol. 41, no. 4, pp. 637–676, 1999.

PLACE
PHOTO
HERE

Yorie Nakahira is a Ph.D. student in Control and Dynamical Systems at California Institute of Technology. She received the bachelor's degree from Tokyo Institute of Technology in 2012. Her research interests include optimization, control, and information theory with applications in neuroscience and biology.

PLACE PHOTO HERE

Lijun Chen (M'05) is an Assistant Professor of Computer Science at University of Colorado at Boulder. He received a Ph.D. in Control and Dynamical Systems from California Institute of Technology in 2007. He was a co-recipient of the Best Paper Award at the IEEE International Conference on Mobile Ad-hoc and Sensor Systems (MASS) in 2007. His current research interests include optimization and control of networked systems, distributed optimization and control, convex relaxation and parsimonious solutions, and game theory and its engineering application. He is an editor for the IEEE Transactions on Communications and editor for the IEEE Control Systems Society Conference Editorial Board. He serves as the chair of the Special Interest Group in ICT-Enabled Transactive Energy and Grid Economics of IEEE ComSoc Smart Grid Communications Emerging Technical Subcommittee.