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IFAC-PapersOnLine 48-22 (2015) 034–039

A linear programming framework for networked control system design

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Abstract: This paper considers stability and performance for a sampled data system with a distributed controller, time-varying delay, quantization, saturation, and external disturbances. We decompose the overall system into two subproblems. The first is a linear subsystem containing a distributed controller satisfying information sharing constraint between sensor and actuator. The second involves nonlinear error dynamics comprised of sampling, quantization, delay and saturation errors. We show how to construct an invariant set of the error dynamics using a linear program. Our method enables the design of distributed/localized controllers and delayed communication channels with linear programming, enhancing both implementation and design scalability. The use of l_∞ signal/l_1 operator norm appears essential to capture communication error and facilitate scalable computation. We also analyze the suboptimality gap and the feasibility condition of the proposed linear program.

Keywords: Sampled data system, Distributed control, Delay, Quantization, Saturation.

1. INTRODUCTION

Networked control systems are becoming increasingly ubiquitous as seen in applications such as vehicle platooning, smart grid, and software defined networking among others. Their distributed nature creates various challenges both in design and implementation. Common elements in these systems include information sharing constraints, quantization, time-varying delays, and sampling. Currently, there is no existing framework that incorporates every one of these components.

Distributed control literature focuses on systems with information sharing constraints. Rotkowitz and Lall (2006) found that quadratically invariant information constraints produce a convex formulation of optimal control problem. Various controller synthesis methods for linear systems extended from their work can be found in Table 1. However, this approach cannot account for band-limited communication channels between sensor and actuator. Naively implementing a distributed controller in the presence of band-limited channels does not guarantee stability.

Another existing body of work considers control under band-limited channels and time-varying delays. Among others, Fagnani and Zampieri (2004); Heemels et al. (2010); Fridman and Dambrine (2009); Nesic and Liberzon (2009) considered stability and performance for a system with nonlinearity arising from saturation, quantization, and time-varying delay. As seen in review of Hespanha et al. (2007); Nair et al. (2007), it is most common to construct an invariant set of the state trajectory by Lyapunov-Krasovskii method. Similar lines of research is by Aysal et al. (2008), Liu et al. (2011) that combine distributed consensus with sampling, quantization, or delay. These methods, dealing with delay and band-limited channels, nonetheless cannot account for complex information sharing constraints.

Our proposed framework resolves the apparent discrepancy between these two bodies of work. We offer design tools for systems with both information sharing constraints and band-limited channels. Additionally, we show how to guarantee stability and performance by linear programming without using Lyapunov-Krasovskii method which uses semidefinite programming. This largely reduces computation complexity and enhances scalability. The intuition behind the proof technique used in this paper can be found in Nakahira et al. (2015), which shows the analytical formula of system performance as a function of channel capacity.

We describe the problem formulation in Section 2. The design process – controller synthesis and channel design – is in Section 3. We present the feasibility condition of the propose program in Section 4, followed by a concluding remark in Section 5.

Table 1. List of controller synthesis literature.

Centralized L1 optimal (linear programming)
Dahleh and Diaz-Bobillo (1994)
Distributed
Lamperski and Doyle (2012b); Lamperski and Lessard (2013) Shah et al. (2011); Lessard and Lall (2012)
Totally distributed (localized)
Wang et al. (2014); Wang and Matni (2014)

Notation

- We denote the set of positive integers by $\mathbb{N}$, the set of real numbers by $\mathbb{R}$, and the set of non-negative real numbers by $\mathbb{R}_+$.

* This research was in part supported by NSF NetSE, AFOSR, the Institute for Collaborative Biotechnologies through grant W911NF-09-0001 from the U.S. Army Research Office.

10.1016/j.ifacol.2015.10.303

- We denote the i-th entry of a vector by x_i, the (i, j)-th entry of a matrix by A_{ij}.
- We use $x(t)$, $t \in \mathbb{R}_+$, for a continuous signal, $x[k]$, $k \in \mathbb{N}$, for a discrete sequence. Bold front lower case is used to represent a sequence, i.e., $\mathbf{x} = \{x(t)\}_{t\in\mathbb{R}_+}$ or $\mathbf{x} = \{x[k]\}_{k\in\mathbb{N}}$.
- We denote the space of continuous signals with bounded infinity norm by $\mathcal{L}_\infty^n \triangleq \{\mathbf{x} : x(t) \in \mathbb{R}^n, \|\mathbf{x}\|_\infty < \infty\}$ where $\|\mathbf{x}\|_\infty \triangleq \sup_{t\in\mathbb{R}_+} \max_i |x_i(t)|$. The space of discrete sequences with bounded infinity norm by $l_\infty^n \triangleq \{\mathbf{x} : x[k] \in \mathbb{R}^n, \|\mathbf{x}\|_\infty < \infty\}$ where $\|\mathbf{x}\|_\infty \triangleq \sup_{k\in\mathbb{N}} \max_i |x_i[k]|$.
- We use $\geq_+$ to represent elementwise inequality, *i.e.*,

$$x \geq_+ y \Longleftrightarrow \forall i,\ x_i \geq y_i \tag{1}$$

- We denote the space of stable rational proper transfer matrices by $\mathcal{RH}_\infty$, and the space of stable rational strictly proper transfer matrices by $\frac{1}{z}\mathcal{RH}_\infty$.

Preliminaries

In this section, we review mathematical preliminaries.

Systems and norms: A stable linear time-invariant causal operator $\mathbf{P} : \mathbf{u} \in l_\infty^n \to \mathbf{y} \in l_\infty^m$ can be written in the form

$$\begin{aligned} x[k+1] &= Ax[k] + Bu[k] \\ y[k] &= Cx[k] + Du[k], \end{aligned} \tag{2}$$

where A has all its eigenvalues inside the open unit circle. We use the notation $\mathbf{P} = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$. The transfer matrix of $\mathbf{P}$ is $\hat{P} = C(zI - A)^{-1}B + D$, where $\hat{\ }$ is used for operators/sequences z-domain. Notice that $\mathbf{P}$ and $\hat{P}$ are different representations of an equivalent object. Additionally, we define

$$P[k] \triangleq \begin{cases} D & k = 0 \\ CA^{k-1}B & k \geq 1 \end{cases} \tag{3}$$

to be its impulse matrix. The l_1 norm is $\|\mathbf{P}\|_1 = \max_i \sum_{j,k} |P_{ij}[k]|$, which is also the induced norm of operator on l_∞. In a similar manner, let $|M|$ be the element-wise absolute value of a matrix $M \in \mathbb{R}^{m\times n}$, *i.e.*, $|M| = \begin{bmatrix} |M_{11}| & \cdots & |M_{1n}| \\ \vdots & & \vdots \\ |M_{m1}| & \cdots & |M_{mm}| \end{bmatrix}$, using which we define the induced matrix as follows.

Definition 1. The *induced matrix* of a stable linear time-invariant causal operator $\mathbf{P} : l_\infty^n \to l_\infty^m$ is defined by

$$|\mathbf{P}|_{e.w} \triangleq \sum_{k=0}^{\infty} |P[k]| \in \mathbb{R}_+^{m\times n}. \tag{4}$$

This induced matrix can be well defined when the linear time-invariant operator is stable. Notice that we can use $|\mathbf{P}|_{e.w}$ to bound the operator output. Recall from notation that $\geq_+$ denotes element-wise inequality. If for any $k \in \mathbb{N}$, $|u[k]| \leq_+ \epsilon$, then

$$|y[k]| = \left|\sum_{i=0}^{k} P[k-i]u[i]\right| \leq_+ \sum_{i=0}^{k} \big|P[k-i]\big|\big|u[i]\big| \leq_+ |\mathbf{P}|_{e.w}\ \epsilon$$

Quantizers: A quantizer partitions the input space into disjoint sets, and maps each set onto its representative point. We considers uniform quantizers.

Definition 2. An uniform quantizer with $L \in \mathbb{N}^n$ level and saturation $X \in \mathbb{R}_+^n$ is a mapping $\mathbb{Q}_{L,X} : x \in \mathbb{R}^n \to y \in \mathbb{R}^n$ defined as following: for $X < \infty$,

$$y_i = \begin{cases} -X_i + \dfrac{X_i}{L_i} & \text{if } x_i \in \left[-\infty, -X_1 + 2\dfrac{X_i}{L_i}\right) \\ -X_i + 3\dfrac{X_i}{L_i} & \text{if } x_i \in \left[-X_i + 2\dfrac{X_i}{L_i}, -X_i + 4\dfrac{X_i}{L_i}\right) \\ \qquad\vdots & \\ X_i - \dfrac{X_i}{L_i} & \text{if } x_i \in \left[X_i - 2\dfrac{X_i}{L_i}, \infty\right], \end{cases}$$

and for $X = \infty$, $\mathbb{Q}_{\cdot,\infty}$ is a identify map, *i.e.*, $x = \mathbb{Q}_{\cdot,\infty}\ x$.

This type of uniform quantizer $\mathbb{Q}_{L,X}$ has a useful property which greatly simplify the analysis: if $|x_i| \leq X_i$, then $|x_i - Q_{L,X}x_i| \leq X_i/L_i$. Let invdiag$(L)$ denote an $n \times n$ square matrix

$$\text{invdiag}(L)_{ij} = \begin{cases} L_i^{-1} & i = j \\ 0 & i \neq j. \end{cases}$$

This property has an alternative expression:

$$\text{if } |x| \leq_+ X, \text{ then } |x - y| \leq_+ \text{invdiag}(L)X. \tag{5}$$

2. PROBLEM DESCRIPTION

We formulate the problem in Section 2.1, and explain its motivation behind using an engineering example in Section 2.2.

2.1 Problem formulation

Consider a sampled data system of the form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B_1w(t) + B_2u(t) \\ z(t) &= C_1x(t) + Du(t) \\ y(t) &= C_2x(t), \end{aligned} \tag{6}$$

where $x(t) \in \mathbb{R}^n$ is the state, $y(t) \in \mathbb{R}^m$ is the sensor measurement, $w(t) \in \mathbb{R}^l$ is the disturbance, and $u(t) \in \mathbb{R}^p$ is the control action. The output $y(t)$ is sent to the controller using a communication channel every T seconds. This channel is a uniform quantizer defined by the mapping $\mathbb{Q}_{L_y,Y} : \mathbb{R}^m \to \mathbb{R}^m$ from definition 2. Let $y[k]$ be the channel output, *i.e.*,

$$y[k] \triangleq \mathbb{Q}_{L_y,Y}\,y(t_k), \tag{7}$$

for any $k \in \mathbb{N}$ and $t_k \triangleq kT$. The controller $\hat{K} = \sum_{i=1}^{\infty} \frac{1}{z^i}K[i]$ is a strictly proper linear time-invariant system. The desired control action $u^\star[k]$ is computed from

$$u^\star[k] = \sum_{i=1}^{k} K[i]y[k-i]. \tag{8}$$

Note that we denote the raw measurement by $y(t_k)$, and quantized measurement by $y[k] = \mathbb{Q}_{L_y,Y}\,y(t_k)$. The desired control action $u^\star[k]$ is sent to the actuators using another communication channel $\mathbb{Q}_{L_u,U} : \mathbb{R}^p \to \mathbb{R}^p$. The control command received by the actuators at time $t = t_k$ is

$$u[k] \triangleq \mathbb{Q}_{L_u,U}\,u^\star[k]. \tag{9}$$

For generality, we consider the case when there are two communication channels – one between sensor and controller, another between controller and actuators. However, the design and analysis technique can be trivially extended to systems with only one channel. The control action is executed with time-varying delay $d_k \in [0, h]$ where $h < T$. Thus, the actual control action is

$$u(t) = \begin{cases} u[k-1] & t \in [t_k, d_k) \\ u[k] & t \in [d_k, t_{k+1}). \end{cases}$$

for any $t \in [t_k, t_{k+1})$. Additionally, we make following assumptions:

A. The controller satisfies $\hat{K} \in \mathcal{S}_c$, where S_c is a subspace of $\frac{1}{z}\mathcal{RH}_\infty$ specifying the information sharing pattern (see remark 1).
B. The pair (A, C) is observable. Equivalently, the pair (e^{AT}, C) is observable.
C. The disturbance satisfies $\|\mathbf{w}\|_\infty \leq 1$.
D. The initial condition of the system is $x(0) = 0$.

E. The delay upper bound h is known to the system designer. However, the controller does not have access to the actual value of d_k.

Given the hardware limitation parametrized by (L_u, L_y, h), our goal is to design a stabilizing control law defined by the triple $(\mathbf{K}, \mathbb{Q}_{L_u,U}, \mathbb{Q}_{L_y,Y})$ such that

$$\sup_{\|w\|_\infty \le 1} \|\mathbf{z}\|_\infty \le \nu. \tag{10}$$

Remark 1. The information sharing constraints are expressed as $\hat{K} \in \mathcal{S}_c$ with $S_c = \begin{bmatrix} \frac{1}{z^{\tau_{11}}}\mathcal{R} & \frac{1}{z^{\tau_{12}}}\mathcal{R} & \cdots \\ \frac{1}{z^{\tau_{21}}}\mathcal{R} & \frac{1}{z^{\tau_{22}}}\mathcal{R} & \\ \vdots & & \ddots \end{bmatrix}$, where $\mathcal{R}$ is the space of proper real rational transfer matrices and $\tau_{ij} \in \mathbb{N}$. Roughly speaking, the information sharing constraints $\mathcal{S}_c$ specify the delay larger than the sampling time T, whereas the time-varying delay d_k specifies delay smaller than the sampling time.

2.2 Motivating example

In large scale systems, each node may not be able to access the current state of the whole system upon deciding its control action. Some multi-agent systems with underwater vehicles or low power sensor networks have limited communication capacity, and the communication between agents is subject to errors arising from quantization, saturation, and delay. The design of these systems is challenging as explained in section 1. We show below, for illustrative purpose, a simple example where the design problem can be formulated in the form presented in section **??**, whereas our problem formulation holds in more generality.

A group of m unmanned vehicle is following a leader. We enumerate each follower by $i = 1, 2, \cdots, m$. Every T seconds, the first vehicle senses its distance from the leader, and the i-th vehicle senses its distance from the $(i-1)$-th. This *sampled data system* has a discrete controller acting on its continuous dynamics: the position/velocity of each vehicle p_i/v_i satisfies $m_i \dot{v}_i(t) = F_i(t), \dot{p}_i = v_i$ where m_i is the mass, F_i is the control action. A naive design without utilizing communication between vehicles – the first vehicle following the leader, and the i-th vehicle following the $i-1$-th vehicle – would have limited performance (the distance between the m-th vehicle and the leader will be large: the first vehicle takes more than T second to sense the leader and reflect on its movement; the second vehicle takes more than $2T$ seconds; and so does the remaining ones). In contrast, if one adds a communication channel between neighbouring vehicles, the m-th vehicle can follow the leader as quickly as it can communicate, achieving faster response. Let the state and control action of the dynamical system be defined by $x = [v_1 - v^\star,\ p_1 - p^\star,\ v_2 - v_1,\ p_2 - p_1,\ \cdots]^T, u = [F_1,\ F_2,\ \cdots]^T$ where $(p^\star, v^\star)$ is the leader's position/velocity. The *information sharing structure* considered in Lamperski and Doyle (2012a) corresponds to the case when i-th vehicle at time t have the 'perfect' information of $\mathcal{I}_i(t) = \{p_i(t) - p_{i-1}(t),\ p_{i-1}(t-T) - p_{i-2}(t-T),\ p_{i+1}(t-T) - p_i(t-T), \cdots\}$. On the other hand, when the communication channel has limited information capacity, the i-th vehicle may not have 'perfect' information, producing errors arising from *quantization* and *saturation*. Moreover, the transmission speed may vary with environmental condition and the distance between each vehicles. These factors are modelled into the *time-varying delay*. Assumption E is made in order to account for this delay that is unknown to other vehicles.

3. DESIGN METHOD

In this section, we present the design method. The communication channel and actuation delay render the overall system nonlinear, and thus a controller stabilizing the discretized system is not enough to guarantee stability. We first extract a linear dynamics on which we can design a controller satisfying information sharing constraints (Section 3.1). Then, we design the communication channel so that the overall system is stable with the presence of quantization, saturation, and delay (Section 3.2).

3.1 Distributed controller for linear dynamics

In this section, we show how to design a discrete controller satisfying information sharing constraints.

Lemma 1. Define the discrete sequences

$$x[k] \triangleq x(t_k) \tag{11}$$

$$w[k] \triangleq e^{AT} \int_{t_k}^{t_{k+1}} e^{-A(\tau - t_k)} B_1 w(\tau) d\tau \tag{12}$$

$$e_u[k] \triangleq e^{AT} \int_{t_k}^{t_k + d_k} e^{-A(\tau - t_k)} B_2 d\tau \Big(u[k-1] - u^\star[k] \Big) + e^{AT} \int_{t_k + d_k}^{t_{k+1}} e^{-A(\tau - t_k)} B_2 d\tau \Big(u[k] - u^\star[k] \Big) \tag{13}$$

$$e_y[k] \triangleq y[k] - y(t_k), \tag{14}$$

where for any $k \in \mathbb{N}$, $t_k = kT$. Let $\bar{A} \triangleq e^{AT}$, $\bar{B}_1 \triangleq \int_0^T \left| e^{-A(\tau - T)} B_1 \right| d\tau$, and $\bar{B}_2 \triangleq e^{AT} \int_0^T e^{-A\tau} B_2 d\tau$. Then, the discrete system dynamics satisfies

$$\begin{aligned} x[k+1] &= \bar{A}x[k] + w[k] + \bar{B}_2 u^\star[k] + e_u[k] \\ y[k] &= C_2 x[k] + e_y[k] \\ |w[k]| &\le_+ \bar{B}_1 \mathbf{1}_l, \end{aligned} \tag{15}$$

where $\mathbf{1}_l \in \mathbb{R}^l$ be a vector of all one entires.

Proof. See appendix. □

Lemma 1 suggests the nonlinearity from communication and delay can be absorbed into the term (e_u, e_y). Now we can design the discrete controller on the linear plant:

$$\mathbf{G} = \left[\begin{array}{c|cc} \bar{A} & \bar{B}_1 & \bar{B}_2 \\ \hline C_1 & 0 & D \\ C_2 & 0 & 0. \end{array} \right] = \left[\begin{array}{c|c} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \hline \mathbf{G}_{21} & \mathbf{G}_{22} \end{array} \right].$$

Existing literature has proposed various controller synthesise methods for a linear system of this form. We list some of them in table 1. Assume now that we have obtained the stabilizing controller for $\hat{G}$:

$$\hat{K} = \sum_{i=1}^{\infty} \frac{1}{z^i} R[i], \tag{16}$$

where $\hat{K}$ is a strictly stable proper transfer matrix. The relation between the sequences $(\mathbf{w}, \mathbf{e}_u, \mathbf{e}_y)$ and $(\mathbf{x}, \mathbf{u})$ permits an explicit formula stated in the next lemma.

Lemma 2. If $\hat{K}$ is a strictly proper stabilizing controller for $\hat{G}$, the input sequences $(\hat{e}_u, \hat{e}_y, \hat{w})$ and output sequences $(\hat{y}, \hat{u}^\star)$ satisfy the linear relation:

$$\begin{bmatrix} \hat{x} \\ \hat{u}^\star \end{bmatrix} = \begin{bmatrix} \hat{R} & \hat{N} & \hat{R} \\ \hat{M} & \hat{Q} & \hat{M} \end{bmatrix} \begin{bmatrix} \hat{e}_u \\ \hat{e}_y \\ \hat{w} \end{bmatrix}, \tag{17}$$

where

$$\begin{aligned} \hat{R} &\triangleq (zI - A - \bar{B}_2\hat{K}C_2)^{-1} & \hat{N} &\triangleq \hat{R}\bar{B}_2\hat{K} \\ \hat{M} &\triangleq \hat{K}C_2\hat{R} & \hat{Q} &\triangleq \hat{K} + \hat{K}C_2\hat{R}\bar{B}_2\hat{K}. \end{aligned} \tag{18}$$

Proof. Since $\hat{K}$ is a stabilizing controller for $\hat{G}$, the term $zI - A - \bar{B}\hat{K}C_2$ is invertible. Combining (15) and (16), we have

$$\begin{aligned} (zI - A)\hat{x} &= \hat{w} + \bar{B}_2\hat{u}^\star + \hat{e}_u \\ \hat{y} &= C_2\hat{x} + \hat{e}_y \\ \hat{u}^\star &= \hat{K}\hat{y}. \end{aligned} \tag{19}$$

We obtain (17) after simple transformation. □

3.2 Invariant set for nonlinear dynamics

Theorem 1 designs a communication channel that constraints the error dynamics inside an invariant set. Combining with corollary 1, we obtain stability of the overall system.

Definition 3. Consider a sequence $\mathbf{e}$ defined by $e[k+1] = F_k(e[0{:}k], w[0{:}k])$ where $\{F_k\}_{k\in\mathbb{N}}$ is a sequence of mapping and $\mathbf{w} \in S$ is the input to the mapping. If for all $k \in \mathbb{N}$, following statement holds: for any $\mathbf{w} \in S$,

$$\forall i \le k-1, e[i] \in E \text{ implies } e[k] \in E,$$

then E is an *invariant set* of $\mathbf{e}$.

Theorem 3. Let (L_u, L_y, h) from Section 2 be fixed parameters representing the capacity and delay of the communication channel, $\hat{K}$ be the stabilizing controller obtained in Section 3.1, and Ψ_1, Ψ_1, p_w be the following matrices:

$$\begin{aligned} \Psi_1(h) &= \sup_{d\in[0,h]} \left| \int_0^h e^{-A(\tau-T)} B_2 d\tau \right| \\ \Psi_2(h) &= \sup_{d\in[0,h]} \left| \int_d^T e^{-A(\tau-T)} B_2 d\tau \right| \\ p_w &= \int_0^T \left| e^{-A(\tau-T)} B_1 \right| \, d\tau \mathbf{1}_l, \end{aligned} \tag{20}$$

where the supremum is taken element-wisely. Define a linear program $\mathcal{P}_{L_u,L_y,h}$ as follows:

$$\begin{aligned} \underset{U,Y,p_u,p_y}{\text{minimize}} \quad & 0 \\ \text{subject to} \quad & U \ge_+ 0, Y \ge_+ 0 & (21) \\ & U = |\mathbf{M}|_{e.w.} (p_u + p_w) + |\mathbf{Q}|_{e.w.} p_y & (22) \\ & Y = |C_2\mathbf{R}|_{e.w.}(p_u + p_w) + |C_2\mathbf{N}|_{e.w.} p_y & (23) \\ & p_u = (2\Psi_1(h) + \Psi_2(h)\text{invdiag}(L_u))\, U & (24) \\ & p_y = \text{invdiag}(L_y)\, Y, & (25) \end{aligned}$$

where $|\cdot|_{e.w.}$ is defined in (4). If the program $\mathcal{P}_{L_u,L_y,h}$ is feasible with solution (U, Y), then under the control law $(\hat{K}, \mathbb{Q}_{L_u,U}, \mathbb{Q}_{L_y,Y})$, the set

$$E = \{(e_u, e_y) : e_u \in \mathbb{R}^p, e_y \in \mathbb{R}^m, |e_u| \le_+ p_u, |e_y| \le_+ p_y\}$$

is an invariant set for the sequences $\mathbf{e}_u$ and $\mathbf{e}_y$.

Proof. Let (U, Y) be the solution of the program $\mathcal{P}_{L_u,L_y,h}$. Assume for any $j \le k-1$, $e_u[j] \le p_u$ and $e_y[j] \le p_y$. We will show below that $e_u[k] \le p_u$ and $e_y[k] \le p_y$. Firstly, from (15), $|w(j)| \le_+ p_w$ for any $j \in \mathbb{N}$. Secondly, we bound the value of $u^*[j]$, $j \le k$, as follows:

$$\begin{aligned} |u^*[j]| &= \left| \sum_{i=0}^{j-1} M[j-i](e_u[i] + w[i]) + Q[j-i]e_y[i] \right| & (26) \\ &\le_+ \sum_{i=0}^{j-1} \left|M[j-i]\right| \Big(\left|e_u[i]\right| + \left|w[i]\right| \Big) + \left|Q[j-i]\right| \left|e_y[i]\right| \\ &\le_+ |\mathbf{M}|_{e.w.}(p_u + p_w) + |\mathbf{Q}|_{e.w.} p_y = U & (27) \end{aligned}$$

Line (26) is from (17), and line (27) is from the assumption $\forall j \le k-1, |e_u[j]| \le p_u, |e_y[j]| \le p_y$ and (22). Using the property (5), we obtain

$$\Big| u[k] - u^*[k] \Big| = \Big| u^*[k] - \mathbb{Q}_{L_u,U} u^*[k] \Big| \le \text{invdiag}(L_u) U \tag{28}$$

Now we are ready to bound $e_u[k]$.

$$\begin{aligned} |e_u[k]| &= \Bigg| \int_{t_k}^{t_k+d_k} e^{-A(\tau-t_k-T)} B_2 d\tau \Big(u[k-1] - u^*[k] \Big) \\ &\quad + \int_{t_k+d_k}^{t_{k+1}} e^{-A(\tau-t_k-T)} B_2 d\tau \Big(u[k] - u^*[k] \Big) \Bigg| & (29) \\ &\le_+ \left| \int_{t_k}^{t_k+d_k} e^{-A(\tau-t_k-T)} B_2 d\tau \right| \Big[\left|u[k-1]\right| + \left|u^*[k]\right| \Big] \\ &\quad + \left| \int_{t_k+d_k}^{t_{k+1}} e^{-A(\tau-t_k-T)} B_2 d\tau \right| \Big| u[k] - u^*[k] \Big| \\ &\le_+ 2\Psi_1(h)U + \Psi_2(h)\text{invdiag}(L_u)U = p_u & (30) \end{aligned}$$

The equality (29) is from (13). Notice $|u^*[j]| \le U$ implies $|u[j]| \le U$. We obtain the inequality (30) is from (20) and (28). Similarly, combining $y(t_k) = C_2 x[k]$, (17) and (23) to have

$$\begin{aligned} |y(t_k)| &= \left| \sum_{i=0}^{k-1} C_2 R[k-i](e_u[i] + w[i]) + C_2 N[k-i] e_y[i] \right| \\ &\le_+ |C_2\mathbf{R}|_{e.w.}(p_u + p_w) + |C_2\mathbf{N}|_{e.w.} p_y = Y & (31) \end{aligned}$$

Combining the property (5) with (25) to have

$$|e_y[k]| = \Big| y(t_k) - \mathbb{Q}_{L_y,Y} y(t_k) \Big| \le_+ \text{invdiag}(L_y) Y = p_y \tag{32}$$

Therefore, the set E is an invariant set of $\mathbf{e}_u, \mathbf{e}_y$. □

Corollary 1. If the linear program $\mathcal{P}_{L_u,L_y,h}$ is feasible with solution (U, Y), then under the control law $(\hat{K}, \mathbb{Q}_{L_u,U}, \mathbb{Q}_{L_y,Y})$, both the state $\mathbf{x}$ and the ouput $\mathbf{z}$ are bounded.

Proof. From Theorem 1 and initial condition $x(0) = 0$ (assumption D), we obtain $\|e_u\|_\infty \le p_u, \|e_y\|_\infty \le p_y$. Since $\hat{K}$ is a stabilizing controller for $\hat{G}$, the transfer matrices $\hat{R}, \hat{N}, \hat{M}, \hat{Q}$ from (18) are stable. From $\hat{x} = \hat{R}\hat{e}_u + \hat{N}\hat{e}_y + \hat{R}w$, $\hat{u} = \hat{M}\hat{e}_u + \hat{Q}\hat{e}_y + \hat{M}w$, we obtain that the sequences $\mathbf{x}$ and $\mathbf{u}$ are bounded. Therefore, the output sequence $\mathbf{z}$ is also bounded. □

3.3 From stability to performance

Next, we bound the sub-optimality gap of the system performance. Recall from (10) that our goal is to achieve $\sup_{\|w\|_\infty \le 1} \|\mathbf{z}\|_\infty \le \nu$. Let ν_p be the value of $\sup_{\|w\|_\infty \le 1} \|z\|_\infty$ when the distributed controller $\hat{K}$ from (16) is used with perfect communication and no delay, *i.e.*, $y[k] = y(t_k), u^*[k] = u[k]$ and $h = 0$. Let $\nu_c = \nu - \nu_p$, the performance criteria (10) is equivalent with

$$\sup_w \|z(t)\|_\infty \le \nu_p + \nu_c. \qquad (33)$$

Intuitively, ν_c captures the performance degradation due to information sharing constraints, and ν_c due to unreliable communication. We assume $\nu_c > 0$.

Theorem 4. Let

$$r_u \triangleq |C_1|\{2\Psi_1(h)|\mathbf{M}|_{e.w} + \Psi_1(T)\big(\text{invdiag}(L_u) + I\big)|\mathbf{M}|_{e.w} + |e^{A\rho}\mathbf{R}|_{e.w.}\} + |C_2\mathbf{M}|_{e.w.}$$
$$r_y \triangleq |C_1|\{2\Psi_1(h)|\mathbf{Q}|_{e.w} + \Psi_1(T)\big(\text{invdiag}(L_u) + I\big)|\mathbf{Q}|_{e.w})| + |e^{A\rho}\mathbf{N}|_{e.w.}\} + |C_2\mathbf{Q}|_{e.w} \qquad (34)$$
$$r_w \triangleq |C_1|\{2\Psi_1(h) + \Psi_1(T)\big(\text{invdiag}(L_u) + I\big)\}|\mathbf{M}|_{e.w.}$$

If the linear program $\mathcal{Q}_{L_u,L_y,h}$:

$$\begin{aligned} &\underset{U,Y,p_u,p_y}{\text{minimize}} && 0 \\ &\text{subject to} && (21) - (25) \\ & && \max\{r_u p_u + r_y p_y + r_w p_w\} \le \nu_p \qquad (35) \end{aligned}$$

is feasible, then the overall performance is bounded by

$$\sup_w \|\mathbf{z}\|_\infty \le \nu_p + \nu_c. \qquad (36)$$

In above theorem, the constraint (21) − (25) is same with the program $\mathcal{P}_{L_u,L_y,h}$, and it guarantees system stability. The additional constraint (35) is used for system performance.

Proof. See appendix. □

4. ANALYSIS

In this section, we consider the issues related to the feasibility of the proposed linear program. If the program is not feasible for a given hardware constraint parametrized by (L_u, L_y, h), then we need to enhance channel capacity by increasing L_u, L_y or to reduce delay by decreasing h. The next theorem states that the proposed program is asymptotically feasible. We mean by 'asymptotically feasible' that enhanced communication or reduced delay will eventually lead to feasibility. The constructive proof of Theorem 5 also suggests the necessary and sufficient condition for the feasibility, which is given in corollary 2.

Theorem 5. Given a controller $\hat{K}$ stabilizing $\hat{G}$, there exists $\bar{L}_u \in \mathbb{N}^p, \bar{L}_y \in \mathbb{N}^m$ and $\bar{h} \in \mathbb{R}_+$ such that the program $\mathcal{P}_{L_u,L_y,h}$ is feasible if all the inequalities below hold: $L_u \ge_+ \bar{L}_u, L_y \ge_+ \bar{L}_y$, and $h \ge \bar{h}$.

Proof. By eliminating p_u, p_y in (22)-(25), we have

$$(I - E)\begin{bmatrix} U \\ Y \end{bmatrix} = \begin{bmatrix} |\mathbf{M}|_{e.w.} \\ |C_2\mathbf{R}|_{e.w.} \end{bmatrix} p_w, \qquad (37)$$

where

$$E = \begin{bmatrix} |\mathbf{M}|_{e.w.} & |\mathbf{Q}|_{e.w.} \\ |C_2\mathbf{R}|_{e.w.} & |C_2\mathbf{N}|_{e.w.} \end{bmatrix} \times \begin{bmatrix} 2\Psi_1(h) + \Psi_2(h)\ \text{invdiag}(L_u) & 0 \\ 0 & \text{invdiag}(L_y) \end{bmatrix}.$$

Recall $\Psi_1(h) = \sup_{d\in[0,h]} \left| \int_0^d e^{-A(\tau-t)} B_2 d\tau \right|$. Using the following properties: $\Psi_1(h)$ is an increasing function of h, $\lim_{h\to 0} \Psi_1(h) = 0$, $\lim_{[L_u]_i\to\infty} [\text{invdiag}(L_u)]_{ii} = 0$, and $\lim_{[L_y]_i\to\infty} [\text{invdiag}(L_y)]_{ii} = 0$. The spectral radius of E is a continuous function of E. Combining with above properties at limit points of h, L_y and L_u, we obtain that for some $\bar{L}_u \in \mathbb{N}^p, \bar{L}_y \in \mathbb{N}^m$ and $\bar{h} \in \mathbb{R}_+$, for any $L_u \ge_+ \bar{L}_u$, $L_y \ge_+ \bar{L}_y$, $h \ge \bar{h}$, the spectral radius of E is strictly less than 1. If the spectral radius of E is strictly less than 1, the square matrix $(I - E)$ is invertible with its inverse be given by $(I-E)^{-1} = \sum_{n=0}^{\infty} E^n$. Thus, there exists $(U^\star, Y^\star)$ such that

$$\begin{bmatrix} U^\star \\ Y^\star \end{bmatrix} = \sum_{n=0}^{\infty} E^n \begin{bmatrix} |\mathbf{M}|_{e.w.} \\ |C_2\mathbf{R}|_{e.w.} \end{bmatrix} p_w$$

Since $|\mathbf{M}|_{e.w.}, |C_2\mathbf{R}|_{e.w.}, E, p_w$ are all element-wisely positive matrices, we obtain

$$U^\star \ge_+ 0, \qquad Y^\star \ge_+ 0.$$

Therefore, $(U^\star, Y^\star)$ is an unique feasible solution of the linear program $\mathcal{P}_{L_u,L_y,h}$. □

Corollary 2. (Feasibility Condition) The program $\mathcal{P}_{L_u,L_y,h}$ is feasible if and only if the spectral radius of E defined in is strictly less than 1.

The argument above also holds for the linear program $\mathcal{Q}_{L_u,L_y,h}$. We present this result below, but omit its proof due to space constraint.

Theorem 6. Given a controller $\hat{K}$ stabilizing $\hat{G}$ and $\mu_c > 0$, there exist $\bar{L}_u \in \mathbb{N}^p, \bar{L}_y \in \mathbb{N}^m$ and $\bar{h} \in \mathbb{R}_+$ such that the program $\mathcal{Q}_{L_u,L_y,h}$ is feasible if all the inequalities below hold: $L_u \ge_+ \bar{L}_u$, $L_y \ge_+ \bar{L}_y$, $h \ge \bar{h}$.

5. CONCLUSION

Our method greatly enhances the scalability in terms of implementation and design. The use of l_∞ signal is essential in capturing quantization, saturation, and time-varying delay as well as producing computationally cheap design methodology.

ACKNOWLEDGEMENTS

The authors would like to thank Dr. Marco Pavone and Dr Fulvio Forni for insightful discussion.

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Appendix A. PROOFS

Proof. (Lemma 1) From (6), we have

$$x(t_{k+1}) = e^{AT}x(t_k) + e^{AT}\int_{t_k}^{t_{k+1}} e^{-A(\tau-t_k)}B_1w(\tau) + $$
$$+e^{AT}\int_{t_k}^{t_{k+1}} e^{-A(\tau-t_k)}B_2u(\tau)d\tau$$

By definition, the first term on the right hand side of the equation equals to $\bar{A}x[k]$, and the second term equals to $w[k]$. The third term satisfies

$$\begin{aligned}
&e^{AT}\int_{t_k}^{t_{k+1}} e^{-A(\tau-t_k)}B_2u(\tau)d\tau \\
&= e^{AT}\int_{t_k}^{t_{k+1}} e^{A(\tau-t_k)}B_2\Big(u^\star[k] + u(\tau) - u^\star[k]\Big)d\tau \\
&= \bar{B}_2u^\star[k] + e_u[k].
\end{aligned}$$

This yields $x[k+1] = \bar{A}x[k] + w[k] + \bar{B}_2u^\star[k] + e_u[k]$. The equation $y[k] = C_2x[k] + e_y[k]$ is immediate from definition. From assumption C, we can bound

$$|w[k]| \leq_+ \int_0^T \Big|e^{-A(\tau-T)}B_1\Big| d\tau\mathbf{1}_l \leq_+ \bar{B}_1\mathbf{1}_l.$$

□

Proof. (Theorem 4) In order to bound the value ν_c, we decompose the discrete state $x[k]$ into two terms: the term due to the disturbance $\mathbf{x}_p$, and the term due to unreliable communication $\mathbf{x}_c$, *i.e.*,

$$x_p[k] \triangleq \sum_{i=0}^{k} R[k-i]w[i] \quad x_c[k] \triangleq x[k] - x_p[k].$$

The controller output $u[k]$ also admits the decomposition:

$$u_p[k] \triangleq \sum_{i=0}^{k} M[k-i]w[i] \quad u_c[k] \triangleq u^*[k] - u_p[k].$$

The continuous counterparts of the four terms can be defined as follows: for any $\rho \in [0,T)$, let

$$\begin{aligned}
x_p(t_k+\rho) &\triangleq e^{A\rho}x_p[k] + \int_{t_k}^{t_k+\rho} e^{-A(\tau-t_k-T)}B_1w(\tau)d\tau \\
&\quad + \int_{t_k}^{t_k+\rho} e^{A(\tau-t_k-T)}B_2u_p[k]d\tau \\
x_c(t_k+\rho) &\triangleq e^{A\rho}x_c[k] \\
&\quad + \int_{t_k}^{t_k+\rho} e^{-A(\tau-t_k-T)}B_2(u(\tau)-u_p[k])d\tau \\
u_p(t_k+\rho) &\triangleq u_p[k] \\
u_c(t_k+\rho) &\triangleq u(t_k+\rho) - u_p[k],
\end{aligned}$$

This formulation satisfies $x(t) = x_p(t) + x_c(t)$ and $u(t) = u_p(t) + u_c(t)$. Now we use these terms to separately bound system output:

$$\begin{aligned}
\sup_w \|z(t)\|_\infty &\leq \sup_w \|C_1x_p(t) + Du_p(t)\|_\infty \qquad \text{(A.1)} \\
&\quad + \sup_w \|C_1x_c(t)\|_\infty + \sup_w \|Du_c(t)\|_\infty.
\end{aligned}$$

First, notice that when the system has perfect communication and no actuation delay, the term $\mathbf{x}_p, \mathbf{u}_p$ remains same while the term $\mathbf{x}_c, \mathbf{u}_c$ becomes zero. Thus, the first term on the right hand side of (A.1) is $\sup_w \|C_1x_p(t)+Du_p(t)\|_\infty = \nu_p$. Next, we bound the second term of (A.1) as follows:

$$\begin{aligned}
|x_c(t_k+\rho)| &\leq_+ \Psi_1(\min(\rho,h))\Big|u[k-1]-u_p[k]\Big| \\
&\quad + \Psi_1(T)\Big|u[k]-u_p[k]\Big| + \Big|e^{A\rho}x_c[k]\Big|.
\end{aligned}$$

We have

$$\begin{aligned}
&\Psi_1(\min(\rho,h))\Big|u[k-1]-u_p[k]\Big| \\
&\leq_+ \Psi_1(h)\Big(\Big|u[k-1]\Big| + \Big|u_p[k]\Big|\Big) \\
&\leq_+ 2\Psi_1(h)U,
\end{aligned}$$

where the first inequality is because each element of $\Psi_1(h)$ is an increasing function of h, the second inequality comes from $|\mathbf{u}| \leq_+ U, \Big|\mathbf{u}_p\Big| \leq_+ U$ (see proof of Theorem 1).

$$\begin{aligned}
&\Psi_1(T)\,|u[k]-u_p[k]| \\
&\quad \leq_+ \Psi_1(T)\Big(|u[k]-u^*[k]| + |u_c[k]|\Big) \\
&\quad \leq_+ \Psi_1(T)\Big(\text{invdiag}(L_u) + I\Big)U \\
&\Big|e^{A\rho}x_c[k]\Big| \leq_+ |e^{A\rho}\mathbf{R}|_{e.w.}p_u + |e^{A\rho}\mathbf{N}|_{e.w.}p_y.
\end{aligned}$$

Combining above, we obtain

$$\begin{aligned}
|C_1x_c(t_k+\rho)| &\leq_+ |C_1|\Big\{|e^{A\rho}\mathbf{R}|_{e.w.}p_u + |e^{A\rho}\mathbf{N}|_{e.w.}p_y \\
&\quad + 2\Psi_1(h)U + \Psi_1(T)\Big(\text{invdiag}(L_u)+I\Big)U\Big\}.
\end{aligned}$$

In a similar manner, we bound the third term of (A.1) by

$$|C_2u_c[k]| \leq_+ |C_2\mathbf{M}|_{e.w.}p_u + |C_2\mathbf{Q}|_{e.w}p_y.$$

Therefore, from definition (34),

$$\sup_w \|C_1x_c(t)\|_\infty + \sup_w \|Du_c(t)\|_\infty \leq_+ r_up_u + r_yp_y + r_wp_w,$$

which combining with (35) yields $\sup_w \|\mathbf{z}\|_\infty \leq \nu_p + \max\{r_up_u + r_yp_y + r_wp_w\} \leq \nu_p + \nu_c$. □