Jacobian Singularities in Optimal Power Flow Problems Caused by Intertemporal Constraints

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***Abstract*—In multi-timestep Optimal Power Flow (OPF) formulations, constraints that are time-dependent such as generator ramp limits and specifically energy storage constraints may cause the Jacobian matrix to become singular as the problem iterates towards the optimal solution. The particular case of this singularity happens when the gradients of the binding intertemporal constraints are linearly dependent. Methods such as using a Moore-Penrose pseudoinverse or modifying the models used in the optimization are discussed along with novel methods developed in this paper that avoid the singular Jacobian by exploiting the specific structure of the problem and constraints.**

I. Introduction

ITERATIVE techniques based on the Newton-Raphson algorithm are widely used to solve Optimal Power Flow (OPF) and Economic Dispatch optimization problems. The Jacobian matrix of the first order optimality conditions used in each Newton-Raphson step is updated as the problem iterates to optimality; however, as will be shown, in some cases the Jacobian matrix is singular at optimality due to specific binding intertemporal constraints. Numerically, this can cause issues not only with inverting the Jacobian at the optimal solution, but with inverting the Jacobian as the problem becomes close to the optimal solution.

Through the Linear Independence Constraint Qualification (LICQ) [1], we can examine how linearly dependent binding constraints cause the Jacobian to become singular. This constraint qualification states that if a set of constraints are binding at the optimal solution and their gradients are linearly dependent, the resulting Lagrange multipliers corresponding to those constraints could have multiple optimal solutions. Solving the set of linear equations that yield the Newton-Raphson update is thus equivalent to solving an underdetermined set of equations.

Previous work has been done in this area as well as closely related areas. Singular Jacobian matrices in Newton-Raphson based load flow calculations were found to be related to situations near voltage collapse [2], [3], and the situations when these cases occur have been analyzed further and solution techniques have been developed [4]. Previous work to avoid the Jacobian singularities caused by intertemporal constraints is discussed in [5]. In this paper, we focus on the Jacobian singularities in optimal power flow, caused by the linear dependence of intertemporal constraints. Further solution techniques and analysis is extended from the solutions in [5], which only provided a solution to certain singularities, namely those occurring whenever a storage device is integrated into a multi-timestep OPF problem and the optimal solution is such that the energy level is at its minimum for multiple time steps at the beginning of the optimization horizon. Here, we provide a solution in the case when these time steps are within the time horizon.

The outline of this paper is as follows: Section II defines the problem formulation and notation that is used throughout the paper. In Section III, the source of the singularity and the relation of this singularity to the intertemporal constraints is explained. In Section IV, techniques such as using the Moore-Penrose pseudoinverse and modifying the models used in the optimization are discussed along with techniques that are developed in this paper. Section V shows simulation results for AC multi-timestep Optimal Power Flow on the IEEE 14 bus test case system.

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II. Problem Formulation

In this section, the notation and formulation for the economic dispatch optimization problem, intertemporal constraints, and KKT conditions are described.

A. Multi-Timestep Optimal Power Flow Problem

In this paper, we are optimizing an economic dispatch objective with AC power flow constraints, wind energy acting as a negative load, and energy storage. The objective function aims to minimize the cost of generation from all generators $i = 1...M$ over a time horizon N:

$$f = \sum_{t=1}^{N} \left(\sum_{i=1}^{M} a_i P_{G_i}^2(t) + b_i P_{G_i}(t) + c_i \right), \tag{1}$$

subject to the power balance and generation constraints at each bus, as well as the constraints on storage [6], [7]:

$$P_k(t) - P_{Gk}(t) + P_{Lk}(t) - P_{Wk}(t) - P_{in,k}(t) + P_{out,k}(t) = 0, \tag{2}$$

$$Q_k(t) - Q_{Gk}(t) + Q_{Lk}(t) = 0, \tag{3}$$

$$E_k(t+T) = E_k(t) + \eta_c T P_{in,k}(t) - \frac{T}{\eta_d} P_{out,k}(t), \tag{4}$$

$$E^{min} \leq \quad E_k(t+T) \quad \leq E^{max}, \tag{5}$$

$$0 \leq \quad P_{in,k}(t) \quad \leq P_{in}^{max}, \tag{6}$$

$$0 \leq \quad P_{out,k}(t) \quad \leq P_{out}^{max}, \tag{7}$$

$$0 \leq \quad P_{Gi}(t) \quad \leq P_{Gi}^{max}, \tag{8}$$

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with variables

P_{Gk} :	Active power generation at bus k
Q_{Gk} :	Reactive power generation at bus k
P_{Wk} :	Available wind at bus k
P_{Lk} :	Active power consumption at bus k
Q_{Lk} :	Reactive power consumption at bus k
P_k :	Active power injected into system at bus k
Q_k :	Reactive power injected into system at bus k
$P_{in,k}$:	Power put into storage at bus k
$P_{out,k}$:	Power withdrawn from storage at bus k
E_k :	Energy level of storage device at bus k

and for $t = 1...N$ and $i = 1...M$. P_k and Q_k represent the power injections for the active and reactive power into the lines connected to bus k, and wind is modeled as a negative load. In Section III, the reason why this storage model, as well as other intertemporal constraints, can cause Jacobian singularities will be explained.

B. General Optimization Formulation

We will use the following notation for the general form of a nonlinear optimization problem:

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & f(x) \\ \text{subject to} \quad & g(x) \; = 0, \\ & h(x) \; \leq 0. \end{aligned} \tag{9}$$

The Lagrangian function is thus formulated as follows:

$$\mathcal{L}(x) = f(x) + \lambda^T g(x) + \mu^T h(x). \tag{10}$$

To solve the first order optimality conditions via Newton-Raphson, the inequality constraints are transformed into equality constraints by introducing non-negative slack variables z:

$$h(x) + z \;\; = \;\; 0, \tag{11}$$

$$z \;\; \geq \;\; 0. \tag{12}$$

Thus, the first order optimality conditions, or the Karush-Kuhn-Tucker (KKT) [8] conditions are given by the following set of equations:

$$\frac{\partial}{\partial x}\mathcal{L}(x^*, z^*, \lambda^*, \mu^*) \;\; = \;\; 0, \tag{13}$$

$$g(x^*) \;\; = \;\; 0, \tag{14}$$

$$h(x^*) + z^* \;\; = \;\; 0, \tag{15}$$

$$\mu^* z^* \;\; = \;\; 0, \tag{16}$$

$$\mu^* \;\; \geq \;\; 0, \tag{17}$$

$$z^* \;\; \geq \;\; 0, \tag{18}$$

where the asterisk (*) denotes that that variable is at its optimum. In the Optimal Power Flow problem, the equality constraints $g(x)$ correspond to the power balance and storage constraints (2)-(4). The inequality constraints $h(x)$ correspond to the upper and lower bounds on the generation and storage variables (5)-(8).

III. Sources of Singularities

In this section we will first describe in general how the Jacobian matrix of the first order optimality conditions (13)-(18) becomes singular and then discuss the specific case of how intertemporal constraints cause this issue to arise.

A. General Discussion

The Linear Independence Constraint Qualification (LICQ) states that at the optimal solution, the gradients of all the binding constraints (including equality constraints) must be linearly independent or there exists no unique solution for the Lagrange multipliers [1], [9]. The KKT conditions may be fulfilled, but no unique solution for the Lagrange multipliers corresponding to the dependent binding constraints exists. An inequality constraint $h_i(x)$ is called "binding" if the corresponding slack variable z_i is zero at the optimum. The consequence is that the Jacobian matrix for the first-order optimality conditions becomes singular.

The Jacobian matrix, formed in the Newton-Raphson algorithm for solving the KKT conditions, has the following structure:

$$\begin{bmatrix} \nabla^2_{xx}\mathcal{L}(x,z,\lambda,\mu) & \nabla g(x)^T & \nabla h(x)^T & 0 \\ \nabla g(x) & 0 & 0 & 0 \\ \nabla h(x) & 0 & 0 & I \\ 0 & 0 & diag\{z\} & diag\{\mu\} \end{bmatrix} \tag{19}$$

The consequences of having dependent binding gradients of the constraints, i.e., having the LICQ unsatisfied, can be seen by analyzing the following rows of the Jacobian:

$$\begin{bmatrix} \nabla g(x) & 0 & 0 & 0 \\ \nabla h(x) & 0 & 0 & I \\ 0 & 0 & diag\{z\} & diag\{\mu\} \end{bmatrix} \tag{20}$$

When the gradients of the constraints are linearly dependent and the constraints are binding, the above rows are linearly dependent. This is due to the fact that when a constraint i is binding, $z_i = 0$ and $\mu_i \neq 0$. Thus, if $\nabla g(x)$ and $\nabla h(x)$ are dependent when binding, this entire matrix block (20) will have dependent rows.

In addition, because of the LICQ, a row of zeros could also be created by the possibility that μ_i and z_i could both become zero. These are two ways how the matrix could become singular due to the violation of LICQ. However, as the following section will discuss, there are multiple ways to modify the problem structure or preemptively prevent the Jacobian from becoming singular.

B. Singularities Caused by Storage Constraints

The specific problem and solutions that are analyzed in this paper are with regards to the storage model described in (4)-(7). When it is optimal to keep the energy level at its minimum or maximum for multiple consecutive time steps, the gradients of the binding constraints become linearly dependent. This is evident by recognizing that there will be more binding constraints than variables. Consider the variable vector

$$x = [E(t) \quad P_{in}(t) \quad P_{out}(t) \quad E(t+T)], \tag{21}$$

and the following constraints on storage:

$$
\begin{aligned}
E(t+T) = \quad & E(t) + \eta_c T P_{in}(t) - \frac{T}{\eta_d} P_{out}(t), && (22)\\
E^{min} \leq \quad & E(t) \quad \leq E^{max}, && (23)\\
E^{min} \leq \quad & E(t+T) \quad \leq E^{max}, && (24)\\
0 \leq \quad & P_{in}(t) \quad \leq P_{in}^{max}, && (25)\\
0 \leq \quad & P_{out}(t) \quad \leq P_{out}^{max}. && (26)
\end{aligned}
$$

If the optimal solution is $x^* = [E_{min} \ \ 0 \ \ 0 \ \ E_{min}]$, i.e., the storage is empty for two consecutive time steps, the matrix of the gradients of the binding constraints (22)-(26) is as follows:

$$
\begin{bmatrix}
0 & \cdots & 0 & -1 & -T\eta_c & \frac{T}{\eta_d} & 1 & 0 & \cdots & 0\\
0 & \cdots & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0\\
0 & \cdots & 0 & 0 & 0 & 0 & -1 & 0 & \cdots & 0\\
0 & \cdots & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0\\
0 & \cdots & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0
\end{bmatrix} \quad (27)
$$

which has linearly dependent rows. Thus, the LICQ is not fulfilled, and the Jacobian matrix will be singular.

IV. Approaches to Avoid Singularities

There are a number of approaches to resolve the singular Jacobian problem. The most optimal choice depends on the desired balance between accuracy, ease of implementation, and computational speed. In this section, various methods to achieve this goal are compared and contrasted to solving a linear system of equations of the form $Ax = b$. The formula for Newton-Raphson with Jacobian matrix J and vector of equations f at iteration k can be written in this form:

$$J(x_k) \cdot \Delta x_k = -f(x_k). \quad (28)$$

To follow the general and familiar form $Ax = b$, the Jacobian matrix J will be referred to as A as the following methods are described.

A. Moore-Penrose Pseudoinverse

One key result from the analysis provided in this paper shows that the Jacobian matrix is actually singular at the optimal solution, and that there are multiple solutions that satisfy the KKT conditions for optimality. That is, the Jacobian is not singular because of a bad problem formulation, a Newton-Raphson step that caused the matrix to be ill-conditioned, or other numerical issues. Because of this fact, the Moore-Penrose pseudoinverse (hereafter referred to as simply the "pseudoinverse") can be used to solve the underdetermined set of equations. Because A is less than full-rank, the Jacobian matrix A must be decomposed using the Singular Value Decomposition (SVD):

$$A = U\Sigma V^T, \quad (29)$$

where U and V are matrices with orthonormal columns, and Σ is a diagonal matrix with singular values (square roots of the eigenvalues of $A^T A$) along its main diagonal; i.e., with r nonzero singular values, $\Sigma = diag(\sigma_1, ...\sigma_r, 0, ..., 0)$. For a singular matrix, some of these values will be zero. The "invertible" part of Σ, that is, the block matrix corresponding to nonzero singular values, is inverted, and the pseudoinverse A^+ is defined as [10]:

$$A^+ = U\Sigma^+ V^T, \quad (30)$$

where $\Sigma^+ = diag(\sigma_1^{-1}, ...\sigma_r^{-1}, 0, ..., 0)$. This A^+ is the same pseudoinverse as given by the MATLAB command `pinv`.

The solution achieved by using this pseudoinverse is given by:

$$x^+ = A^+ b, \quad (31)$$

where this solution, x^+, solves the underdetermined linear least squares (minimum norm) problem, which finds the optimum to:

$$
\begin{aligned}
\underset{x}{\text{minimize}} \quad & \|x\|_2\\
\text{subject to} \quad & Ax = b.
\end{aligned} \quad (32)
$$

The optimal solution to this problem, that is, the minimum norm solution, is x^*. Since the system is underdetermined and multiple solutions exist, at a solution x^*, $Ax^* = b$. It can be shown [11] that $x^+ = x^*$ and thus using this pseudoinverse to solve (28) will yield a solution to the KKT conditions; i.e., the solution is a local minimum fulfilling (13)-(18).

B. Storage Standby Losses

Another approach is to avoid the Jacobian becoming singular altogether. In the considered problem, this can be achieved by introducing standby storage losses into equations (4)-(7). There are two possible methods by which these losses could be included:

i. Subtractive Standby Losses

A constant standby loss term ϵ can be subtracted from the energy balance equation to represent energy losses from elapsed time rather than just including charging/discharging losses. This loss could represent inertia losses from a flywheel or charge leakage from a lithium-ion battery, for example. The new storage formulation can be written as

$$E(t+T) = E(t) + \eta_c T P_{in}(t) - \frac{T}{\eta_d} P_{out}(t) - \epsilon_L. \quad (33)$$

Equations (5)-(7) in the model will stay the same. This value is dependent on what storage technology (battery, pumped hydro, flywheel, etc.) is included in the model. This loss will prevent all of the intertemporal constraints related to this storage device to be simultaneously binding. Consider the case where the storage is at its minimum capacity at time t. Because of the standby losses, at time $t+T$, the energy level would dip below E_{min} unless $P_{in}(t)$, the power into the storage, is nonzero; i.e., one of the previously binding constraints is now non-binding. Similarly, when the energy level $E(t)$ is at its maximum, $E(t+T)$ cannot also be at its maximum in the next time step unless $P_{in}(t) \neq 0$. Hence, because of the standby losses, all of these storage constraints are prevented from being simultaneously binding.

ii. Multiplicative Standby Losses

Depending on the storage technology, the losses can also be modeled as a nonlinear, percentage loss:

$$E(t+T) = \epsilon_N E(t) + \eta_c T P_{in}(t) - \frac{T}{\eta_d} P_{out}(t). \tag{34}$$

However, it is important to note that if the minimum storage level E_{min} is zero, this model can still result in simultaneously binding constraints. This is because when $E(t) = 0$, and all other storage constraints are binding ($P_{in}(t) = P_{out}(t) = 0$), the term $\epsilon_N E(t)$ will still be zero; i.e., the storage will not need to feed in power to account for standby losses and all of the storage constraints can become binding. A solution to this problem is to enforce a non-zero lower limit on the minimum energy level.

In both of the above methods for incorporating standby losses, rare cases can occur that still result in Jacobian singularities. For example, if $E(t) = E_{min}$, $P_{in}(t) = P_{in}^{max}$, $P_{out}(t) = 0$, and $E(t+T) = E^{max}$; i.e., the storage is initially empty and wants to charge at its maximum rate for the current step. The value for the maximum charging rate must exactly result in the storage being at E_{max} after the time interval T. In this case, standby losses do not help, and the constraints (4) - (7) can all be binding. However, cases such as this are rare and presumably will not be frequently encountered.

C. Constraint/Variable Removal as Intertemporal Constraints Approach Binding

The third approach discussed in this paper to avoid singular Jacobian matrices is to remove the rows that correspond to linearly dependent constraints and solve the resulting linearly independent system of equations. A priori, it is not known which, if any, of the intertemporal equations will be binding. By analyzing the structure of the Jacobian, however, we can deduce that once these constraints become binding, they will stay binding.

Analyzing the structure of (28), we see that the Newton-Raphson step has the following form:

$$\begin{bmatrix} \nabla^2_{xx}\mathcal{L} & \nabla g(x)^T & \nabla h(x)^T & 0 \\ \nabla g(x) & 0 & 0 & 0 \\ \nabla h(x) & 0 & 0 & I \\ 0 & 0 & diag\{z\} & diag\{\mu\} \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta\lambda \\ \Delta\mu \\ \Delta z \end{bmatrix} = \begin{bmatrix} \nabla_x\mathcal{L} \\ g(x) \\ h(x)+z \\ diag\{\mu\}\cdot z \end{bmatrix}$$

For a particular slack variable z_i, we see that:

$$z_i \cdot \Delta\mu_i + \mu_i \cdot \Delta z_i = \mu_i \cdot z_i. \tag{35}$$

Assuming μ_i at the current step is nonzero, if $z_i = 0$, then in order for this equation to hold, Δz_i must be zero. Thus, z_i will not change during future iterations. This fact will be used to identify the storage constraints which already become binding at their optimum during the Newton-Raphson iterations. We then use this to remove constraints to ensure that the Jacobian matrix will not become singular as we continue the iterations.

To see the implications of a binding constraint in this context, we can examine the KKT conditions relating to the storage constraints. In the case of the storage being empty, for example, we can examine the transformed inequality constraints,

$$-E(t) + E^{min} + z_l = 0, \tag{36}$$
$$-P_{in}(t) + z_o = 0, \tag{37}$$
$$-P_{out}(t) + z_p = 0, \tag{38}$$

and see that if any of the slack variables are zero, it implies that that variable is at its minimum. To ensure that these constraints are satisfied when the slack variables become zero, a check must also be done to determine if the variables $P_{in}(t)$, and $P_{out}(t)$ are at their minimum, and $E(t+T)$ and $E(t)$ are both at their minimum or both at their maximum as well. Numerically, these variables may not actually reach exactly zero, so the comparison for implementation purposes is done with a small number ϵ that is close to zero. For the rare case when both $\mu_i(t)$ and $z_i(t)$ become zero at the same time, the Jacobian will become singular because a row of zeros is created. To avoid this case, certain measures can be taken such as running the optimization from a different starting point or using a smaller damping value on the Newton-Raphson step.

Overall, the steps to indicate whether or not to remove the rows corresponding to the storage device constraints and actually remove corresponding rows and columns are as follows:

1) Determine if ($|E(t)-E_{min}|$ or $|E(t)-E_{max}|$), ($|E(t+T) - E_{min}|$ or $|E(t+T) - E_{max}|$), $|P_{in}(t)|$, and $|P_{out}(t)| < \epsilon$.
2) Determine if the slack variables $z(t)$ corresponding to the above variables are all less than ϵ.
3) If 1) and 2) are true, remove the following elements of the Jacobian:
   a) The rows and columns in the $\nabla^2_{xx}\mathcal{L}$ block that correspond to the partial derivatives of $E(t)$, $P_{in}(t)$, and $P_{out}(t)$.
   b) The rows and columns corresponding to the gradients of the binding storage constraints at time t.
   c) The rows and columns in $diag\{\mu\} \cdot z$ that include $\mu(t)$ and $z(t)$ for the corresponding inequality constraints.
4) Replace instances of variables $E(t)$, $P_{in}(t)$, and $P_{out}(t)$ where they appear in the rest of the Jacobian matrix with their optimal values.
5) Adjust the KKT conditions in the right-hand side vector of the update to no longer include constraints (4)-(7), partial derivatives of the Lagrangian with respect to the storage variables, or complementary slackness conditions $diag\{\mu(t)\} \cdot z(t)$.
6) Repeat for each time instance considered in the optimization; i.e., for the entire problem time horizon

$t = 0, ..., N-1$.

7) Perform the Newton-Raphson step with the reduced Jacobian and right-hand side vector.

D. Discussion of Methods

Depending on the application and purpose, some of these methods may be more appropriate than others. For example, using a Moore-Penrose pseudoinverse may be more computationally complex and require more computation time than other methods. It also requires that either the rank or condition number of the Jacobian is checked each iteration to determine if the Jacobian is close to singular. Integrating storage standby losses may not only fix the singularity of the Jacobian, but may also provide a more realistic model of a storage device.

However, this does require a modification of the model, and if the standby losses are too small, the matrix may still be close to singular and numerically unstable. Also, in some rare cases, even with standby losses, all of the storage inequalities can still be binding, as discussed above. The technique of removing the binding constraints/variables has the benefit of reducing the size of the Jacobian matrix and hence potentially reducing the number of computations per Newton step; however, this method also has the downside of deciding the tolerance parameter ϵ. If the constraints are removed prematurely and they actually are not binding at the optimal solution, the KKT conditions may not be satisfied, and if they are removed too late, the Jacobian may already be close to singular, resulting in numerical issues.

V. Simulation Results

In this section, results are shown for AC OPF simulations on the IEEE 14 bus system [12]. Wind generators, modeled as negative loads, have been added at buses 5 and 14, and a storage device has been added at bus 5 as seen in Figure 1. The objective is to minimize the quadratic cost of generation from the generators at buses 1, 2, and 3. Simulations were done over a period of 24 hours, with a 5-minute discretization and prediction horizons $N = 5$ and $N = 10$. The receding horizon concept is used, where the optimization is performed over the time horizon N, variables are updated, and the time window is shifted and the process is repeated.

The formulation of the KKT conditions in Section II was modified to incorporate the Unlimited Point Algorithm [13], which is a technique to ensure the non-negativeness of μ and z by raising these variables to an even power. In our simulations, we have squared μ and z where they appear in the KKT conditions, keeping the complementary slackness condition $diag\{\mu\} \cdot z = 0$ the same, because it is equivalent to $diag\{\mu^2\} \cdot z^2 = 0$. The modified Jacobian is shown below:

$$\begin{bmatrix} \nabla^2_{xx}\mathcal{L}(x,z,\lambda,\mu) & \nabla g(x)^T & 2\nabla h(x)^T \cdot diag\{\mu\} & 0 \\ \nabla g(x) & 0 & 0 & 0 \\ \nabla h(x) & 0 & 0 & 2diag\{z\} \\ 0 & 0 & diag\{z\} & diag\{\mu\} \end{bmatrix} \quad (39)$$

This does not change the singularity problem, as the dependent binding constraints still result in these rows being linearly dependent. Other methods to account for the positivity of μ and z, such as using an interior point or barrier method, result in the same issues. Thus the given methods to fix the singularities were explained for the general KKT conditions in (13)-(18), but it is important to note that these singularities still exist even when using the Unlimited Point or Interior Point method. In the following simulations, we use the method of removing the rows of the Jacobian which cause the singularity issue.

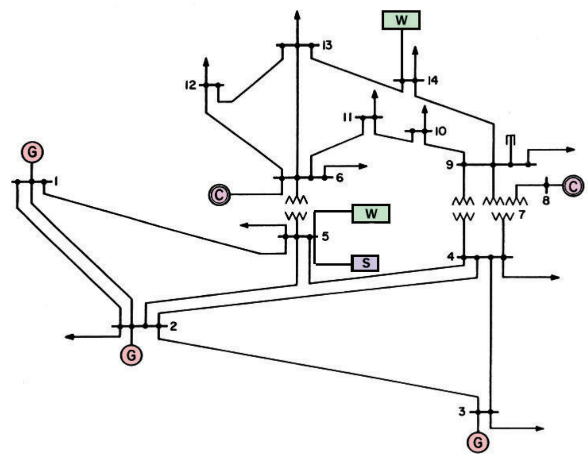

Fig. 1. Modified IEEE 14-bus System

Data for the wind and load curves were taken from the Bonneville Power Administration [14]. One simulation output for the energy level of the storage device over a 24-hour period with $N = 5$ is seen in Figure 2. The storage device has a minimum required capacity of 0.2p.u. and a maximum capacity of 1.2p.u. The storage level for this horizon length never reaches its maximum; however, with longer horizons, the storage is utilized more and does reach its maximum value, as seen in Figure 3. The storage is utilized to balance out the intermittency of the wind generators in attempts to keep controllable generators at a more constant level without having to ramp up and down.

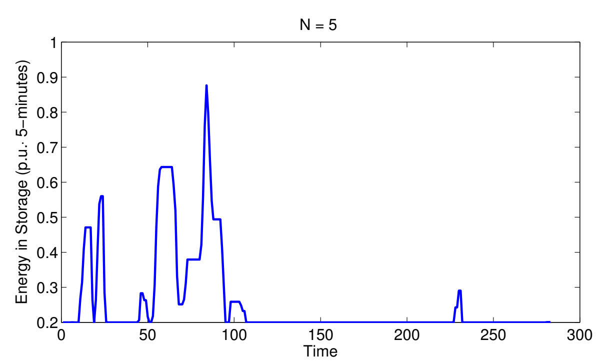

Fig. 2. Storage Energy Level for N = 5

For the cases of horizons $N = 5$ and $N = 10$, instances where the constraints (4)-(7) were found to be simultaneously binding for at least one time instance within the horizon and constraints have been removed are identified with a '1' in Figures 4 and 5. This occurs whenever the storage level is

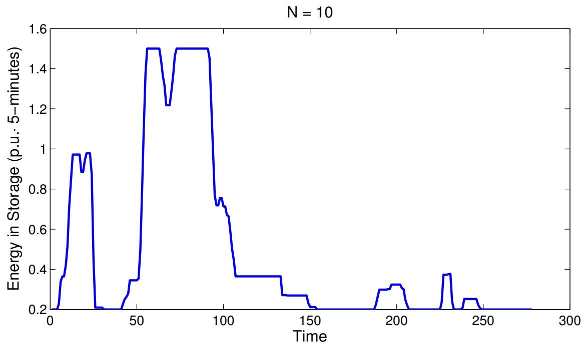

Fig. 3. Storage Energy Level for N = 10

at its maximum or minimum value for multiple consecutive time steps. A binding constraint is indicated by using the tolerance parameter $\epsilon = 10^{-9}$. It is important to note that one cannot simply look at Figures 2 and 3 to know when constraints have been removed because these figures only show the actual energy level in the storage, not the optimal output of the prediction horizon considered in the original optimization problem. At 161 out of 288 points in the simulation, one set of storage constraints in the $N = 5$ horizon was found to be binding. For $N = 10$, 129 time points in the simulation had the case with dependent rows. Thus, it is a very common occurrence in the considered problem setup.

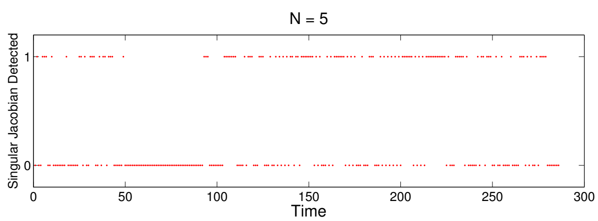

Fig. 4. Instances where storage constraints are binding and dependent

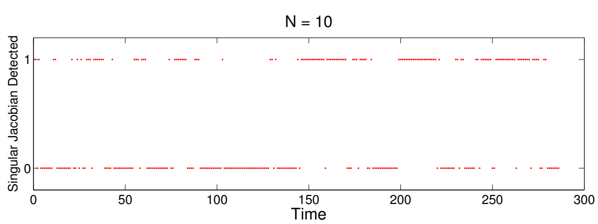

Fig. 5. Instances where storage constraints are binding and dependent

VI. Conclusion

This paper addresses the case of a singular Jacobian matrix due to linearly dependent binding intertemporal constraints in optimal power flow formulations and demonstrates that the Jacobian matrix is actually rank-deficient at the optimal solution. The specific case of the integration of storage devices into an OPF problem was studied. It was shown that singular Jacobian matrices can be avoided with a variety of techniques, such as introducing standby losses that prevent all of the relevant constraints from being simultaneously binding.

Alternatively, methods are discussed that do not require modifying the structure of the storage model. One technique is by using a Moore-Penrose pseudoinverse. Another technique considers analyzing the slack variables related to the intertemporal constraints. When the relevant constraints are simultaneously binding, as indicated by the slack variable being zero, the rows and columns corresponding to these constraints and related variables can be removed from the Jacobian. This technique has the benefit of reducing the size of the Jacobian so that the computation time for each iteration could potentially decrease. Identifying the cause of Jacobian singularities and knowing how to continue to solve the optimization problem once the Jacobian has become singular are two very important issues to be aware of when solving OPF problems. As seen from the simulation results, this can be a frequently encountered problem, especially when using certain energy storage models.

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