18-799F Algebraic Signal Processing Theory
Spring 2007
Assignment 4
Due Date: Feb. 21th 2:30pm (at the beginning of class)

1. (40 pts) Let \( p(x) = \sum_{i=0}^{n} \beta_i x^i \), \( \beta_n \neq 0 \), be a polynomial of degree \( n \).
   
   (a) Prove that the mapping
   \[
   \phi : \mathbb{C}[x]/p(x) \to \mathbb{C}[x]/p(x) \\
   q(x) \mapsto xq(x) \quad (\text{mod } p(x))
   \]
   is linear.
   
   (b) Represent this mapping as a matrix \( B_\phi \) with respect to the canonical basis \( \{1, x, x^2, \ldots, x^n\} \).
   
   (c) Determine \( B_\phi \) in the special case \( p(x) = x^n - 1 \). Do you know this matrix?

2. (30 pts) Let \( p(x) \) as in question 1. Assume that it has pairwise distinct zeros; i.e. \( p(x) = \beta_n \cdot \prod_{i=0}^{n-1} (x - \alpha_i) \), such that \( i \neq j \Rightarrow \alpha_i \neq \alpha_j \). In the class you learned that the mapping
   \[
   \phi : \mathbb{C}[x]/p(x) \to \bigoplus_{k=0}^{n-1} \mathbb{C}[x]/(x - \alpha_k) \\
   q(x) \mapsto (q(\alpha_0), \ldots, q(\alpha_{n-1}))
   \]
   is an isomorphism of algebras.
   
   (a) Represent this mapping as a matrix \( B_\phi \) with respect to the canonical bases \( b = \{1, x, x^2, \ldots, x^n\} \) and \( c = \{e_0, \ldots, e_{n-1}\} \) in \( \bigoplus_{k=0}^{n-1} \mathbb{C}[x]/(x - \alpha_k) \). Here, \( e_k \) is a canonical vector that has 0 in all positions except the \( k \)-th and 1 in the \( k \)-th position.
   
   (b) Determine \( B_\phi \) in the special case \( p(x) = x^n - 1 \). Do you know the mapping \( v \mapsto B_\phi v, v \in \mathbb{C}^n \)?

3. (30 pts) Let \( i = \sqrt{-1} \).
   
   (a) Determine the matrix representation of the mapping \( \mathbb{C}[x]/(x^4 - 1) \to \bigoplus_{k=0}^{3} \mathbb{C}[x]/(x - i^k) \) with respect to canonical bases as in the previous problem. Explain your answer.
   
   (b) Determine the matrix representation of the mapping \( \mathbb{C}[x]/(x^4 - 1) \to \mathbb{C}[x]/(x^2 - 1) \oplus \mathbb{C}[x]/(x^2 + 1) \), where in each summand on the right side we choose \( \{1, x\} \) as a basis.

4. (20 pts) **Extra-credit** Let’s continue the decomposition in 3(b).
   
   (a) Completely decompose both summands
   \[
   \mathbb{C}[x]/(x^2 - 1) \oplus \mathbb{C}[x]/(x^2 + 1) \to (\mathbb{C}[x]/(x - 1) \oplus \mathbb{C}[x]/(x + 1)) \oplus (\mathbb{C}[x]/(x - i) \oplus \mathbb{C}[x]/(x + i)).
   \]
   and determine the matrix representation of this mapping.
   
   (b) Use the results from parts (b) and (c) to obtain a factorization of the matrix in part (a) into a product of (sparse) matrices. Write this factorization as a (correct) matrix equation. Note: make sure you follow the order of the summands in the direct sum in 3(a).