1. (30 pts) Describe the structure of the following sets $S$ with respect to addition and multiplication. Possible answers may include (but are not limited to):
- $(S, +)$ is a group;
- $(S, +)$ is a commutative group;
- $(S, +, \cdot)$ is a ring;
- $(S \setminus \{0\}, \cdot)$ is a group;
- $(S \setminus \{0\}, \cdot)$ is a commutative group;
- $(S, \cdot)$ is a commutative group;
- $(S, +, \cdot)$ is a field.

Only state the "most structure." Briefly explain why the set has the structure and give counterexamples to show that it has not more structure. (The comments in the parentheses make a loose connection to signal processing.)

*Additional information:* As you know, $\alpha_i$ is a zero of a polynomial $f(x)$ if $f(\alpha_i) = 0$; in this case, if $\deg(f) = n$ and $\alpha_1, \ldots, \alpha_n$, are the zeros of $f(x)$, you can write the polynomial as $f(x) = \prod_{i=0}^{n} (x - \alpha_i)$.

(a) The set of real invertible matrices of size $n \times n$: $GL_n(\mathbb{R}) = \{ A \in \mathbb{R}^{n \times n} \mid \text{exists } A^{-1} \in \mathbb{R}^{n \times n} \}$.
(b) (stable IIR filters) Set of complex rational functions $S = \{ \frac{p(x)}{q(x)} \mid p(x), q(x) \in \mathbb{C}[x], q(x) \neq 0 \}$, for every zero $\alpha$ of $q(x): |\alpha| \leq 1$.
(c) (minimum-phase filters) Set of complex rational functions $S = \{ \frac{p(x)}{q(x)} \mid p(x), q(x) \in \mathbb{C}[x], q(x) \neq 0 \}$, for every zero $\alpha$ of $p(x)$ or $q(x): |\alpha| \leq 1$.
(d) (shifts) $S = \{ x^n \mid n \in \mathbb{Z} \}$.

2. (21 pts)

(a) Show that $\phi : \mathbb{R}[x] \to \mathbb{C}, \ p(x) \mapsto p(\sqrt{-1})$

is a ring homomorphism.
(b) Show that $\phi$ is surjective.
(c) Determine the kernel of $\phi$ and apply the homomorphism theorem to $\phi$.

3. (14 pts) Recall that for square matrices $A, B \in \mathbb{R}^{n \times n}$, $\det(AB) = \det(A) \det(B)$ and $\det(A^{-1}) = \det(A)^{-1}$ (provided $A$ is invertible).

Let $SL_n(\mathbb{R}) = \{ A \in \mathbb{R}^{n \times n} \mid \det(A) = 1 \}$. Show that

(a) $(SL_n(\mathbb{R}), \cdot) \preccurlyeq (GL_n(\mathbb{R}), \cdot)$.
(b) $(GL_n(\mathbb{R})/SL_n(\mathbb{R}), \cdot) \simeq (\mathbb{R} \setminus \{0\}, \cdot)$. (Hint: define a suitable homomorphism and apply the homomorphism theorem).

4. (35 pts) In the class we asserted that $\mathbb{C}[x]$ is a Euclidean ring with respect to the usual polynomial division with rest and $\delta = \deg (\text{the degree, defined as } \deg(\sum_{i=0}^{n} a_i x^i) = n)$.

(a) Determine $\mathbb{C}[x]^*$. 
(b) Find \( \text{gcd}(x^3 - x^2 + 2x - 2, x^2 - 1) \) using the Euclidean algorithm. Write \((x^3 - x^2 + 2x - 2)\mathbb{C}[x] + (x^2 - 1)\mathbb{C}[x]\) as a principal ideal.

(c) Explain why \( \mathbb{C}[x]/p(x)\mathbb{C}[x] \) is a ring with respect to addition and multiplication for any \( p(x) \in \mathbb{C}[x] \).

(d) Recall that we can write \( \mathbb{Z}/n\mathbb{Z} = \{0, \ldots, [n-1]\} \) simply as the set \( \{0, \ldots, n-1\} \) with addition and multiplication \( \text{mod} \ n \). Similarly, we can view \( \mathbb{C}[x]/p(x)\mathbb{C}[x] \) as the set of polynomials \( \{q(x) \in \mathbb{C}[x] \mid \deg(q) < \deg(p)\} \) with addition and multiplication performed \( \text{mod} \ p(x) \).

(i) Compute \( x^i \) (for \( i \geq 0 \)) in \( \mathbb{C}[x]/(x^4 - 1)\mathbb{C}[x] \).

(ii) Describe \( (\mathbb{C}[x]/(x^4 - 1)\mathbb{C}[x])^\times \). What can you say about the zeros of its elements?

5. **Extra credit problem** (20 pts)

(a) Consider rings \( (R_1, +, \cdot), \ldots, (R_n, +, \cdot) \). Show that their Cartesian product \( R = R_1 \times \cdots \times R_n = \{(a_1, \ldots, a_n) \mid a_i \in R_i\} \) is also a ring with respect to component-wise addition and multiplication. What are its neutral elements with respect to addition and multiplication (i.e. its zero and one)?

(b) Consider \( p(x) = \prod_{i=1}^{n} (x - \alpha_i) \in \mathbb{C}[x] \) with pairwise distinctive zeros (i.e. \( i \neq j \Rightarrow \alpha_i \neq \alpha_j \)). Prove that the mapping

\[
\phi : \mathbb{C}[x]/(p(x)\mathbb{C}[x]) \rightarrow \mathbb{C}[x]/(x - \alpha_1)\mathbb{C}[x] \times \cdots \times \mathbb{C}[x]/(x - \alpha_n)\mathbb{C}[x]
\]

\[
q(x) \mapsto (q(\alpha_1), \ldots, q(\alpha_n))
\]

is a ring isomorphism. This fact is known as **Chinese Remainder Theorem**.