1. (15 pts) Let \( C_6 = \langle x \mid x^6 = 1 \rangle \) be the cyclic group of order 6.
   (a) Determine all subgroups (including the trivial ones \( \langle 1 \rangle \) and \( C_6 \)) of \( C_6 \).
   (b) For each subgroup, determine the minimum set of generators.
   (c) Express each subgroup using generators and relations.

2. (12 pts) The function \( \cos : \mathbb{R} \to [-1, 1] \) induces the following equivalence relation on \( \mathbb{R} \):
   \[ x \sim y \iff \cos x = \cos y. \]
   (a) Determine the partition \( \mathbb{R}/\sim \) induced by \( \cos \).
   (b) Determine the canonical factorization of \( \cos \) and draw the associated commutative diagram.

3. (17 pts) Which of the following is well-defined (give a counterexample or prove)?
   (a) The operation \( [x] \cdot [y] = [xy] \) on \( \mathbb{Z}/n\mathbb{Z} \).
      Is \( (\mathbb{Z}/n\mathbb{Z}, \cdot) \) a group? Explain your answer.
   (b) The function \( f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \) \( [x] \mapsto [x^2] \)
   (c) The function \( f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \) \( [x] \mapsto [x + 1] \)

4. (10 pts) Consider the set of all rational numbers (excluding zero) with multiplication: \( (\mathbb{Q} \setminus \{0\}, \cdot) \).
   (a) Show that \( (\mathbb{Q} \setminus \{0\}, \cdot) \) is a group.
   (b) What are the generators of this group? Explain your answer.

5. (20 pts) Consider the dihedral group of order 8: \( D_8 = \langle x, y \mid x^4 = y^2 = 1, xy = yx^{-1} \rangle \). As we discussed in the class, this is the group of symmetries of a square (\( x \) is a 90-degree rotation, and \( y \) is a reflection.) Obviously, one of its subgroups is a cyclic group of order 4: \( C_4 = \langle x \mid x^4 = 1 \rangle \leq D_8 \).
   (a) What is the index of \( C_4 \) in \( D_8 \)?
   (b) Express \( D_8 \) as a union of left cosets of \( C_4 \).
   (c) What are the elements of \( D_8 \) (expressed in \( x \) and \( y \))? 
   (d) Prove that \( C_4 \leq D_8 \).

6. (15 pts) \( \mathbb{R}[x] = \{ p(x) = \sum_{i=0}^{n} a_i x^i \mid n \in \mathbb{N}_0, a_i \in \mathbb{R} \} \) is the set of all polynomials with real coefficients \( (\mathbb{N}_0 = \{0, 1, \ldots \}) \).
   For two polynomials \( p(x), q(x) \in \mathbb{R}[x] \) we say that \( p(x) \) divides \( q(x) \) if \( q(x) = p(x)r(x) \) for some \( r(x) \in \mathbb{R}[x] \). We write this as \( p(x) \mid q(x) \).
   (a) Let us fix some \( p(x) \in \mathbb{R}[x] \). Prove that the following is an equivalence relation on \( \mathbb{R}[x] \):
      \[ r(x) \sim s(x) \iff p(x) | (r(x) - s(x)) \]
      (b) Consider the group \( (\mathbb{R}[x], +) \). Find a subgroup \( H \leq \mathbb{R}[x] \), such that for \( \sim \) defined above,
      \[ \mathbb{R}/\sim = \mathbb{R}[x]/H \]
      and show this holds.
(c) Is $(\mathbb{R}[x]/H, +)$ a group? Explain your answer.

7. (11 pts) Let $G = \{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mid x \in \mathbb{R} \}$. 

(a) Show that $G$ is a group with respect to the multiplication $\cdot$ of matrices.

(b) Show that 

$$
\phi : (\mathbb{R}, +) \to (G, \cdot) \\
x \mapsto \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}
$$

is an isomorphism.

8. Extra credit problem (20 pts)

(a) Let $H \leq G$ is of index 2 in $G$. Prove that $H \triangleleft G$.

(b) Prove that $C_n = \langle x \mid x^n = 1 \rangle$ does not have non-trivial subgroups if and only if $n$ is prime.

Note: This shows that $C_p$, $p$ prime, are simple groups, the only abelian ones in fact.