Subgroups

Let \( G \) be a group. \( H \leq G \) is a "subgroup" of \( G \) if \( H \) is a group (w.r.t. the same operation).

Notation: \( H \leq G \)

Test for subgroup:
1. \( H \subseteq G \)
2. \( xy \in H \Rightarrow xy^{-1} \in H \)

Coset decomposition

Let \( H \leq G \).

\[ x \sim y \iff x^{-1}y \in H \iff y \in xH \]

is an equivalence relation.

Proof:
- \( x \sim x \) since \( x^{-1}x = e \in H \)
- \( x \sim y \Rightarrow x^{-1}y \in H \Rightarrow (x^{-1}y)^{-1} = x^{-1}y^{-1} \in H \Rightarrow y \sim x \)
- \( x \sim y, y \sim z \Rightarrow x^{-1}y \in H, y^{-1}z \in H \Rightarrow x^{-1}y \cdot y^{-1}z \in H \Rightarrow x \sim z \)

Note: the proof needs \( e_1 \), inverse \( e_1 \) and closed under \( \cdot \)

hence only subgroups make \( \cdot \) an equivalence relation.

- \( \text{ex.} = xH \) is called "left coset"
- notation: \( G/H = G/H \)
- \( |xH| = |yH| = |H| \) for \( x, y \in G \)
- if \( |G/H| \) finite: \( G = xH \cup \ldots \cup xH \) (one of these "coset decompositions"

analogously: define right cosets via \( x \sim y \iff xy^{-1} \in H \)

Lemma: \( |G| = |G/H| \cdot |H| \)

index of \( H \) in \( G \)

We have decomposed \( G \) into the subgroup \( H \) and \( G/H \). Ideally, \( G/H \) is also a group w.r.t.:

\[ [x] \cdot [y] = [xy] \]
\[ xH \cdot yH = xyH \]

Need to assure well-defined. Other group properties are straightforward.
Check: \( a \cdot x, b \cdot y \Rightarrow a \cdot b \sim xy \)

\[ a = x^4, \quad b = y^6, \quad x, y \in H \]

\[ \Rightarrow \quad ab = x^4 y^6 = xy = y^{-1} x \quad y^{-1} \quad y \in H \]

Result: \( * \) well-defined \( \Leftrightarrow \) for all \( h \in H, \; x \in G : \; y^{-1} h y \in H \)

\[ \Leftrightarrow \quad \text{for all } \; y \in G, \; y^{-1} H y \subseteq H \]

\[ \Leftrightarrow \quad \text{or } \; H y = y H \]

Definition: \( H \leq G \) is called a "normal subgroup" if for all \( y \in G, \; y^{-1} H y = H \). Notation: \( H \trianglelefteq G \).

Lemma: If \( N \trianglelefteq G \), then \( G/N \) is a group called a "factor group."

If \( G \) is abelian then every subgroup is normal.

\[ G/H \text{ a group } \Leftrightarrow H \trianglelefteq G \]

Brief discussion on the classification of groups and especially simple groups (groups without normal subgroups) which was finished in 2000.

Examples:

a.) \( C_4 = <x \mid x^4 = 1> = \{1, x, x^2, x^3\}, \; H = <x^2> = \{1, x^2\} \trianglelefteq C_4 \)

\[ \Rightarrow \quad \text{for } \; \text{if } \; H \text{ is a } C_2 \text{ and } C_4/H \text{ is a group (of size 2)} \]

b.) \( G = \mathbb{R} \) (with addition), \( H = \mathbb{Z} \trianglelefteq \mathbb{R} \), \( xH \equiv x \quad \text{mod } \mathbb{Z} \)

\[ \mathbb{R}/\mathbb{Z} \text{ is the group } \{x + y / y \in \mathbb{Z} \} \]

\[ \xrightarrow{1} \quad \text{or better } \circ \]

Since \( 0 \)-1

\[ \circ \]

c.) \( (\mathbb{Z}, +) \trianglelefteq (\mathbb{Q}, +) \Rightarrow (\mathbb{Q}/\mathbb{Z}, +) \text{ is a group,} \]

in fact a cyclic group of size \( 1 \).
**Group homomorphism**

**Definition:** A group homomorphism is a mapping \( \Phi : G \rightarrow H \) (\( G, H \) groups) s.t. \( \Phi(xy) = \Phi(x)\Phi(y) \), \( x, y \in G \).

**Lemma:** a) \( e, e' \) neutral elements of \( G, H \), resp.

\[ \Rightarrow \quad \Phi(e) = e' \]

b) \( \Phi(x)^{-1} = \Phi(x^{-1}) \)

**Examples:**
- \( \Phi : (\mathbb{R}, +) \rightarrow (\mathbb{R}, +) \)
  \[ x \mapsto \log_2(x) \]
- \( H \leq G \), \( \Sigma : H \rightarrow G \)
  \[ x \mapsto \Sigma(x) \text{ called "embedding" (injective)} \]
- \( N \trianglelefteq G \), \( \kappa : G \rightarrow G/N \)
  \[ x \mapsto xN \text{ called "canonical (surjective) projection"} \]

**Kernel**

\( \Phi : G \rightarrow H \) hom.\, \text{Ker}(\Phi) = \{ x \in G / \Phi(x) = e' \} \)

is called "kernel of \( \Phi \)"

**Lemma:**

a) \( \text{Ker}(\Phi) \trianglelefteq G \)

b) Every \( N \trianglelefteq G \) is kernel of a suitable hom.

c) \( G/N \cong \mathbb{Z}/\text{Ker}(\Phi) \)

**Proof:**

a) \( \text{Ker}(\Phi) \trianglelefteq G \)

- \( x, y \in \text{Ker}(\Phi) \Rightarrow \Phi(xy^{-1}) = \Phi(x)\Phi(y)^{-1} = e'e' = e' \Rightarrow xy^{-1} \in \text{Ker}(\Phi) \)

- \( x \in \text{Ker}(\Phi), y \in G \Rightarrow \Phi(y^{-1}xy) = \Phi(y)^{-1}\Phi(x)\Phi(y) \Rightarrow \text{Ker}(\Phi) \trianglelefteq G \).

- \( \Phi(y)^{-1}e'\Phi(y) = e' \Rightarrow \text{Ker}(\Phi) \trianglelefteq G \).

b) Choose \( \Phi : G \rightarrow G/N \)

c) Equivalence induced by \( \Phi : x \mapsto y \Leftrightarrow \Phi(x) = \Phi(y) \)
As a consequence of c.) we get the canonical factorization:

\[ \begin{array}{cccc}
G & \xrightarrow{\phi} & H \\
\downarrow{\text{surj.}} & & \uparrow{\text{inj.}} \\
\phi/\ker \phi & \xrightarrow{\tilde{\phi}} & \phi(G) \\
\end{array} \]

\[ \text{bijective} \]

and \( \tilde{\phi} \) is also a hom:

\[ \begin{array}{c}
\text{map: } \tilde{\phi}(cxj), \tilde{\phi}(cyj) \xrightarrow{\text{def}} \tilde{\phi}(cxj)\tilde{\phi}(cyj) \\
\text{map: } [xyj] \xrightarrow{\text{def}} \tilde{\phi}(cxj)\tilde{\phi}(cyj) = \tilde{\phi}(xyj) \end{array} \]

since \( \text{is a hom} \)

**Theorem:** \( \psi: G \to H \) a hom. Then

\[ \frac{G}{\ker \psi} \cong \psi(G) \]

**Summary:** What we did

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**Little history of group theory**

- **One origin:** study of symmetries by the Moors (13th century)
- **Groups to solve polynomial equations:** Évariste Galois (1811–1832) in 1829, Terned "group" and "normal subgroup"
- **Study of groups:** Arthur Cayley (1821-1855) in the 1850s
- **First complete def. of an abstract group (as taught today):** Heinrich Weber (1842–1813) in 1882, Terned "asb'amu".