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Cellular Economies of Scale and Why Disparities in Spectrum Holdings are Detrimental

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Abstract

Now that traffic volumes are increasing rapidly, the cost of expanding capacity has become a large portion of expenditures for Mobile Network Operators (MNOs). This paper uses an engineering-economic model to show that there are strong economies of scale when expanding capacity, because an MNO with more spectrum benefits more from every new cell tower, and an MNO with more towers benefits more from every new MHz of spectrum. While it is technically possible to expand capacity by increasing either towers or spectrum holdings, we find that the cost-effective approach is to increase both types of assets at a similar rate. In the absence of countervailing policies, the big MNOs are well positioned to get bigger, in terms of spectrum holdings, towers, and ultimately market share. For policymakers, this economy of scale creates a trade-off between two important objectives: reducing the cost of cellular capacity, and increasing competition. This paper derives the Pareto optimal division of spectrum with respect to these two competing objectives, and shows that any Pareto optimal assignment will split the spectrum fairly evenly among competing MNOs. This is not simply a method of ensuring that there are many competitors; spectrum should be divided fairly evenly regardless of whether the number of competitors is large or small. A large disparity in spectrum holdings may yield poor results with respect to both objectives, i.e. the lower cost-effectiveness of a larger number of MNOs, and the lower competitive pressure of a smaller number of MNOs. One effective way to achieve a division of spectrum that is close to Pareto optimal is a spectrum cap, provided that this cap is set at a level consistent with other policies and policy objectives, including antitrust policy.

Highlights

- There are large economies of scale when MNOs are rapidly increasing capacity.
- MNOs minimize cost by growing spectrum and tower holdings at similar rates.
- There is a trade-off between cost-effectiveness and market competition.
- It is Pareto optimal to divide spectrum fairly evenly among MNOs.
- Spectrum caps are one way to reach Pareto optimal divisions of spectrum.

Keywords: cellular, spectrum, competition, economy of scale, market concentration, spectrum cap

[1] A version of this paper will be presented at the 2017 TPRC (Peha, 2017).

1 Introduction

Until recently, cell phones were primarily for phone calls and texts. For someone who uses her cell phone primarily for phone calls, coverage is extremely important. Driving through even a small hole in coverage can terminate an important conversation. To attract such customers, Mobile Network Operators (MNOs) needed spectrum throughout the areas that the MNO intended to serve, and preferably some of that spectrum would be at low frequencies. This probably explains why Ofcom found (Ofcom, 2012, Jan. 12; Ofcom, 2012, July 24) that in the years before 2012 an MNO had been a "credible" competitor if it held 10% or more of the cellular spectrum in that MNO's country. Although Ofcom still justifies spectrum policy decisions on that conclusion (Ofcom, 2016), the situation has changed as smartphones became pervasive, and smartphone applications emerged that transferred large amounts of data. Today's cell phone users still want coverage, but now many want to know whether their share of an MNO's capacity will be enough. The capacity required to meet the demands of cell phone users is doubling worldwide every 20 months (Cisco 2016). As a result, expanding macrocellular capacity has become a major driver of annual expenditures for cellular MNOs.

As this paper will show, this need to rapidly increase capacity has yielded new economies of scale. The largest MNO can expand its capacity at a lower cost than its smaller competitors, which means that the large can easily get larger. This result may help to explain why large MNOs around the world have generally been growing, and many small MNOs have been seeking opportunities to merge. It is also consistent with the profitability often observed in large MNOs, e.g. in the U.S. the two largest MNOs (Verizon and AT&T) have significantly greater margins than the third and fourth largest (T-Mobile and Sprint) (Forbes 2015). The fact that scale decreases costs has advantages for users of cellular services, as long as the large MNOs choose to pass these cost savings along to their customers. However, if the large MNOs increase market share, there will be less competition, giving MNOs less incentive to lower prices and improve quality of service.

This paper explores how best to divide spectrum resources among MNOs given this economy of scale. There are two potentially competing objectives for policymakers: increasing competition and lowering the cost of capacity. This paper will show that in any result that is Pareto optimal with respect to these two objectives, spectrum is divided fairly evenly among MNOs, regardless of whether the number of competing MNOs is large or small. Large disparities in spectrum holdings are therefore not in the public interest.

One simple way to divide spectrum in a market economy in a way that is close to Pareto optimal is through some form of *spectrum cap,* where an MNO is free to put together a portfolio of frequencies that meets the company's needs, provided that the total does not exceed some upper bound. Proposed mergers, such as AT&T with T-Mobile in the U.S. and O2 with Three in the U.K., have sparked intense debates over whether the goal of antitrust policy should be to have at least three competing MNOs or at least four. This paper shows that spectrum policy is also important, and that spectrum policy and antitrust policy should be in sync on this issue.

The analysis in this paper assumes that there is no market failure in spectrum, so that spectrum is available at the price where supply meets demand. This means we assume that MNOs are not engaging in a strategy of "foreclosure," even though a common argument for spectrum caps is as a means of preventing foreclosure (Ergas & Ralph, 1998; Baker, 2007; Cramton et al, 2011; U.S. Dept. of Justice (DOJ), 2013) rather than addressing an economy of scale. Because there is a limited amount of spectrum available to MNOs and few opportunities to obtain spectrum from sources other than a direct competitor, an MNO that is determining the price at which it will buy or sell spectrum "will include in its private value not only its use value of the spectrum but also the value of keeping the spectrum from a competitor" (Cramton et al, 2011). As defined by the U.S. Department of Justice, "the latter might be called 'foreclosure value' as distinct from 'use value.' The total private value of spectrum to any given provider is the sum of these two types of value" (DOJ, 2013). If some MNOs do consider foreclosure value in their transactions, this would only increase the need for policies that limit large disparities in spectrum holdings.

This paper is organized as follows. Section 2 describes the most important assumptions of our analytic model with respect to both the cost and capacity of cellular infrastructure. Section 3 explores the resource decisions that would maximize profit for MNOs. This section shows how MNOs reduce cost by balancing spectrum acquisitions and tower acquisitions, and the economies of scale for both. Section 4 explores the resource decisions that would serve the public interest, which might involve lowering the cost of capacity, increasing competition, or both. This section derives the Pareto optimal strategies with respect to these two competing objectives. While Sections 3 and 4 assume that one MNO's cost per cell tower does not depend on the choices of that MNO's competitors, which is often but not always the case, Section 5 relaxes that assumption by considering the effects of explicit tower-sharing arrangements between MNOs. Finally, we summarize conclusions and discuss their policy implications in Section 6.

2 Model Assumptions

This section presents some of the most fundamental assumptions underlying our analysis, and the implications of any simplifications made. Section 2.1 describes our assumptions about how MNOs maximize profit. Section 2.2 explains our focus on capacity-limited macrocells in both cost and capacity calculations. Sections 2.3 and 2.4 discuss key assumptions in the cost and capacity analyses, respectively. All of our analysis considers only facilities-based MNOs, rather than Mobile Virtual Network Operators (MVNOs) that only resell services. It is facilities-based MNOs that operate infrastructure, and make decisions about spectrum and tower acquisitions to meet capacity needs. Moreover, it is competition among facilities-based MNOs that motivates service providers to pass savings in infrastructure cost on to consumers.

2.1 Profit Maximization

This paper assumes that MNOs are in an equilibrium state where profit has been maximized, e.g. where cost is minimized for a given capacity or capacity is maximized for a given cost, and where costly resources have not been spent to increase capacity that is not (yet) needed to carry customer traffic. This simplification ignores the unique history that produced each MNO's infrastructure. For example, immediately after a merger, an MNO's infrastructure and spectrum holdings are unlikely to be as cost-effective as possible, but the MNO will strive to move towards a maximally cost-effective state over time. This simplification also ignores dynamic elements. For example, where our assumptions might cause an MNO to gradually obtain spectrum and gradually build towers to meet the gradually-increasing customer demand for capacity, a real MNO might occasionally obtain and use a large block of spectrum that is expected to meet both current needs and anticipated needs for the next few years, thereby creating capacity that is temporarily unneeded. Thus, at any instant in time, one MNO may have more excess capacity than another. Over the long term, however, these temporary effects should have little impact.

We also assume that all MNOs are deriving similar revenues per unit of capacity. Thus, if two MNOs have the same capacity but one has much higher cost, the high-cost MNO will be less profitable. Moreover, this means that an MNO's share of overall cellular capacity should be similar to its share of the overall cellular market, at least in regions where there is no excess capacity that is not in use. These assumptions are reasonable where MNOs offer similar services and compete primarily based on price, thereby becoming more like commodities, which has been typical in recent years. A somewhat more complicated model is needed if MNOs offer diverse services that can command very different prices per GB, such as where one MNO offers a service that is far more reliable and more expensive, or where an MNO offers cellular services bundled with unique content.

2.2 Capacity-Limited Macrocells

In both our cost analysis and our capacity analysis, we consider a region where all MNOs are capacity-limited, i.e. where an MNO that merely deploys the minimum number of cell towers to bring adequate coverage to the region will not have enough capacity. Of course, there are also many sparsely-populated regions that are coverage-limited rather than capacity-limited, i.e. where simply employing the minimum number of towers to bring adequate coverage will also provide more than enough capacity. We assume that a region that is coverage-limited for one MNO is coverage-limited for all MNOs, and that the value of spectrum in these sparely-populated regions is negligible compared to the value of spectrum in the capacity-limited regions. Thus, we do not count the cell towers deployed in coverage-limited regions in our analysis. Although these assumptions might not be valid for a new company wishing to enter the cellular market by building an entirely new cellular infrastructure, and therefore in need of spectrum in coverage-limited as well as capacity-limited regions, these assumptions are appropriate when the entities seeking spectrum are nationwide MNOs that are trying to expand their existing capacity.

We also consider only macrocells in our cost and capacity analysis. The load on macrocells can be reduced by offloading traffic to inexpensive short-range devices, such as Wi-Fi hotspots, residential femtocells, and even roadside units that provide Internet access to users in moving cars via DSRC (Ligo et al, 2015; Ligo et al, 2016; Ligo et al, 2017). However, because these devices are short-range, they can only cover a small fraction of the capacity-constrained regions. Even as the volume of traffic offloaded to such devices is increasing, total volume is increasing even faster, so the volume of traffic that cannot be offloaded is also increasing. As a result, MNOs must continue to expand the capacity of their macrocells to meet user needs and expectations, and this capacity expansion will be a large part of the MNOs' expenditures. Thus, the assumptions underlying our analysis are valid even with offloading.

Finally, we assume that the macrocellular networks operated by all of the MNOs are comparable in the sense that they use a similar mix of technologies and frequencies, and therefore have a similar spectral efficiency, and a similar number of sectors per cell. Of course, MNOs upgrade their technology at different times, so at any given time this may not be the case. For example, in the U.K. today, MNO Three has no 2G technologies in use while its competitors do, so Three has an advantage with respect to spectral efficiency. However, advantages of this kind are temporary, and tend to benefit different MNOs in different years. They should not matter when one is considering the long-term sustainability of competition.

2.3 Capacity

We calculate capacity in a way that is traditional for a capacity-limited cellular network, where an MNO's capacity increases linearly with the number of cell towers that the MNO uses, and linearly with the amount of spectrum it holds. Technology is also a factor, including whether it uses second, third, or fourth generation cellular technology.

While it is possible to increase macrocellular capacity by increasing either the number of cell towers or the amount of spectrum, the two do not actually have identical impact as we assume here. The advantages of increasing spectrum are actually somewhat greater than our traditional model would imply. As a result, our analysis may underestimate the disadvantages of MNOs with especially low spectrum holdings. One reason is that resources can be allocated to individual mobile devices in a way that takes advantage of multiuser diversity (Capozzi et al, 2013). If all devices in a cell were at a single location, then as Shannon's theorem dictates, achievable downstream capacity would increase linearly with downstream spectrum bandwidth, and would depend on SINR at that specific location. However, devices are not all in the same location. Path loss and interference vary from location to location and frequency to frequency, and consequently so does SINR. Thus, device 1 may achieve greater throughput using frequency f_1 than using frequency f_2, while device 2 achieves greater throughput with frequency f_2 rather than f_1. Algorithms that take advantage of this will increase throughput. When such algorithms are used, as is the case with LTE technology, capacity increases with spectrum bandwidth at a rate that is more than linear. The same cannot be said when increasing the number of cell towers.

Another reason that our model may undervalue increasing spectrum holdings relative to increasing tower density is that signal path loss characteristics are somewhat different over short distances than over long distances. As a result, when cell size becomes sufficiently small, capacity with tower density at a rate that is less than linear (Sousa, Velez & Peha, 2017).

2.4 Cost

There are two principal costs associated with cellular infrastructure: spectrum and cell towers. MNOs can expand capacity by increasing expenditures on either resource. To make them comparable, we consider the net present value of all such costs over time. Thus, we do not distinguish one-time costs from annual costs, as both are accounted for.

We assume that all MNOs can obtain spectrum when they wish, and at the same price per MHz, as would be appropriate in the absence of market failure. This may not always be the case in today's spectrum market, especially if some players pursue a foreclosure strategy. Thus, this assumption somewhat understates the benefits of holding spectrum in reserve. Moreover, we assume that each

MNO must make deployment decisions using spectrum prices that are stable and exogenously given, such that no individual MNO can change the price per MHz of spectrum.

We consider three categories of tower costs. Some tower costs are constant, regardless of spectrum holdings or data rates. This category includes the cost of building a new tower, rental charges for placing a multiband antenna on an existing tower, and the capital expenses for a backup generator. Many tower costs fall in this category. A second category is costs that depend on spectrum holdings, but not data rates. For example, adding a new spectrum band requires additional equipment, with the associated capital and operating expenses. Also, increasing total spectrum bandwidth means increasing the number of power amplifiers and associated operating costs. We assume that costs in the second category increasing linearly with spectrum bandwidth. The third category of costs depend on data rates. The obvious example is backhaul. While a large fraction of backhaul costs fall in the first category of fixed costs, there are also backhaul costs that increase with data rate. We assume that the latter increase linearly, i.e. backhaul costs for a tower are *a*+ *b**(data rate), for some constants *a* and *b*. (Note that if the cost increase that depends on data rate is sub-linear due to even steeper volume discounts, this means that there is even more benefit to having more spectrum and therefore greater data rate per tower as compared to more towers.)

In cases where towers are shared among MNOs, it also matters how the costs are shared. In this paper, we employ two different models. In the first model, which we explore in Sections 3 and 4, we assume that one MNO's tower costs depend only on the choices of that MNO, and not the choices of its competitors. This is obviously the case if each MNO owns and operates its own towers. It can also be the case if MNOs primarily rent space on towers that are owned and operated by third-party providers. In the second model, which we explore in Section 5, we assume that each MNO establishes a cooperative relationship with one of its competitors for the sharing of towers. In this model, an MNO wanting more towers can work with its partner to add shared towers, or can work alone to deploy its own towers, depending on which strategy best serves its own interests.

3 Strategies to Meet MNO Objectives: Economies of Scale

In this section, we present the strategies that best serve MNOs. We assume that MNOs seek to maximize their profit, which means that they will accumulate the cell tower and spectrum assets that minimize cost for whatever capacity that infrastructure provides.

More specifically, each MNO i wishes to provide capacity C_i Mb/s per square km using S_i MHz of spectrum, where capacity and spectrum holdings may differ from MNO to MNO. MNO i deploys N_i cell towers per square km in the capacity-limited regions. (Towers in the coverage-limited regions are excluded from our analysis.) We assume all MNOs have a spectral efficiency averaged throughout their cells of *e* bps/Hz, as would be appropriate if they all employ a similar mix of technologies (e.g. 2G, 3G, and 4G cellular) and a similar mix of frequencies. (The value of spectrum can also depend on its frequency (Peha 2013, Sept.; Tan & Peha 2015), but we ignore frequency-dependent differences in this analysis.) We assume that all MNOs have r sectors per cell, and a frequency reuse of f.

$$C_i = N_i \, r \, e \, S_i / f$$

As discussed in Section 2.4, most infrastructure costs are simply proportional to the number of towers, but we consider three categories of tower costs. Let T_0 be the net present value (NPV) of the fixed costs of deploying and operating a tower, so it includes both CAPEX and OPEX. Let T_{bw}be the NPV of costs that are proportional to bandwidth, and T_{bps} be the NPV of costs that are proportional to data rate. Let *M* be the cost per MHz divided by the area of the capacity-limited regions. We can then derive MNO i's cost per square km $K_i$as a function of its spectrum holdings S_i.

$$K_i = M \, S_i + N_i \left(T_0 + \; T_{bw} \, S_i + \; T_{bps} \, \frac{r \, e \, S_i}{f} \right)$$

If MNO *i* is rational, it will choose the amount of spectrum and number of towers to minimize cost K_i for a given capacity C_i. By combining these equations, we get the following.

$$K_i = M \, S_i + \frac{C_i \, f \, T_0}{r \, e} \frac{1}{S_i} + \frac{C_i \, f \, T_{bw}}{r \, e} + \; C_i \, T_{bps}$$

$$0 = \frac{dK_i}{dS_i} = M - \frac{C_i \, f \, T}{r \, e} \frac{1}{{S_i}^2}$$

$${S_i}^2 = \frac{f \, T_0}{M \, r \, e} \, C_i$$

This shows that capacity does not increase linearly with spectrum holdings as many people believe when an MNO minimizes its costs. In this case, capacity increases with the square of spectrum holdings. This is because a rational MNO that wishes to expand capacity over time will increase spectrum holdings and the number of towers in the capacity-constrained region simultaneously and at a similar rate, assuming that the ratio of cost per MHz to cost per tower remains stable. Indeed, when S_i and N_i are selected to minimize MNO i's costs, the number of towers in the capacity-limited regions is proportional to spectrum holdings, as shown by the equation below. (Note that this contrasts with the coverage-limited regions, where the number of towers depends highly on whether an MNO holds spectrum at low frequencies, but depends little on how much spectrum each MNO has.)

$$N_i = \frac{M}{T_0} \, S_i$$

The fact that capacity increases with the amount of resources squared shows that there is a large economy of scale. The MNO with the most towers will benefit the most from another 10 MHz of spectrum, and the MNO with the most spectrum will benefit the most from another tower. As a result, an MNO with more spectrum has a smaller cost per capacity, as shown by the following equation. Thus, if a large portion of overall cost comes from providing adequate capacity, as might be expected when utilization per user is increasing rapidly, it will be hard for the MNOs with less spectrum to compete when there is a large disparity in spectrum holdings. For example, the market price might make the cost of increasing capacity less than the revenue derived from that additional capacity for the large MNOs, making expansion profitable. That same market price might make the cost of expansion less than the revenue derived from that capacity for small MNOs, making expansion too expensive to contemplate. In this case, the small MNOs will stop growing. With data rate per user doubling every 16 months, an MNO that does not expand will rapidly lose market share. Eventually, it may be acquired by a rival, go bankrupt, or simply become a small niche player that is no longer a competitive factor.

$$\frac{K_i}{C_i} = \frac{M\,S_i}{C_i} + \frac{f\,T_0}{r\,e}\,\frac{1}{S_i} + \frac{f\,T_{bw}}{r\,e} + \ T_{bps} = \frac{f\,T_0}{r\,e}\,\frac{1}{S_i} + \frac{f\,T_0}{r\,e}\,\frac{1}{S_i} + \frac{f\,T_{bw}}{r\,e} + \ T_{bps}$$
$$= \frac{2\,f\,T_0}{r\,e}\,\frac{1}{S_i} + \frac{f\,T_{bw}}{r\,e} + \ T_{bps}$$

Moreover, once a large disparity exists, this disparity is likely to grow, because the MNO with more spectrum is willing to pay more for additional spectrum. If revenues are roughly proportional to capacity, then the value to MNO *i* of additional spectrum is $\frac{dC_i}{dS_i}$, which increases linearly with MNO *i*'s spectrum S_i as follows.

$$\frac{dC_i}{dS_i} = \frac{2\,M\,r\,e}{f\,T_0}\,S_i$$

4 Strategies to Meet Policy Objectives: Competition vs. Capacity

In this section, we present the best strategies for spectrum policymakers who are tasked with serving the public interest. This is reflected in two different objectives. First, for a given set of resources dedicated to MNOs, the public interest is served when these resources are used to make a great deal of capacity available to cellular users. Second, the public interest is served when competition among MNOs is high, as this gives profit-seeking MNOs incentive to offer better services at lower prices.

More specifically, we assume that the total amount of spectrum S_{tot} and the total number of towers N_{tot} across all MNOs is fixed. S_{tot} describes the resources allocated directly by the spectrum policymaker. N_{tot} describes the investment from MNOs. Whereas in Section 3 we assumed that MNOs seek to minimize the cost for achieving a certain capacity, in this section we assume that they seek to maximize the capacity for a given amount of spectrum and number of towers, which is roughly the same as maximizing capacity for a given cost. As discussed in Section 2.4, most tower costs are fixed, and therefore are directly proportional to N_{tot}. Even more importantly, MNOs would be happy to reduce the costs that are fixed (as reflected in T_0), but would not want to reduce the costs that are proportional to the capacity that they are struggling to increase (as reflected in T_{bw} and T_{bps}).[2] Thus, the number of towers is an appropriate measure of non-spectrum costs.

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[2] This is why a carrier would consider T_0 when deciding how many towers to employ, but not T_{bw} or T_{bps}, as shown in Section 3.

We seek the allocation of these spectrum and tower resources that optimizes the spectrum policymaker's objectives: total capacity C_{tot}, which is the sum of capacities of all MNOs, and cellular competition as measured using the Herfindahl–Hirschman Index (HHI), which is the standard measure of market concentration. For the HHI calculation, we assume that market share is proportional to capacity in the capacity-constrained region, for reasons discussed in Section 2.1. The allocation of spectrum is defined by the spectrum shares s_i and tower shares n_i of each MNO, where $s_i = S_i \ / S_{tot}$ and $n_i = N_i \ / N_{tot}$.

$$C_i = (n_i \ N_{tot})\, r\, e\, (s_i\, S_{tot})\, / f = \alpha\, n_i\, s_i \qquad \text{where } \alpha = r\, e\, N_{tot} \ S_{tot} \ / \ f$$

$$C_{tot} = \sum_{i=1}^{\infty} C_i = \alpha \sum_{i=1}^{\infty} n_i\, s_i$$

$$HHI = \frac{\sum_{i=1}^{\infty} C_i^2}{C_{tot}^2} = \frac{\alpha^2}{C_{tot}^2} \sum_{i=1}^{\infty} n_i^2 \ s_i^2$$

Ideally, C_{tot} should be as large as possible, and HHI should be as small as possible. Of course, these objectives are inherently in conflict, because of the economies of scale that were demonstrated in Section 3. Capacity C_{tot} is optimized by giving one MNO all of the spectrum and all of the towers, which yields the worst possible HHI of 1, i.e. a monopoly.

The most cost-effective allocation of spectrum and tower resources is for each MNO to choose the same spectrum share and tower share, i.e. to let $n_i = s_i$. Indeed, Section 3 showed that a profit-seeking MNO would choose to let $n_i = s_i$ by showing that $N_i \ / \ S_i$ is the same constant for all *i*. This means that $n_i \ / \ s_i$ is also the same for all *i*, which is only possible when $\sum_{i=1}^{\infty} n_i = \sum_{i=1}^{\infty} s_i = 1$ if $n_i = s_i$. We can show that letting $n_i = s_i$ is also the most cost-effective approach within the policymaker framework of this section as follows. Consider a resource allocation in which spectrum share and tower share are not the same for all MNOs. In this case, there must be at least one MNO *j* for which $n_j > s_j$ and at least one MNO *k* for which $n_k < s_k$. If we shift the same amount *x* of tower share from MNO *j* to MNO *k* and spectrum share from MNO *k* to MNO *j*, where *x* is less than both $n_j - s_j$ and $s_k - n_k$, then we reduce the differences between tower share and spectrum share for both MNOs. The equations below show that we also improve the capacity of both MNOs. Thus, everyone benefits from reducing the difference between spectrum share and tower share. Let C_j^* and C_k^* be capacities after the shift.

$$C_j^* = \alpha\,(n_j - x)(s_j + x) = \alpha\,(n_j\, s_j + n_j x - s_j x - x^2) = C_j + \alpha\, x\,(n_j - s_j - x) \qquad > C_j$$
$$C_k^* = \alpha\,(n_k + x)(s_k - x) = \alpha\,(n_k\, s_k - n_k x + s_k x - x^2) = C_k + \alpha\, x\,(s_k - n_k - x) \quad > C_k$$

With this result, we can quantify the capacity advantages of market concentration. Consider the case where there are *m* MNOs with equal amounts of spectrum, so $s = 1/m$, and with the efficient allocation of $n_i = s_i$ for all *i*.

$$C_{tot} = \sum_{i=1}^{\infty} C_i = \alpha \sum_{i=1}^{m} \frac{1}{m} * \frac{1}{m} = \frac{\alpha}{m}$$

Thus, if spectrum and towers are evenly divided, then total capacity is inversely proportional to the number of MNOs. For example, five MNOs would have 20% less total capacity than four MNOs for the same number of towers and the same amount of spectrum, but this increase in capacity would come at the expense of reduced competition.

But is it important for spectrum to be evenly divided? Spectrum policymakers should prefer the division of spectrum resources that is best for the competing objectives of maximizing capacity and maximizing competition when $n_i = s_i$. While reasonable minds may disagree on how to balance the two competing objectives, we should all agree that a good division of spectrum would be Pareto optimal, i.e. if spectrum holdings s_i for all *i* are good, then it should not be possible to change spectrum holdings in a way that makes results better with respect to one objective without making results worse with respect to the other objective. We will now show that any Pareto optimal solution requires spectrum to be divided fairly evenly among the MNOs. More specifically, if there are *d* MNOs with spectrum (i.e. for which $s_i > 0$), then *d*-1 of these MNOs should have the same amount of spectrum. The last MNO gets whatever is left in S_{tot}, which should be the same as or less than what the other *d*-1 MNOs have.

Another way to describe this Pareto optimal result, which will be proven below, is to let the number of competitors *b* be any real number ≥ 1. If *b* is not an integer, then the fraction represents the extent to which the last MNO gets a smaller share of spectrum than the others. For example, if *b*=4.7, then the Pareto optimal market would have 4 MNOs with equal amounts of spectrum, equal number of towers in the capacity-limited regions and equal market share, while the fifth MNO had .7 times as much spectrum as the first four. Figure 1 shows how spectrum would be divided among MNOs as a function of *b.*

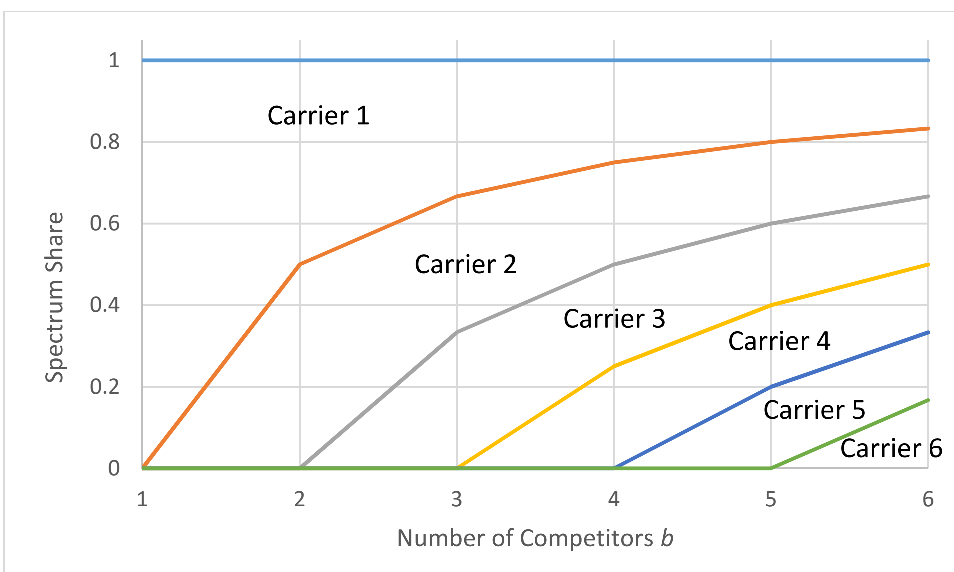

Figure 1: Division of spectrum versus number of competitors *b*

To prove this assertion, we consider a case where the condition above does not hold, and show that this division of spectrum cannot be Pareto optimal. If this condition does not hold, then there must be a MNO that has more spectrum than two other MNOs. Without loss of generality, we number the MNOs such that $s_1 > s_2 \geq s_3 > 0$. To prove that this is not Pareto optimal, we show that it is possible to shift spectrum from MNOs 1 and 3 to MNO 2 in a way that improves (decreases) *HHI* without reducing total capacity C_{tot}. In particular, we increase s_2 by a small amount ds_2 while decreasing s_1 and s_3 as follows. Because $\frac{ds_1}{ds_2} + \frac{ds_2}{ds_2} + \frac{ds_3}{ds_2} = 0$, $\sum s_i$ remains constant at 1 even as spectrum holdings change.

$$\frac{ds_1}{ds_2} = -\frac{(s_2 - s_3)}{(s_1 - s_3)}$$

$$\frac{ds_3}{ds_2} = -\frac{ds_2}{ds_2} - \frac{ds_1}{ds_2} = -\frac{(s_1 - s_2)}{(s_1 - s_3)}$$

Let $C_{tot}(s_2)$ and $HHI(s_2)$ be total capacity and *HHI* as a function of s_2, respectively. If spectrum is initially divided such that increasing s_2 from its initial value will decrease $HHI(s_2)$ while $C_{tot}(s_2)$ remains constant, i.e. if $\frac{dHHI((s_2)}{ds_2} < 0$ and $\frac{dC_{tot}(s_2)}{ds_2} = 0$, then that initial division of spectrum cannot be Pareto optimal. We assume here that $s_2 > s_3$, as $s_2 = s_3$ is a minor special case.[3]

$$\begin{aligned}
C_{tot}(s_2 + ds_2) - C_{tot}(s_2) &= \alpha \left(s_1 + \frac{ds_1}{ds_2}\, ds_2\right)^2 + \alpha\,(s_2 + ds_2)^2 + \alpha \left(s_3 + \frac{ds_3}{ds_2}\, ds_2\right)^2 - \alpha \sum_{i=1}^{3} s_i^2 \\
&= \alpha \left(s_1^2 - 2\, s_1 \frac{(s_2 - s_3)}{(s_1 - s_3)}\, ds_2 + \frac{(s_2 - s_3)^2}{(s_1 - s_3)^2}\, ds_2^2\right) + \alpha\,(s_2^2 + 2\, s_2\, ds_2 + ds_2^2) \\
&\quad + \alpha \left(s_3^2 - 2\, s_3 \frac{(s_1 - s_2)}{(s_1 - s_3)}\, ds_2 + \frac{(s_1 - s_2)^2}{(s_1 - s_3)^2}\, ds_2^2\right) - \alpha \sum_{i=1}^{3} s_i^2 \\
&= 2\alpha \left(-s_1 \frac{(s_2 - s_3)}{(s_1 - s_3)} + s_2\, ds_2 - s_3 \frac{(s_1 - s_2)}{(s_1 - s_3)}\right) ds_2 + \alpha\, \frac{(s_2 - s_3)^2 + (s_1 - s_3)^2 + (s_1 - s_2)^2}{(s_1 - s_3)^2}\, ds_2^2 \\
&= 0\, ds_2 + \alpha\, \frac{(s_2 - s_3)^2 + (s_1 - s_3)^2 + (s_1 - s_2)^2}{(s_1 - s_3)^2}\, ds_2^2
\end{aligned}$$

$$\frac{dC_{tot}(s_2)}{ds_2} = 0 + \alpha\, \frac{(s_2 - s_3)^2 + (s_1 - s_3)^2 + (s_1 - s_2)^2}{(s_1 - s_3)^2}\, ds_2 = 0$$

[3] Where $s_2 = s_3$, the slope of both $HHI(s_2)$ and $C_{tot}(s_2)$ with respect to s_2 are both 0. A slight transfer of spectrum from carrier 3 to carrier 2 therefore has a negligible impact on both objectives, and then the assumption that $s_1 - s_2 > s_3 > 0$ applies.

Thus, total capacity remains constant as spectrum holdings shift using this formula.
HHI changes as follows.

$$
\begin{aligned}
&HHI(s_2 + ds_2) - HHI(s_2) \\
&\quad = \frac{\alpha^2}{C_{tot}^2}\left(s_1 + \frac{ds_1}{ds_2} ds_2\right)^4 + \frac{\alpha^2}{C_{tot}^2}(s_2 + ds_2)^4 + \frac{\alpha^2}{C_{tot}^2}\left(s_3 + \frac{ds_3}{ds_2} ds_2\right)^4 - \frac{\alpha^2}{C_{tot}^2}\sum_{i=1}^{3} s_i^4 \\
&\quad = \frac{4\,\alpha^2}{C_{tot}^2}\left(-\frac{(s_2 - s_3)}{(s_1 - s_3)}\, s_1^3 \,+\, s_2^3 - \frac{(s_1 - s_2)}{(s_1 - s_3)}\, s_3^3\right) ds_2 \,+\, O(ds_2^2) +\, O(ds_2^3) \\
&\quad = \frac{4\,\alpha^2}{C_{tot}^2}\, \frac{-(s_2 - s_3)\, s_1^3 + (s_2 - s_3)\, s_2^3 + (s_1 - s_2)\, s_2^3 - (s_1 - s_2)\, s_3^3}{(s_1 - s_3)} ds_2 \,+\, O(ds_2^2) +\, O(ds_2^3) \\
&\quad = \frac{4\,\alpha^2}{C_{tot}^2}\, \frac{(s_1 - s_2)\,(s_2 - s_3)\,[(s_3^2 - s_1^2) + s_2(s_3 - s_1)]}{(s_1 - s_3)} ds_2 \,+ O(ds_2^2) +\, O(ds_2^3)
\end{aligned}
$$

$$
\frac{dHHI(s_2)}{ds_2} = \frac{4\,\alpha^2}{C_{tot}^2}\, \frac{(s_1 - s_2)\,(s_2 - s_3)\,[(s_3^2 - s_1^2) + s_2(s_3 - s_1)]}{(s_1 - s_3)} \;<\; 0
$$

Thus, the derivative of *HHI* is always negative at a division of spectrum where $s_1 > s_2 > s_3 > 0$, so moving spectrum from MNOs 1 and 3 to MNO 2 according to the formula above improves *HHI* without changing total capacity. Indeed, this can continue until either s_3 falls to 0 or s_2 rises to equal s_1. This means that the initial division of spectrum cannot be Pareto optimal.

5 Impact of Tower Sharing on Spectrum Strategies

One of the ways for MNOs to reduce the costs of cell towers is to enter into agreements with other MNOs to share towers, or even share towers with government systems intended for public safety agencies (Hallahan & Peha, 2011; Peha 2013, Summer], and thereby share some of the fixed costs. In Sections 5.1 and 5.2, we consider tower sharing and how it might affect the results of Sections 3 and 4, respectively.

We assume in this section that each MNO has an arrangement to share towers with exactly one other MNO, so MNOs share in pairs. When two MNOs share a tower, each pays a fixed cost per tower T_{0sh} that is significantly lower than the fixed cost T_0 without sharing. The cost per bandwidth and cost per data rate are the same. If both MNOs in a pair consider it cost-effective to add a tower when paying their portion of the cost, they do so. If one MNO does not consider it worth paying its portion of the cost, but the other considers it cost-effective to add a tower at full cost, the latter adds a tower that is not shared. This model is somewhat similar to the current market in the UK, where there are four large MNOs; British Telecom-EE and Three have a tower sharing agreement, and Telefonica O2 and Vodafone have a tower sharing agreement.

5.1 The MNO Perspective

If two cooperating MNOs have similar capacity requirements, then they can both benefit from relying on shared towers, and acquiring the same amount of spectrum. In this case, the analysis is similar to what was presented in Section 2, except with lower fixed cost per tower of T_{0sh} instead of T_0. Thus, the MNOs would choose to have more towers and less spectrum when towers are shared, i.e. $N_{sh}/S_i = M/T_{0sh}$ instead of the M/T_0 derived in Section 2, where N_{sh} is the number of shared towers per square km in the capacity-limited region.

If MNO *i* needs more capacity than MNO *j*, then MNO *i* may be not be able to rely entirely on towers shared with MNO *j*. The most cost-effective approach for MNO *i* is to have more spectrum and more towers than MNO *j* is willing to share. Consider the case where MNO *j* is willing to pay for up to N_{sh} shared towers. Because MNO *i* will choose to have N_{sh} towers or more, i.e. $N_i \geq N_{sh}$, its total cost is simply the cost derived in Section 3 minus the savings from sharing which is a constant.

$$K_i = M\, S_i + N_{sh}\left(T_{0sh} + T_{bw}\, S_i + T_{bps}\,\frac{r\, e\, S_i}{f}\right) + (N_i - N_{sh})\left(T_0 + T_{bw}\, S_i + T_{bps}\,\frac{r\, e\, S_i}{f}\right)$$
$$= M\, S_i + N_i\left(T_0 + T_{bw}\, S_i + T_{bps}\,\frac{r\, e\, S_i}{f}\right) - N_{sh}\,(T_0 - T_{0sh})$$

This only differs from the K_i derived in Section 3 by a constant $N_{sh}(T_0 - T_{0sh})$, so $\frac{dK_i}{dS_i}$ is the same as in Section 3, which means that MNO *i* will choose to acquire the same amount of spectrum as derived in Section 3, and the same total number of towers as well.

$$S_i^{\,2} = \frac{f\, T_0}{M\, r\, e}\, C_i$$

$$N_i = \frac{M}{T_0}\, S_i$$

Thus, tower sharing does not change the fundamental observations from Section 3. With or without sharing, (i) there are economies of scale in the provision of adequate capacity in capacity-limited regions, (ii) MNOs who have more capacity can add capacity at lower cost, and (iii) MNOs with more spectrum should be willing to pay more per MHz for additional spectrum than MNOs with less spectrum.

5.2 The Policymaker Perspective

This section explores the Pareto optimal allocation of resources when MNOs share all their towers, and the objectives are maximizing both capacity and competition (as quantified through the *HHI*), as in Section 4.

In any Pareto optimal solution, two MNOs sharing all of their towers should have the same amount of spectrum. Since they have the same number of towers N_{sh}, their combined capacity does not depend on how spectrum is divided between them, and *HHI* is best if they have equal spectrum and therefore equal capacity. The question is then how to divide spectrum among each pair of MNOs that share towers. Let s_i is the spectrum share for the *i*th pair, with each of the MNOs in that pair getting half that spectrum. Let n_i be the number of towers that these two MNOs are sharing.

$$C_{tot} = \sum_{i=1}^{\infty} \left(\alpha \, n_i \, \frac{s_i}{2} + \alpha \, n_i \, \frac{s_i}{2} \right) = \alpha \sum_{i=1}^{\infty} n_i \, s_i$$

$$HHI = \frac{1}{C_{tot}^2} \sum_{i=1}^{\infty} ((\alpha \, n_i \, s_i \,/2)^2 + (\alpha \, n_i \, s_i \,/2)^2) = \frac{\alpha^2}{2 \, C_{tot}^2} \sum_{i=1}^{\infty} n_i^2 \, s_i^2$$

The equation for total capacity is identical to that of Section 4, and the equation for *HHI* is simply that of Section 4 divided by 2. Thus, spectrum divisions among MNOs that were Pareto optimal in Section 4 are identical to the spectrum divisions among pairs of MNOs that are optimal in the scenario considered here. In any Pareto optimal resource allocation with *d* MNOs, each of which share their towers with one other MNO, *d*-2 of these MNOs will have the same amount of spectrum, and the last pair of MNOs will split what is left of the spectrum, and their share will be the same as or less than the first *d*-2.

6 Conclusions and Policy Implications

Since smartphones like the iPhone have become widespread, MNOs have had to rapidly increase capacity. With current cellular technology, e.g. LTE, this means that macrocellular capacity must be increased, by upgrading technology, increasing tower density, or adding spectrum. This has created a high demand for spectrum. Note that technical innovation created this need to rapidly expand macrocellular capacity, and technical innovation may someday reduce it. For example, if MNOs gain access to far more spectrum at very high frequencies, and the technology to make use of that spectrum, then growing macrocellular capacity at lower frequencies may someday be less critical. However, that is not the situation faced by MNOs or regulators today, and is beyond the scope of this paper.

We find that during periods when increasing the capacity of macrocells to meet customer expectations is a significant cost for MNOs, there are strong economies of scale. An MNO with more cell towers benefits more from any spectrum it has, and an MNO with more spectrum benefits more from every tower it has. Thus, the most cost-effective strategy for an MNO is to increase both spectrum holdings and number of towers together over time. Consequently, an MNO with large spectrum holdings will generally also be an MNO with many towers, and an MNO with enough capacity to support a large customer base will generally have large holdings of both spectrum and towers. This is radically different from markets where MNOs are driven primarily to improve coverage rather than capacity. If it is coverage alone that attracts customers, then an MNO needs some spectrum, especially at lower frequencies, but particularly large spectrum holdings are not important. If capacity is what attracts customers, then the amount of spectrum matters a great deal.

Moreover, the big MNOs are well positioned to get bigger. Since revenue tends to be proportional to capacity in a capacity-limited region, and large MNOs can increase their capacity more than small MNOs with every MHz of spectrum they obtain, then in general, large MNOs are likely to bid more in spectrum auctions than their small competitors. The British regulator Ofcom has suggested that MNOs with little spectrum who are unable to outbid their rivals in spectrum auctions could compensate by building more towers (Ofcom 2012, Jan.; Ofcom, 2012, July]. This is certainly possible technically, but our analysis shows that this would be a highly unprofitable strategy. For an MNO with far more towers and far less spectrum than its competitors, obtaining more spectrum is typically the least expensive way to expand capacity. If such an MNO cannot afford additional spectrum, then the MNO should simply stop expanding capacity rather than adopt an even more costly approach. When data usage per user is increasing rapidly, to stop expanding it so surrender market share to the large MNOs. In the absence of countervailing forces, possibly from spectrum policymakers, this would make competition increasingly difficult to sustain over time. Eventually, an MNO with small market share but valuable spectrum and infrastructure assets might be acquired by a rival.

These economies of scale create a trade-off between two important objectives for policymakers: increasing competition and lowering the cost of capacity. A small number of large MNOs can exploit economies of scale to reduce costs, but with little competition, these MNOs have little incentive to pass those cost-savings on to consumers. Policymakers must determine the right balance, and this should be reflected in both spectrum policy and antitrust policy. Whatever the number of MNOs, we find that the public interest is best served when spectrum is split fairly evenly among them. In particular, without tower sharing, any division of spectrum among *d* MNOs that is Pareto optimal with respect to our two policy objectives would give the same amount of spectrum to *d*-1 MNOs, and spectrum disparities among them are contrary to the public interest. The last MNO would get whatever spectrum is left, and less than the first *d*-1. Our results with tower sharing are only slightly different. Moreover, these results were obtained using assumptions that actually understate the harm from large disparities in spectrum holdings. For example, we assumed that MNOs never pursue a strategy of foreclosure, and that capacity increases linearly with spectrum bandwidth.

One policy that will naturally produce a division of spectrum like this is a spectrum cap. As shown in Section 2, if MNO *i* has far more spectrum than MNO *j*, then MNO *i* has great advantages in increasing its share of spectrum relative to MNO *j*. This is true unless or until MNO *i*'s holdings are near a spectrum cap. If the economies of scale described in this paper persist for an extended period, then we are less and less likely to see two credible nationwide MNOs with very different spectrum holdings that are both well under the cap. This is equivalent to saying that all of the major MNOs except one would have spectrum holdings close to the cap, which is precisely the division of spectrum found to be Pareto optimal in Section 4. For example, if a spectrum caps prevents any MNO from acquiring more than 30% of the spectrum, then in addition to guaranteeing that there will always be at least four MNOs with spectrum, this policy makes it more likely that the division of spectrum will be similar to the Pareto optimal division described above. An advantage of reaching this result through a spectrum cap rather than through more specific regulations is that under a cap each MNO is free to determine which spectrum it wishes to hold based on its own unique needs. There are efficiencies to be gained by acquiring a mix of low-frequency spectrum which is generally more cost-effective for coverage-limited regions and high-frequency spectrum which is generally less expensive per MHz (Peha 2013, Sept.; Alotalbi et al, 2015), and by ensuring that the MNO can use the same reasonably small number of spectrum bands everywhere it operates (Peha, 1998; Peha, 2013, Sept.).

Reasonable minds may differ on the level of competition that a spectrum cap should protect. Whatever that number is, it should be consistent across policies, which is not always the case. For example, Ofcom currently proposes that the cap should be 42%, and that some bands that will be useful for macrocells should be exempt from the cap (Ofcom, 2016). With such a policy, the UK may eventually see just three MNOs with significant capacity and market share in the capacity-limited regions. Yet, Ofcom supported the European Commission's decision to block the proposed merger of O2 and Three to maintain the number of competitors at four. As Ofcom stated (Ofcom, 2016), "we believe that the existence of at least four credible MNOs is important for the UK mobile market. ... We agree with the EC's conclusions." The U.S. Government similarly showed its preference for having four or more MNOs when it blocked the merger of AT&T and T-Mobile. If four MNOs is indeed the long-term goal, then a 42% cap is high. Worse yet, if a nation has a spectrum policy allows three but not four MNOs to obtain the spectrum they need to expand capacity at costs that are consistent with competitive prices, and an antitrust policy that prevents that fourth MNO from merging with one of the other three, then we may see a poor result with respect to *both* policy objectives: effective competition and efficiency. We may get the effective competition one would expect with just three significant MNOs, and the lower efficiency of four. Achieving a poor result with respect to both objectives is precisely what it means to yield results that are not Pareto optimal, as discussed in Section 4.

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