Analyzing r-Robustness of Random K-out Graphs for the Design of Robust Networks

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***Abstract*— We consider a graph property known as r-robustness, a robustness metric that plays a key role in analyzing consensus dynamics. It was previously shown that in the presence of adversarial nodes, consensus can be reached in an r-robust network for sufficiently large r. Further, r-robustness is a stronger property than r-connectivity, hence it is also useful in many applications where robustness of networks to disruptions such as adversarial attacks or node failures is of practical interest. In this paper, we study r-robustness of random K-out graphs, which have been used in many applications including random (pairwise) key predistribution in wireless sensor networks, anonymous message routing in crypto-currency networks, and differentially-private federated averaging. Significantly improving an earlier result, we provide a set of conditions for K and n that ensure, *with high probability* (whp), the r-robustness of the random K-out graph. Simulation results are used to verify the results. To demonstrate the viability of our results in practical applications, we compare our results with the results from Erdős-Rényi and the Barabási-Albert random graph models.**

***Index Terms*— Robustness, random graphs, random K-out graphs, consensus dynamics, resilience**

I. Introduction

The advances in networking and low-cost, high performance devices has resulted in the emergence of various forms of complex distributed networks, such as communication networks [1], power grids [2], and economic networks [3]. Since such large-scale distributed systems have many potential vulnerable points for failures or attacks, there has been a considerable effort in studying the robustness of such networks. Robustness of a network can be simply defined as the ability to withstand failures and perturbations, and due to the various forms these failures and perturbations can occur, various metrics have been defined for robustness.

One such classical metric is *connectivity*. A graph is connected if there exists a path of edges between every pair of vertices. A more generalized form of the connectivity property is r-connectivity. A graph is said to be r-connected if it remains connected after the removal of any set of $r-1$ (or, fewer) nodes. r-connectivity is used in many applications, such as in *minimum r-connectivity maintenance methods*, where it is desired for a robot team to perform various behaviors at best while maintaining a global and redundant connected network [4]. Further, in an r-connected network, there are at least r disjoint paths between each pair of nodes. Since higher number of disjoint paths signifies higher fault tolerance, r-connectivity is also useful in applications where higher fault tolerance is desired, such as wireless sensor networks [5].

Another metric pertaining to the robustness of a graph is *r-robustness*, which was introduced in [6]. A graph is said to be *r-robust* if, for every disjoint subset pairs of the graph, at least one node in one of these subsets is adjacent to at least r nodes outside that set. The r-robustness property is stronger than r-connectivity since it was shown in [7] that if a graph is r-robust, it is at least r-connected. As will be explained below, the r-robustness property is especially useful in *consensus dynamics*.

Consensus dynamics is the process of alignment of some parameters of several agents through a sufficiently long period of local interactions. It is one of the most popular and simplest multi-agent dynamics and has been studied in many applications in communication systems. For example, in Internet of Things (IoT), it has been used in a variety of applications such as federated learning [8], block-chain [9], and smart grid [10]. For example, in [11] a leader-follower consensus algorithm was used to synchronize heterogeneous energy storage devices and bring plug-and-play capability smart grid systems. In social sciences, it has been studied in *wisdom of crowds*, where aggregate opinion is used to estimate unknown quantities [12]. It has also been used in control theory applications of multi-agent systems such as flocking, swarming, synchronization of coupled oscillators, load balancing in networks [13].

Since consensus dynamics relies on interactions with other nodes, one key consideration is the resilience against adversarial nodes. In [14], conditions on the connectivity of the network required to be resilient to misbehaving or faulty agents were derived for a linear consensus network. However, it was shown in [7] that for consensus networks in general, when adversarial nodes are present, network connectivity is not sufficient to characterize consensus. Instead, it was shown that when there are up to F adversarial nodes in the neighborhood of every correctly-behaving node, consensus can be reached if the network is $(2F+1)$-robust. Hence, r-robustness property is especially useful for consensus dynamics, and has been studied in numerous applications. For example, in [15], control laws were derived to group robots in formations of r-robust graphs to allow for consensus in the presence of malicious robots. Hence, it is of practical interest to analyze the r-robustness of different graph models in order to find graph models that satisfies this property as

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efficiently (i.e. with fewest number of edges) as possible.

One graph model that is known to be edge efficient in terms of connectivity and r-connectivity properties is the random K-out graph model [16]. Random K-out graph, denoted as $\mathbb{H}(n;K)$, is a random graph constructed over a set of n nodes as follows. Each node selects K distinct nodes uniformly at random, and then an undirected edge is formed between any pair of nodes if at least one selects the other. The orientation of the edges is ignored, resulting in an *undirected* graph. Random K-out graph is one of the earliest random graph models studied in literature [17], and has been used in applications such as the random *pairwise* key predistribution scheme in wireless sensor networks [18]–[22], and anonymity preserving crypto-currency networks [23]. It has also recently been used in federated learning applications of consensus dynamics, to construct a communication graph in a differentially-private federated averaging scheme [24].

One reason for the widespread use of the random K-out graph model is its ability to generate a connected topology even with a small number of edges per node. It was previously shown in [17], [25] that random K-out graphs are connected *whp* if $K \geq 2$ and not connected if $K = 1$; i.e.,

$$\lim_{n\to\infty} \mathbb{P}\left[\mathbb{H}(n;K) \text{ is connected}\right] = \begin{cases} 1 & \text{if} \quad K \geq 2, \\ 0 & \text{if} \quad K = 1. \end{cases} \tag{1}$$

Further, it was shown in [17] that random K-out graphs are r-connected whp when $K \geq r$. The thresholds for both these properties are much smaller than the respective thresholds for Erdős-Rényi graphs, one of the most commonly studied random graph models [26], demonstrating the edge efficiency of random K-out graph model in terms of connectivity and r-connectivity. Since r-connectivity and r-robustness are related as described above, one question is whether random K-out graphs also satisfy r-robustness efficiently.

With these motivations in mind, in this paper, we study the r-robustness property of random K-out graphs. The r-robustness property has been studied on several random graph models such as the Erdős-Rényi graph and the Barabási-Albert graph model [27], and by us on random K-out graphs [28]. In our previous work, we have shown that for large n, $K = O(r\log(r))$ is needed to ensure r-robustness *with high probability* (whp). In this paper, using a novel proof technique, we show that r-robustness can be achieved with a much smaller threshold, and find that $r^\star(K) \geq \lfloor K/2 \rfloor$, in other words, that $K \geq 2r$ is sufficient to ensure that the random K-out graph $\mathbb{H}(n;K)$ is r-robust *w*hp. We use computer simulations to verify this result, and compare this result with those obtained for an Erdős-Rényi graph and Barabási-Albert graph with same average node degree, and determine that random K-out graphs attain r-robustness at a significantly lower mean node degree value compared to Erdős-Rényi graphs, and within a factor of 2 compared to Barabási-Albert graphs.

The rest of the paper is organized as follows. In Section II, we introduce the notations, the random K-out graph model and the r-robustness property. In Section III, we present the main results and provide a discussion of its utility in practical applications. In Section IV, we provide the proof of our result. Conclusions are provided in Section V.

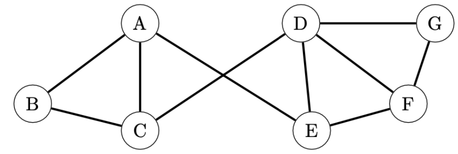

Fig. 1. *An example for a 2-connected, and 1-robust graph.*

II. NOTATIONS AND r-ROBUSTNESS OF A GRAPH

All random variables are defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and probabilistic statements are given with respect to the probability measure $\mathbb{P}$. The complement of an event A is denoted by $A^{\rm c}$. The cardinality of a discrete set A is denoted by $|A|$. The indicator function is denoted by $\mathbb{1}\{\}$. If the probability of an event tends to one as $n \to \infty$, we say that it occurs with high probability (whp). The asymptotic equivalence $a_n \sim b_n$ is used to denote $\lim_{n\to\infty} \frac{a_n}{b_n} = 1$. We let $\langle d \rangle$ denote the mean node degree of a graph.

Definition 2.1 (Random K-out Graph): [25], [29], [30] *The random K-out graph is defined on the vertex set* $V := \{v_1, \ldots, v_n\}$ *as follows. Let* $\mathcal{N} := \{1, 2, \ldots, n\}$ *denote the set vertex labels. For each* $i \in \mathcal{N}$*, let* $\Gamma_{n,i} \subseteq \mathcal{N} \setminus i$ *denote the set of* K *labels, selected uniformly at random, corresponding to the nodes selected by* v_i*. It is assumed that* $\Gamma_{n,1}, \ldots, \Gamma_{n,n}$ *are mutually independent. Distinct nodes* v_i *and* v_j *are adjacent, denoted by* $v_i \sim v_j$ *if at least one of them picks the other. Namely,* $v_i \sim v_j \quad \text{if} \quad [j \in \Gamma_{n,i}] \vee [i \in \Gamma_{n,j}]$*. The random graph defined on* V *through this adjacency condition is called a random K-out graph and is denoted by* $\mathbb{H}(n;K)$*. The set of neighbors of a node* i *is denoted by* $\mathcal{V}_i := \{j \in \mathcal{N} \setminus i : v_i \sim v_j\}$

Definition 2.2 (r-connectivity): A graph is r-connected if it remains connected after the removal of *any* set of $r-1$ (or, fewer) nodes or edges.

The random K-out graph $\mathbb{H}(n;K)$ is r-connected *whp* for large n when $K \geq r$ [17].

Definition 2.3 (r-reachable Set): [6, Definition 6] *For a graph* $\mathbb{G}$ *and a subset* S *of nodes* $S \subset \mathcal{N}$*, we say* S *is* r*-reachable if* $\exists i \in S : |\mathcal{V}_i \setminus S| \geq r$*, where* $r \in \mathbb{Z}^+$*. In other words,* S *is an* r*-reachable set if it contains a node that has at least* r *neighbors outside* S*.*

Definition 2.4 (r-robust Graph): [6, Definition 7] *A graph* $\mathbb{G}$ *is* r*-robust if for every pair of nonempty, disjoint subsets of* $\mathcal{N}$*, at least one of these subset pairs is* r*-reachable, where* $r \in \mathbb{Z}^+$*.*

For example, for the graph given in Fig. 1, if we construct the subset pair $S_1 = \{v_A, v_B, v_C\}$ and $S_2 = \{v_D, v_E, v_F, v_G\}$, both v_A and v_C in S_1 have only one neighbor in S_2, while v_D and v_E in S_2 have only one neighbor in S_1, meaning both S_1 and S_2 are 1-reachable. It can be seen that all other subset pairs are also at least 1-reachable, hence the graph is 1-robust. Further, the removal of any one node does not disconnect the graph, but removal

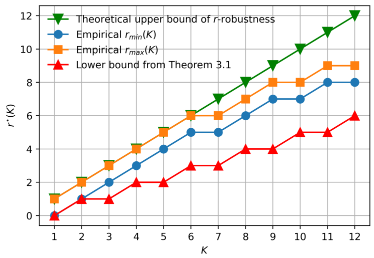

Fig. 2. *Empirically observed minimum $r^*(K)$ value in 500 experiments for $n = 20$, along with the theoretical minimum r value asserted by Theorem 3.1 and the theoretical threshold for r-connectivity.*

of nodes D and E disconnects the graph, hence this graph is 2-connected, and 1-robust. We also note that r-robustness is a stronger property than r-connectivity since it was previously shown in [7] that if a graph is r-robust, it is also r-connected.

III. Main Results and Discussion

Our main result is presented as Theorem 3.1 below.

Theorem 3.1: *Define*

$$r^\star(K) = \max_{r=1,2,3\ldots} \{\lim_{n\to\infty} P(\mathbb{H}(n;K) \text{ is } r\text{-robust}) = 1\}$$

Then, we have

$$r^\star(K) \geq \lfloor K/2 \rfloor$$

In Theorem 3.1, we establish a threshold for one-law of r-robustness in random K-out graphs, and find that $r^\star(K) \geq \lfloor K/2 \rfloor$. In other words, we find that *with high probability*, a random K-out graph is r-robust when $K \geq 2r, r \geq 2, r \in \mathbb{Z}^+$, and $n \to \infty$. This threshold is much smaller than the previously established threshold [28] of $K > \frac{2r(\log(r)+\log(\log(r)+1)}{\log(2)+1/2-\log\left(1+\frac{\log(2)+1/2}{2\log(r)+5/2+\log(2)}\right)}$ which scales with $r\log r$. Hence, Theorem 3.1 constitutes a sharper one-law for r-robustness. The proof of this result is given in Section IV.

A. Simulation Results

Determining r-robustness of a graph involves checking all subsets of a graph, and it was shown in [31] that it is a co-NP-complete problem, hence we were only able to run simulations for small n values. In the simulations, we generate instantiations of the random graph $\mathbb{H}(n;K)$, with $n = 20$, and K in the range $[1, 12]$. For each K value, we generate 500 experiments and in each experiment we record the empirical $r^*(K)$, which is $r^\star_{emp}(K) = \max\{r = 1, 2, :$ generated graph $\mathbb{H}(n;K)$ is r-robust$)\}$. The minimum, and maximum empirical $r^*(K)$ observed in these 500 experiments for each K value, $r_{min}(K)$ and $r_{max}(K)$, are plotted in Fig. 2 along with the corresponding theoretical plot obtained from Theorem 3.1. Also the theoretical upper bound for r-robustness is plotted. This plot is obtained from the upper bound of $r \leq K$ for r-connectivity [17], since r-robustness is a stronger property than r-connectivity, it can also be used as an upper bound on r-robustness. Hence, combining this with the lower bound of Theorem 3.1, we can write $\lfloor K/2 \rfloor \leq r^\star(K) \leq K$. As can be seen from Fig. 2, both empirical plots, $r_{min}(K)$ and $r_{max}(K)$, are between the plots of theoretical lower and upper bounds for all tested K values, validating both the lower bound asserted by Theorem 3.1 and the upper bound obtained from [17] when the number of nodes is small.

B. Discussion

To demonstrate the utility of random K-out graphs in practical applications compared to other random graph models, we compare our results with the results from other graph models. First, we compare with an Erdős-Rényi graph $G(n,p)$, one of the most commonly studied random graph models. In [31], it was shown that an Erdős-Rényi graph $G(n,p)$ is r-robust *whp* if $p_n = \frac{\log(n)+(r-1)\log(\log(n))+\omega(1)}{n}$, which translates to an average node degree of $< k >\sim \log(n) + (r-1)\log(\log(n))$. Since the random K-out graph is r-robust *whp* when $K \geq 2r$, which means an average node degree of $< k >= 4r$, we can conclude that the average node degree required for a random K-out graph to be r-robust is significantly smaller than the average node degree required for an Erdős-Rényi graph, demonstrating that the random K-out graph is more edge efficient for r-robustness.

Secondly, we compare our result with the Barabási-Albert preferential attachment graph model, which is a graph model where starting with a seed (initial graph), at each time a new node with a given number of edges is added to the graph. This new node forms an edge with another existing node in the graph with probability that is proportional to the number of links that the existing node already has. It was found in [31] that a Barabási-Albert graph is r-robust if the seed (initial graph) G is an r-robust graph, and if the degree of each added node is at least r. Hence, an r-robust Barabási-Albert graph can be constructed with an average node degree of $< k >= 2r$. This result is less than our result of $< k >= 4r$ for the random K-out graph model, hence the Barabási-Albert graph is more efficient on the number of edges to attain r-robustness compared to random K-out graphs. This can be attributed to the fact that the edge selection for a node in a random K-out graph is uniformly random, but nodes are more structured in the Barabási-Albert graph model, nodes that have a higher degree have a higher probability of being selected. The drawback of this model is that the degree of each node needs to be known at each step of the generating process, which might be a concern in applications that require privacy or anonymity of the nodes.

Since random K-out graphs satisfy the r-robustness property with much fewer edges compared to Erdős-Rényi graphs, and only within a factor of 2 with the Barabási-Albert graphs, and as discussed in the introduction, they have already been used in many applications for their property of being connected efficiently (with as few edges as possible, compared to Erdős-Rényi graphs) [32], this demonstrates that the random K-out graph model is a viable option for applications that require r-robustness with as few edges as possible such as robust control of multi-agent systems, and robust and differentially-private federated learning.

Another point worth mentioning is the comparison of the thresholds for r-robustness and r-connectivity. It was shown in [31] that the thresholds for r-robustness and r-connectivity are the same in Erdős-Rényi graphs. However, the random K-out graph $\mathbb{H}(n;K)$ is not r-connected *whp* for large n when $K < r$, and is r-connected *whp* for large n when $K \geq r$ [17]; and this is different from our threshold of $K \geq 2r$ for r-robustness. This raises the question as to whether our threshold of $2r$ is the tightest possible for r-robustness. Since in Figure 2, the empirical plots, $r_{min}(K)$ always lies below the theoretical upper bound for r-robustness, this shows that having a threshold of $r^*(K) \geq K$ is not possible based on this simulation. Still, determining whether a threshold tighter than $\lfloor K/2 \rfloor \leq r^*(K) \leq K$ exists for r-robustness is currently an open problem and a direction for future work.

IV. A Proof of Theorem 3.1

Due to space limitations, we only provide a shortened version of the proof of Theorem 3.1 here. We refer the readers to [33] for the detailed version of the proof.

To prove Theorem 3.1, we need to show that $r^\star(K) \geq \lfloor K/2 \rfloor$, i.e. that for large n, the random K-out graph $\mathbb{H}(n;K)$ is r-robust *whp* when $K \geq 2r$, for $r \geq 2 \in \mathbb{Z}^+$. To do this, similar to the proof given in [31] for Erdős-Rényi graphs, we first find an upper bound on the probability of a subset of given size being not r-reachable, and then use this to show that the probability of not being r-robust goes to zero when $n \to \infty$ and $K \geq 2r$. Different from prior work which relied on the commonly used upper bounds for the binomial coefficients $\binom{n}{k} \leq \left(\frac{en}{k}\right)^k$ and the union bound [28], [31] to bound the probability of a subset of given size being not r-reachable, our proof uses tighter upper bounds for the binomial coefficient, and we also use the Beta function to derive tight upper bounds on the CDF of the Binomial distribution, enabling us to establish a tighter threshold than what was previously possible; e.g., see [28]. First, we provide a few definitions and properties that will be useful throughout the proof, starting with the following standard bounds.

$$\binom{n}{m} \leq \left(\frac{en}{m}\right)^m, \ \forall m = 1, \dots, n \tag{2}$$

Upper bound for binomial coefficients when $n \to \infty$ and $1 \leq m \leq \lfloor n/2 \rfloor$ [34]:

$$\binom{n}{m} \leq (1 + o(1))\sqrt{\frac{n}{2\pi m(n-m)}} \cdot \frac{n^n}{m^m (n-m)^{(n-m)}} \tag{3}$$

Let $B(a,b)$ denote the beta function, $B_x(a,b)$ denote the incomplete beta function, and $I_x(a,b)$ denote the regularized incomplete beta function, where a and b are non-negative integers. These functions are defined as follows [35]:

$$\begin{aligned} B(a,b) &= \int_0^1 t^{a-1}(1-t)^{b-1}dt = \frac{(a-1)!(b-1)!}{(a+b-1)!} \\ B_x(a,b) &= \int_0^x t^{a-1}(1-t)^{b-1}dt, \quad 0 \leq x \leq 1 \\ I_x(a,b) &= \frac{B_x(a,b)}{B(a,b)}, \quad 0 \leq x \leq 1 \end{aligned} \tag{4}$$

Using these definitions, it can be shown that when $r > 0$, $I_{1/2}(r,r) = 1/2$ since $B(r,r) = 2B_{1/2}(r,r)$.

The cumulative distribution function $F(a;n,p)$ of a Binomial random variable $X \sim B(n,p)$ can be expressed as:

$$\begin{aligned} F(a;n,p) &= \mathbb{P}[X \leq a] = I_{1-p}(n-a, a+1) \\ &= (n-a)\binom{n}{a}\int_0^{1-p} t^{n-a-1}(1-t)^a dt \end{aligned} \tag{5}$$

Lemma 4.1: [35, Eq. 8.17.20]: For $a, b > 0, \ 0 \leq x \leq 1$,

$$I_x(a+1,b) = I_x(a,b) - \frac{x^a(1-x)^b}{aB(a,b)} \tag{6}$$

Lemma 4.2: [35, Eq. 8.17.21]: For $a, b > 0, \ 0 \leq x \leq 1$,

$$I_x(a,b) = I_x(a,b+1) - \frac{x^a(1-x)^b}{bB(a,b)} \tag{7}$$

Lemma 4.3: : For $r > 0, \ 0 \leq x \leq 1/2$,

$$I_x(r+1,r+1) \leq I_x(r,r) \tag{8}$$

Proof: Using (6) and (7), for $r > 0, \ 0 \leq x \leq 1$, we have:

$$\begin{aligned} I_x(r+1,r+1) &= I_x(r,r) \\ &+ \frac{x^r(1-x)^r}{r} \cdot \left[\frac{x}{B(r+1,r)} - \frac{1}{B(r,r)}\right] \end{aligned} \tag{9}$$

Using $B(r,r) = 2B(r+1,r)$ completes the proof:

$$I_x(r+1,r+1) = I_x(r,r) + \frac{x^r(1-x)^r}{r \cdot B(r+1,r)} \cdot \left[x - \frac{1}{2}\right]$$

Using this property, we will use the following upper bound in the rest of the proof since $I_x(2,2) = 3x^2 - 2x^3$:

$$I_x(r,r) \leq 3x^2 - 2x^3, \quad r \geq 2 \tag{10}$$

First, let $\mathcal{E}_n(K_n, r; S)$ denote the event that $S \subset V$ is an r-reachable set as per Definition 2.3. The event $\mathcal{E}_n(K_n, r; S)$ occurs if there exists at least one node in S that is adjacent to at least r nodes in S^c, the subset comprised of nodes outside the subset S. Thus, we have

$$\mathcal{E}_n(K_n, r; S) = \bigcup_{i \in \mathcal{N}_S} \left[\left(\sum_{j \in \mathcal{N}_{S^c}} \mathbb{1}\{v_i \sim v_j\}\right) \geq r\right]$$

with $\mathcal{N}_S$, $\mathcal{N}_{S^c}$ denoting the set of labels of the vertices in S and S^c, respectively. Note that at least one subset in every disjoint subset pairs needs to be r-reachable according to the definition of r-robustness, hence the event $\mathcal{E}_n(K_n, r; S) \bigcup \mathcal{E}_n(K_n, r; S')$ needs to hold *with high probability* for every disjoint subset pairs $S, S' \subset V, S \cap S' = \emptyset$. Now, let $\mathcal{E}_1(K_n, r)$ denote the event that S or S' is r-reachable for every disjoint subset pairs $S, S' \subset V$, and let $\mathcal{P}_n$ denote the collection of all non-empty subsets of V, then:

$$\mathcal{E}_1(K_n, r) := \bigcap_{\substack{S, S' \in \mathcal{P}_n \\ S \cap S' = \emptyset}} [\mathcal{E}_n(K_n, r; S) \cup \mathcal{E}_n(K_n, r; S')]$$

Note that the event that all $S \subset V$ such that $|S| \leq \lfloor \frac{n}{2} \rfloor$ are r-reachable implies the event $\mathcal{E}_1(K_n, r)$ since for the disjoint subset pairs $S, S' \subset V$, either $|S| \leq \lfloor n/2 \rfloor$ or $|S'| \leq$

$\lfloor n/2 \rfloor$ must hold true. This is because when all $S \subset V$ such that $|S| \leq \lfloor \frac{n}{2} \rfloor$ are counted, it includes at least one of the sets S or S' for every disjoint subset pair $S, S' \subset V$. Using this, $\mathcal{E}_1(K_n, r) \supseteq \bigcap_{S \in \mathcal{P}_n:\ |S| \leq \lfloor \frac{n}{2} \rfloor} \mathcal{E}_n(K_n, r; S)$. Now, let $\mathcal{Z}(K_n, r) := \mathcal{E}_1^{\rm c}(K_n, r)$ denote the event that both S and S' are not r-reachable. Thus, we have $\mathcal{Z}(K_n, r) \subseteq \bigcup_{S \in \mathcal{P}_n:\ |S| \leq \lfloor \frac{n}{2} \rfloor} \mathcal{E}_n^{\rm c}(K_n, r; S)$. Using union bound, we get

$$P_Z \leq \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{S \in \mathcal{P}_{n,m}} \mathbb{P}[\mathcal{E}_n^{\rm c}(K_n, r; S)], \tag{11}$$

where $\mathcal{P}_{n,m}$ denotes the collection of all subsets of V with exactly m elements. Further, $\mathbb{P}\left[\mathcal{Z}(K_n, r)\right]$ is abbreviated as $P_Z := \mathbb{P}\left[\mathcal{Z}(K_n, r)\right]$. From the exchangeability of the node labels and associated random variables, we have

$$\sum_{S \in \mathcal{P}_{n,m}} \mathbb{P}[\mathcal{E}_n^{\rm c}(K_n, r; S)] = \binom{n}{m} \mathbb{P}[\mathcal{E}_n^{\rm c}(K_n, r; S_m)] \tag{12}$$

since $|\mathcal{P}_{n,m}| = \binom{n}{m}$, as there are $\binom{n}{m}$ subsets of V with m elements. Note that $S_m \in \mathcal{P}_{n,m}$ denotes a subset of the vertex set V with size m, i.e. $S_m \subset V$ and $|S_m| = m$. Substituting this into (11), we obtain $P_Z \leq \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{m} \mathbb{P}[\mathcal{E}_n^{\rm c}(K_n, r; S_m)]$.

Before evaluating this expression, we will start with evaluating the probability that the set S_m is not r-reachable, abbreviated as $\mathbb{P}[\mathcal{E}_n^{\rm c}(K_n, r; S_m)] := P_{S_m}$. In a random K-out graph, there can be an edge between nodes v_i and v_j, $v_i \sim v_j$, if node v_i picks node v_j (among its selection of K nodes), or similarly if v_j picks v_i. Hence, a node $v \in S_m$ can have neighbors in S_m^c if it forms an edge with nodes in S_m^c or if nodes in S_m^c forms edges with node v. Let $\mathcal{E}_{S_{m,1}}$ denote the event that all nodes $v \in S_m$ form an edge with less than r nodes in S_m^c, and let $P_{S_{m,1}}$ denote its probability. Also let $\mathcal{E}_{S_{m,2}}$ denote the event that for each node $v \in S_m$, nodes in S_m^c form less than r edges with it, and let $P_{S_{m,2}}$ denote its probability. $\mathcal{E}_n^{\rm c}(K_n, r; S_m) \subseteq \mathcal{E}_{S_{m,1}} \cap \mathcal{E}_{S_{m,2}}$, and $\mathcal{E}_{S_{m,1}}$ and $\mathcal{E}_{S_{m,2}}$ are independent events, hence $P_{S_m} \leq P_{S_{m,1}} \cdot P_{S_{m,2}}$. Further, let $P_{v_{m,1}}$ denote the probability that a node $v \in S_m$ forms an edge with less than r nodes in S_m^c, and let $P_{v_{m,2}}$ denote the probability that nodes in S_m^c form less than r edges with the node $v \in S_m$.

Lemma 4.4: [33]: The probability that the node $v \in S_m$ chooses less than r nodes in the set S_m^c, denoted as $P_{v_{m,1}}$, can be upper bounded by the cumulative distribution function $F(r-1; K_n, p)$ of a binomial random variable with K_n trials and success probability $p = \frac{n-m-r+1}{n-r}$.

Using this upper bound, and plugging in $n = K_n$ and $p = \frac{n-m-r+1}{n-r}$ to (5), then we have

$$P_{v_{m,1}} \leq F\left(r-1; m, 1 - \frac{m-1}{n-r}\right) = I_{\frac{m-1}{n-r}}(K_n - r + 1, r)$$

The selections of each node in S_m are independent, hence we can use $(P_{S_{m,1}}) = (P_{v_{m,1}})^m$. Now, in order to find $P_{v_{m,2}}$, a node in S_m^c forming an edge with the node v can be modeled as a Bernoulli trial with probability $p = K_n/(n-1)$ so the event that nodes in S_m^c forming less than r edges with the node v can be represented by a Binomial model with $n-m$ trials and $p = \frac{K_n}{n-1}$. Hence, $P_{v_{m,2}} = F\left(r-1; n-m, \frac{K_n}{n-1}\right) = I_{\frac{n-K_n-1}{n-1}}(n-m-r+1, r)$.

Since the nodes in S_m^c forming edges with nodes in S_m are not independent of the other nodes in S_m, it is not necessarily the case that $(P_{S_{m,2}}) \leq (P_{v_{m,2}})^m$. To find $(P_{S_{m,2}})$, we decompose it into the following conditional probabilities.

$$P_{S_{m,2}} = \mathbb{P}[d_{v_1} < r] \cdot \mathbb{P}[d_{v_2} < r | d_{v_1} < r] \cdot \\ \ldots \mathbb{P}[d_{v_m} < r | d_{v_1} < r, d_{v_2} < r, \ldots, d_{v_{m-1}} < r]$$

where $v_1, v_2, \ldots, v_m \in S_m$ represent all the nodes in S_m, and d_{v_i} is used to denote the number of nodes in S_m^c that form an edge with the node v_i. To find an upper bound on $P_{S_{m,2}}$, we consider the worst case. In the worst case, all the preceding nodes are selected by nodes in S_m^c exactly $r-1$ times. This reduces the number of available edges in S_m^c that can make connections with the remaining nodes in S_m, hence increases the probability of nodes in S_m^c forming less than r edges with the remaining nodes in S_m. Hence,

$$P_{S_{m,2}} \leq \mathbb{P}[d_{v_1} < r] \cdot \mathbb{P}[d_{v_2} < r | d_{v_1} = r-1] \cdot \\ \ldots \mathbb{P}[d_{v_m} < r | d_{v_1} = r-1, \ldots, d_{v_{m-1}} = r-1]$$

Consider the general case for $\mathbb{P}[d_{v_{a+1}} < r | d_{v_1} = r-1, \ldots, d_{v_a} = r-1]$ where $1 \leq a \leq m-1$. Assume that q_1 nodes in S_m^c formed an edge with only one node among the nodes $v_1, \ldots, v_a$. Similarly, assume q_2 nodes in S_m^c formed an edge with only two nodes, and so on (q_{K_n} nodes in S_m^c formed edges with K_n nodes in $v_1, \ldots, v_a$). Also define $q = q_1 + q_2 + \ldots + q_{K_n}$. It can be seen that $a = \frac{q_1 + 2q_2 + \ldots + K_n q_{K_n}}{r-1}$. Here, we have $n_0 = n - m - q$ nodes that did not use any of their selections on $v_1, \ldots, v_a$, so their probability of choosing the node v_{a+1} is $p_0 = \frac{K_n}{n-a-1}$. Similarly, we have $n_1 = q_1$ nodes that used one of their selections on $v_1, \ldots, v_a$, so their probability of choosing v_{a+1} is $p_1 = \frac{K_n - 1}{n-a-1}$ (for $n_{K_n} = q_{K_n}$ nodes, $p_{K_n} = \frac{K_n - K_n}{n-a-1} = 0$).
Claim: $\mathbb{P}[d_{v_{a+1}} < r | d_{v_1} = r-1, \ldots, d_{v_a} = r-1] \leq 1/2$ for all $a = 1, \ldots, m$ with $1 \leq m \leq \lfloor n/2 \rfloor$ when $K \geq 2r$.

Proof: Considering the event that a node in S_m^c picking the node v_{a+1} as a Bernoulli trial, since nodes in with different probabilities, the total number of times the node v_{a+1} is picked by the nodes in S_m^c in all the trials, N_{a+1}, is distributed as a Poisson Binomial distribution $N_{a+1} \sim PB([p_0]^{n_0}, \ldots, [p_{K_n}]^{n_{K_n}})$, where $p_i = \frac{K_n - i}{n-a-1}, 0 \leq i \leq K_n$; $n_i = q_i, 1 \leq i \leq K - n$, and $n_0 = n - m - q$ as defined above. The mean of this distribution is:

$$\mu_{N_{a+1}} = (n - m - q) * \frac{K_n}{n-a-1} + \sum_{i=1}^{K_n} q_i \frac{K_n - i}{n-a-1} \\ = \frac{2n - 2m - a}{n-a-1} * r + \frac{(K_n - 2r) * (n-m) + a}{n-a-1}$$

Note that $2n - 2m \geq n$ since $m \leq \lfloor n/2 \rfloor$, hence first term is larger than r. Thus, $\mu_{N_{a+1}} > r$ when $K_n \geq 2r$. Since the median of a Poisson Binomial distribution, M_X; satisfies $\lfloor \mu_X \rfloor \leq M_X \leq \lceil \mu_X \rceil$ [36], we have that $M_{N_{a+1}} \geq r$ for any $1 \leq a \leq m-1$ if $K_n \geq 2r$.

Combining these, we have that $P_{S_{m,2}} \leq \left(\frac{1}{2}\right)^m$ when $K_n \geq 2r$. Now, using $P_{S_m} \leq P_{S_{m,1}} \cdot P_{S_{m,2}}$, we have:

$$P_{S_m} \leq (P_{v_{m,1}})^m \cdot \left(\frac{1}{2}\right)^m \leq \left(\frac{1}{2} I_{\frac{m-1}{n-r}}(K_n - r + 1, r)\right)^m$$

We will divide the summation into three parts as follows:

$$P_Z \leq \sum_{m=1}^{\lfloor n/2 \rfloor} \binom{n}{m} P_{S_m} = \sum_{m=1}^{\lfloor \log(n) \rfloor} P_m + \sum_{m=\lceil \log(n) \rceil}^{\lfloor 0.3n \rfloor} P_m + \sum_{m=\lceil 0.3n \rceil}^{\lfloor n/2 \rfloor} P_m := P_1 + P_2 + P_3 \quad (13)$$

where $P_m := \binom{n}{m} P_{S_m}$. Starting with the first summation P_1 and using (2), we have:

$$\begin{aligned}
P_m &= \binom{n}{m} P_{S_m} \leq \binom{n}{m} P_{S_{m,1}} \cdot P_{S_{m,2}} \leq \binom{n}{m} P_{S_{m,1}} \\
&\leq \left(\frac{en}{mB(K_n - r + 1, r)} \int_0^{\frac{m-1}{n-r}} t^{K_n - r}(1-t)^{r-1} dt\right)^m \\
&\leq \left(\frac{e \cdot \frac{n}{m} \cdot \frac{m-1}{n-r}}{B(K_n - r + 1, r)(K_n - r + 1)} \left(\frac{m-1}{n-r}\right)^{K_n - r}\right)^m \\
&\leq \left(\frac{e(1+r)\left(\frac{\log(n)-1}{n-r}\right)^{K_n - r}}{B(K_n - r + 1, r)(K_n - r + 1)}\right)^m := (a_n)^m
\end{aligned}$$

For $K_n > r$, since $B(K_n - r + 1, r)$ and r are finite values, we have $\lim_{n\to\infty} a_n = 0$ by virtue of $\lim_{n\to\infty} \left(\frac{\log(n)-1}{n-r}\right)^{K_n - r} = 0$. Using this, we can express the summation as:

$$P_1 \leq \sum_{m=1}^{\lfloor \log(n) \rfloor} (a_n)^m \leq \sum_{m=1}^{\infty} (a_n)^m \leq \frac{a_n}{1 - a_n} \quad (14)$$

The geometric sum converges since $\lim_{n\to\infty} a_n = 0$, leading to P_1 converging to zero as $n \to \infty$. Now, similarly consider the second summation P_2. Using (2), and (10), we have

$$\begin{aligned}
P_m &\leq \binom{n}{m} P_{S_{m,1}} \cdot P_{S_{m,2}} \leq \left(\frac{en}{2m} I_{\frac{m-1}{n-r}}(K_n - r + 1, r)\right)^m \\
&\leq \left(\frac{en}{2m} I_{\frac{m-1}{n-r}}(r, r)\right)^m \quad (15) \\
&\leq \left(\frac{en}{2m}\left[3\left(\frac{m-1}{n-r}\right)^2 - 2\left(\frac{m-1}{n-r}\right)^3\right]\right)^m := (b_{n,m})^m
\end{aligned}$$

where (15) is obtained from the fact that $I_{\frac{m-1}{n-r}}(K_n - r + 1, r) \leq I_{\frac{m-1}{n-r}}(r, r)$ as a consequence of the property (6). It can be seen that $b_{n,m} < 0.98$ for all $\lceil \log(n) \rceil \leq m \leq \lfloor 0.3n \rfloor$ when $n \to \infty$, and defining $b_n = \limsup_{n\to\infty, \lceil \log(n) \rceil \leq m \leq \lfloor 0.3n \rfloor} b_{n,m}$

$$P_2 = \sum_{m=\lceil \log(n) \rceil}^{\lfloor 0.3n \rfloor} (b_{n,m})^m \leq \sum_{m=\lceil \log(n) \rceil}^{\infty} (b_n)^m \leq \frac{b_n^{\log(n)}}{1 - b_n}$$

The geometric sum converges since $b_n < 1$, leading to P_2 converging to zero for large n. Note that the selection of $\lfloor 0.3n \rfloor$ as the upper limit P_2 is arbitrary, and this particular value was chosen to have a finite ratio for $\frac{m}{n}$ on the lower limit of the third summation, and to satisfy $b_n < 1$.

Lastly, consider the third summation P_3. Using (3), and (10); and assuming $K_n \geq 2r$, we have

$$\begin{aligned}
P_m &\leq (1 + o(1))\sqrt{\frac{n}{2\pi m(n-m)}} \cdot \left(\left(\frac{n}{n-m}\right)^{\frac{n}{m}-1} \cdot \frac{n}{2m} \cdot I_{\frac{m-1}{n-r}}(K_n - r + 1, r)\right)^m \\
&:= (1 + o(1))\sqrt{\frac{n}{2\pi m(n-m)}} (c_{n,m})^m \quad (16)
\end{aligned}$$

Define $d_{n,m} := \frac{n}{2m} \cdot \left(\frac{n}{n-m}\right)^{\frac{n}{m}-1} I_{\frac{m-1}{n-r}}(r, r)$. As a consequence of property (6), when $K_n \geq 2r$, $n \to \infty$, $\lceil 0.3n \rceil \leq m \leq \lfloor n/2 \rfloor$, and r is finite; we have that

$$\begin{aligned}
I_{\frac{m-1}{n-r}}(K_n - r + 1, r) &= I_{\frac{m-1}{n-r}}(r, r) - \sum_{i=r+1}^{K_n - r + 1} \frac{\left(\frac{m-1}{n-r}\right)^{i-1} \cdot \left(1 - \frac{m-1}{n-r}\right)^r}{(i-1) \cdot B(i-1, r)} \\
&\leq I_{\frac{m-1}{n-r}}(r, r) - \sum_{i=r+1}^{K_n - r + 1} \frac{(0.3)^{i-1} \cdot (0.5)^r}{(i-1) \cdot B(i-1, r)} \\
&\leq I_{\frac{m-1}{n-r}}(r, r) - \frac{(0.15)^r}{r \cdot B(r, r)} := I_{\frac{m-1}{n-r}}(r, r) - \epsilon(r) \quad (17)
\end{aligned}$$

Thus, it can be seen that $c_{n,m} = d_{n,m} \cdot \frac{I_{\frac{m-1}{n-r}}(K_n - r + 1, r)}{I_{\frac{m-1}{n-r}}(r, r)} \leq d_{n,m} \cdot \frac{I_{\frac{m-1}{n-r}}(r, r) - \epsilon(r)}{I_{\frac{m-1}{n-r}}(r, r)}$, and since $\epsilon(r) > 0$ when $r \geq 2$, $K_n \geq 2r$ and $\lceil 0.3n \rceil \leq m \leq \lfloor n/2 \rfloor$; we have that $c_{n,m} < d_{n,m}$.

$$d_{n,m} \leq \frac{n}{2m} \cdot \left(\frac{n}{n-m}\right)^{\frac{n}{m}-1} \cdot \left[3\left(\frac{m-1}{n-r}\right)^2 - 2\left(\frac{m-1}{n-r}\right)^3\right]$$

It can be seen that $\limsup_{n\to\infty, \lceil 0.3n \rceil \leq m \leq \lfloor n/2 \rfloor} d_{n,m} = 1$ since r is finite (This can easily be verified by writing $d_{n,m}$ in terms of a single variable $\beta = \frac{m}{n}$, and finding its value for all $0.3 \leq \beta \leq 0.5$). Hence, we have $c_{n,m} < 1$ for all $\lceil 0.3n \rceil \leq m \leq \lfloor n/2 \rfloor$ when $r \geq 2$, $K_n \geq 2r$, $n \to \infty$.

$$P_3 \leq \sum_{m=\lceil 0.3n \rceil}^{\lfloor \frac{n}{2} \rfloor} (1 + o(1))\sqrt{\frac{n}{2\pi m(n-m)}} (c_{n,m})^m \quad (18)$$

the summation converges to zero when $n \to \infty$ since $c_{n,m} < 1$ for all terms of the summation, leading to P_3 converging to zero as n gets large. Since P_1, P_2, and P_3 all converge to zero in large n, P_Z also converges to zero as n gets large. This concludes the proof of Theorem 3.1.

V. Conclusions

In this work, we establish that for large n, $r^\star(K) \geq \lfloor K/2 \rfloor$, in other words, $K \geq 2r$ ensures, *with high probability*, the r-robustness of the random K-out graph $\mathbb{H}(n; K)$. We use computer simulations to verify our result. Using our result, we compare the mean node degree of a random K-out graph with the mean node degree of an Erdős-Rényi graph and a Barabási-Albert graph at the threshold value required to ensure r-robustness *whp*, and determine that random K-out graphs attain r-robustness at a significantly lower mean node degree value compared to Erdős-Rényi graphs, and within a factor of 2 compared to Barabási-Albert graphs. Hence, our result shows that the random K-out graph model is a viable option in a wide range of applications that require r-robustness with as few edges as possible such as robust control of multi-agent systems, and in communication systems such as robust and differentially-private federated learning, and wireless sensor networks.

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