

Reliability of Wireless Sensor Networks under a Heterogeneous Key Predistribution Scheme

Rashad Eletreby and Osman Yağın

Department of Electrical and Computer Engineering and CyLab,

Carnegie Mellon University, Pittsburgh, PA, 15213 USA

reletreby@cmu.edu, oyagan@ece.cmu.edu

Abstract—We consider the network reliability problem in wireless sensor networks secured by the heterogeneous random key predistribution scheme. This scheme is recently introduced by Yağın, as a generalization of the Eschenauer-Gligor scheme, to account for the cases when the network comprises sensor nodes with varying level of resources; e.g., regular nodes vs. cluster heads. It induces the *inhomogeneous* random key graph, denoted $\mathbb{G}(n; \boldsymbol{\mu}, \mathbf{K}, P)$, where each of the n nodes are independently assigned to one of r classes according to a probability distribution $\boldsymbol{\mu} = (\mu_1, \dots, \mu_r)$ and then a class- i node is assigned K_i keys uniformly at random from a pool of size P ; two nodes that share a key are then connected by an edge. We analyze the *reliability* of $\mathbb{G}(n; \boldsymbol{\mu}, \mathbf{K}, P)$ against random link failures. Namely, we consider $\mathbb{G}(n; \boldsymbol{\mu}, \mathbf{K}, P, \alpha)$ formed by deleting each edge of $\mathbb{G}(n; \boldsymbol{\mu}, \mathbf{K}, P)$ independently with probability $1 - \alpha$, and study the probability that the resulting graph i) has no isolated node; and ii) is connected. We present scaling conditions on \mathbf{K} , P , and α such that both events take place with probability zero or one, respectively, as the number of nodes gets large. We present numerical results to support these in the finite-node regime.

Index Terms—Wireless Sensor Networks, Security, Inhomogeneous Random Key Graphs, Reliability, Connectivity.

I. INTRODUCTION

Wireless sensor networks (WSNs) consist of low-cost, low-power, small sensor nodes that are typically deployed randomly in large numbers, with application areas as diverse as military, health, environmental monitoring, etc [1]. In most cases, WSNs are deployed in hostile environments (e.g., battlefields) making it crucial to use cryptographic protection to secure sensor communications. To that end, significant efforts have been devoted to developing methods for securing WSNs, and *random key predistribution schemes* have been widely accepted as feasible solutions in the face of unique challenges WSNs present: limited computational capabilities and transmission power, lack of a priori knowledge of deployment configuration, and vulnerability to node capture attacks; e.g., see [2]–[5] for a detailed discussion on security challenges in WSNs and solutions based on key predistribution.

In this paper, we consider a *heterogeneous* key predistribution scheme introduced recently by Yağın [6] as a variation of the *classical* Eschenauer-Gligor (EG) scheme [2]. The heterogeneous key predistribution scheme accounts for the cases when the network comprises sensor nodes with varying level of resources (e.g., regular nodes vs. cluster heads), which is likely to be the case for many WSN applications [7]. According to this scheme, each sensor belongs to one of r

priority classes that controls the *number* of cryptographic keys assigned to them. More specifically, each of the n sensors is independently assigned to class- i with probability $\mu_i > 0$, for each $i = 1, \dots, r$; obviously we have $\sum_{i=1}^r \mu_i = 1$. Sensors from class- i are each given K_i keys selected uniformly at random from a pool of size P . Then, pairs of sensors that have at least one key in common can communicate securely after deployment. With $\boldsymbol{\mu} = \{\mu_1, \dots, \mu_r\}$ and $\mathbf{K} = \{K_1, \dots, K_r\}$, we let $\mathbb{G}(n; \boldsymbol{\mu}, \mathbf{K}, P)$ denote the random graph induced by the heterogeneous key predistribution scheme. This model was referred to as the *inhomogeneous* random key graph in [6], wherein, zero-one laws for absence of isolated nodes and connectivity are established.

The main goal of this paper is to investigate the *reliability* of secure WSNs under the heterogeneous key predistribution scheme. In particular, to account for the possibility that links between two sensor nodes may fail (e.g., due to random failures, adversarial attacks, etc.), we apply a Bernoulli link-failure model to the inhomogeneous random key graph $\mathbb{G}(n; \boldsymbol{\mu}, \mathbf{K}, P)$. Namely, we assume that each link in $\mathbb{G}(n; \boldsymbol{\mu}, \mathbf{K}, P)$ is operational with probability α and fails with probability $1 - \alpha$, independently from others. This models random attacks as well as random failures due to sensor malfunctioning or harsh environmental conditions.

Let $\mathbb{G}(n; \boldsymbol{\mu}, \mathbf{K}, P, \alpha)$ denote the resulting random graph that contains all operational links in $\mathbb{G}(n; \boldsymbol{\mu}, \mathbf{K}, P)$. The network reliability problem is concerned [8], [9, Section 7.5] with deriving the probability that $\mathbb{G}(n; \boldsymbol{\mu}, \mathbf{K}, P, \alpha)$ exhibits certain desired properties – that captures the ability of the network to continue its services – as a function of the link failure probability $1 - \alpha$. Here, we focus on two standard and related properties that the network i) has no *isolated* node, and ii) is connected. For arbitrary graphs with fixed size n , deriving these probabilities are known [10], [11] to be $\#P$ -complete, meaning that no polynomial algorithm exists for their solution, unless $P = NP$. Given that it is not feasible to derive them, we study the *asymptotic* behavior of these probabilities as n gets large, when the model parameters are scaled with n ; the finite-node case is also considered via simulations.

Our contributions are as follows. We present conditions on how to scale \mathbf{K}, P and the link failure probability α such that the network $\mathbb{G}(n; \boldsymbol{\mu}, \mathbf{K}_n, P_n, \alpha_n)$ is connected with probability approaching to one and zero, respectively, as n grows unboundedly large. We establish an analogous zero-one

law for $\mathbb{G}(n; \boldsymbol{\mu}, \mathbf{K}_n, P_n, \alpha_n)$ to have no node that is isolated (i.e., that has zero edge). These sharp results are likely to be useful in *dimensioning* the heterogeneous scheme, namely *controlling* the key ring parameters $(\boldsymbol{\mu}, \mathbf{K})$, and key pool size P such that the network has a desired level reliability against link failures. A particularly surprising conclusion derived from our results is that network reliability is tightly dependent on the *smallest* key ring size used in the network; see Section III-A for details.

Our results complement and generalize several previous work in the literature. In particular, we complement the work [12] that studies the reliability of secure WSNs against a fixed number k of link failures; in our case the number of failed links can be unboundedly large. Our results also contain as special cases the zero-one laws for connectivity in inhomogeneous random key graphs [6] and reliability results in homogeneous random key graphs [13]; see Section III-B for details.

All limiting statements, including asymptotic equivalences are considered with the number of sensor nodes n going to infinity. The indicator function of an event E is denoted by $\mathbf{1}[E]$. We say that an event holds with high probability (whp) if it holds with probability 1 as $n \rightarrow \infty$. In comparing the asymptotic behavior of the sequences $\{a_n\}, \{b_n\}$, we use the standard Landau notation, e.g., $a_n = o(b_n)$, $a_n = w(b_n)$, $a_n = O(b_n)$, $a_n = \Omega(b_n)$, and $a_n = \Theta(b_n)$. We also use $a_n \sim b_n$ to denote the asymptotic equivalence $\lim_{n \rightarrow \infty} a_n/b_n = 1$.

II. THE MODEL

The heterogeneous random key predistribution scheme introduced in [6] works as follows. Consider a network of n sensors labeled as v_1, v_2, \dots, v_n . Each sensor node is classified into one of the r classes (e.g., priority levels) according to a probability distribution $\boldsymbol{\mu} = \{\mu_1, \mu_2, \dots, \mu_r\}$ with $\mu_i > 0$ for $i = 1, \dots, r$ and $\sum_{i=1}^r \mu_i = 1$. Then, a class- i node is assigned K_i cryptographic keys selected uniformly at random and *without replacement* from a key pool of size P . It follows that the key ring Σ_x of node v_x is a random variable (rv) with

$$\mathbb{P}[\Sigma_x = S \mid t_x = i] = \binom{P}{K_i}^{-1}, \quad S \in \mathcal{P}_{K_i},$$

where t_x denotes the class of v_x and \mathcal{P}_{K_i} is the collection of all subsets of $\{1, \dots, P\}$ with size K_i . The classical key predistribution scheme of Eschenauer and Gligor [2] constitutes a special case of this model with $r = 1$, i.e., when all sensors belong to the same class and receive the same number of keys; see also [14].

Let $\mathbf{K} = \{K_1, K_2, \dots, K_r\}$ and assume without loss of generality that $K_1 \leq K_2 \leq \dots \leq K_r$. Consider a random graph \mathbb{G} induced on the vertex set $\mathcal{V} = \{v_1, \dots, v_n\}$ such that a pair of nodes v_x and v_y are adjacent, denoted by $v_x \sim_G v_y$, if they have at least one cryptographic key in common, i.e.,

$$v_x \sim_G v_y \quad \text{if} \quad \Sigma_x \cap \Sigma_y \neq \emptyset. \quad (1)$$

The adjacency condition (1) defines the inhomogeneous random key graph denoted by $\mathbb{G}(n; \boldsymbol{\mu}, \mathbf{K}, P)$ [6]. This model is also known in the literature as the *general random intersection*

graph; e.g., see [12], [15], [16]. The probability p_{ij} that a class- i node and a class- j node are adjacent is given by

$$p_{ij} = \mathbb{P}[v_x \sim_G v_y \mid t_x = i, t_y = j] = 1 - \frac{\binom{P-K_i}{K_j}}{\binom{P}{K_j}} \quad (2)$$

as long as $K_i + K_j \leq P$; otherwise if $K_i + K_j > P$, we have $p_{ij} = 1$. Let λ_i denote the *mean* probability that a class- i node is connected to another node in $\mathbb{G}(n; \boldsymbol{\mu}, \mathbf{K}, P)$. We have

$$\lambda_i = \mathbb{P}[v_x \sim_G v_y \mid t_x = i] = \sum_{j=1}^r p_{ij} \mu_j. \quad (3)$$

To account for the possibility that links between two sensor nodes may fail (e.g., due to random failures, adversarial attacks, etc.), we apply a Bernoulli link-failure model to the inhomogeneous random key graph $\mathbb{G}(n; \boldsymbol{\mu}, \mathbf{K}, P)$: With $\alpha \in (0, 1)$ let $\{B_{ij}(\alpha), 1 \leq i < j \leq n\}$ denote independent Bernoulli rvs, each with success probability α . Then the link between sensors v_x and v_y is deemed to be operational (i.e., *up*) if $B_{xy}(\alpha) = 1$, and not operational (i.e., *down*) if $B_{xy}(\alpha) = 0$. Put differently, every edge in $\mathbb{G}(n; \boldsymbol{\mu}, \mathbf{K}, P)$ is deleted independently with probability $1 - \alpha$ ¹.

Let $\mathbb{G}(n; \boldsymbol{\mu}, \mathbf{K}, P, \alpha)$ denote the resulting random graph that contains all the operational links in $\mathbb{G}(n; \boldsymbol{\mu}, \mathbf{K}, P)$. To simplify notation, we let $\boldsymbol{\theta} = (\mathbf{K}, P)$, and $\boldsymbol{\Theta} = (\boldsymbol{\theta}, \alpha)$. In $\mathbb{G}(n; \boldsymbol{\mu}, \boldsymbol{\Theta})$, distinct nodes v_x and v_y are adjacent, denoted $v_x \sim v_y$, if and only if they are adjacent in $\mathbb{G}(n; \boldsymbol{\mu}, \mathbf{K}, P)$ and the edge $v_x \sim_G v_y$ is operational (i.e., has not failed). By independence, the probability of an edge between a class- i node and a class- j node in $\mathbb{G}(n; \boldsymbol{\mu}, \boldsymbol{\Theta})$ is then given by

$$\mathbb{P}[v_x \sim v_y \mid t_x = i, t_y = j] = \alpha p_{ij}.$$

Similar to (3), we denote the mean edge probability for a class- i node in $\mathbb{G}(n; \boldsymbol{\mu}, \boldsymbol{\Theta})$ as Λ_i . It is clear that

$$\Lambda_i = \sum_{j=1}^r \mu_j \alpha p_{ij} = \alpha \lambda_i, \quad i = 1, \dots, r. \quad (4)$$

Throughout, we assume that the number of classes r is fixed and does not scale with n , and so are the probabilities μ_1, \dots, μ_r . All other parameters are scaled with n .

III. MAIN RESULTS AND DISCUSSION

We refer to a mapping $\boldsymbol{\Theta} = K_1, \dots, K_r, P, \alpha : \mathbb{N}_0 \rightarrow \mathbb{N}_0^{r+1} \times (0, 1)$ as a *scaling* if

$$1 \leq K_{1,n} \leq K_{2,n} \leq \dots \leq K_{r,n} \leq P_n/2 \quad (5)$$

for all $n = 2, 3, \dots$. We note that under (5), the edge probability p_{ij} is given by (2).

¹An interesting direction for future work would be to consider a heterogeneous link-failure model, where the link between a type- i and type- j node fails with probability $1 - \alpha_{ij}$.

A. Results

We first present a zero-one law for the absence of isolated nodes in $\mathbb{G}(n; \boldsymbol{\mu}, \boldsymbol{\Theta}_n)$.

Theorem 3.1: Consider a probability distribution $\boldsymbol{\mu} = \{\mu_1, \mu_2, \dots, \mu_r\}$ with $\mu_i > 0$ for $i = 1, \dots, r$ and a scaling $\boldsymbol{\Theta} : \mathbb{N}_0 \rightarrow \mathbb{N}_0^{r+1} \times (0, 1)$ such that

$$\Lambda_1(n) = \alpha_n \lambda_1(n) \sim c \frac{\log n}{n} \quad (6)$$

for some $c > 0$. We have

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\begin{array}{l} \mathbb{G}(n; \boldsymbol{\mu}, \boldsymbol{\Theta}_n) \text{ has} \\ \text{no isolated nodes} \end{array} \right] = \begin{cases} 0 & \text{if } c < 1 \\ 1 & \text{if } c > 1 \end{cases}$$

The scaling condition (6) will often be used in the form

$$\Lambda_1(n) = c_n \frac{\log n}{n}, \quad n = 2, 3, \dots \quad (7)$$

with $\lim_{n \rightarrow \infty} c_n = c > 0$.

Next, we present an analogous result for connectivity.

Theorem 3.2: Consider a probability distribution $\boldsymbol{\mu} = \{\mu_1, \mu_2, \dots, \mu_r\}$ with $\mu_i > 0$ for $i = 1, \dots, r$ and a scaling $\boldsymbol{\Theta} : \mathbb{N}_0 \rightarrow \mathbb{N}_0^{r+1} \times (0, 1)$ such that (6) holds for some $c > 0$. Then, we have

$$\lim_{n \rightarrow \infty} \mathbb{P}[\mathbb{G}(n; \boldsymbol{\mu}, \boldsymbol{\Theta}_n) \text{ is connected}] = \begin{cases} 0 & \text{if } c < 1 \\ 1 & \text{if } c > 1 \end{cases}$$

under the extra conditions that

$$P_n \geq \sigma n, \quad n = 1, 2, \dots \quad (8)$$

for some $\sigma > 0$ and

$$p_{11}(n) = \omega \left(\frac{1}{n\alpha_n} \right). \quad (9)$$

Theorem 3.1 (resp. Theorem 3.2) states that $\mathbb{G}(n; \boldsymbol{\mu}, \boldsymbol{\Theta}_n)$ has no isolated node (resp. is connected) whp if the mean degree of class-1 nodes (that receive the smallest number $K_{1,n}$ of keys) is scaled as $(1 + \epsilon) \log n$ for some $\epsilon > 0$. On the other hand, if this minimal mean degree scales as $(1 - \epsilon) \log n$ for some $\epsilon > 0$, then whp $\mathbb{G}(n; \boldsymbol{\mu}, \boldsymbol{\Theta}_n)$ has an isolated node, and hence not connected. These results indicate that the minimum key ring size in the network has a surprisingly significant impact on the reliability of $\mathbb{G}(n; \boldsymbol{\mu}, \boldsymbol{\Theta}_n)$. This is more clearly seen under the additional assumption that $\lambda_1(n) = o(1)$ which gives [6, Lemma 4.2]

$$\lambda_1(n) \sim \frac{K_{1,n} K_{\text{avg},n}}{P_n}$$

where $K_{\text{avg},n} = \sum_{j=1}^r \mu_j K_{j,n}$ denotes the mean key ring size. Using this in (6), we see that for fixed mean number $K_{\text{avg},n}$ of keys per sensor, network reliability is directly affected by the minimum key ring size $K_{1,n}$. For example, reducing $K_{1,n}$ by half means that the smallest α for which the network remains connected is increased by two-fold, which then reduces the largest link failure probability $1 - \alpha$ that can be sustained by a similar order.

The resemblance of the results presented in Theorem 3.1 and Theorem 3.2 indicates that the absence of isolated nodes

property and connectivity property are asymptotically equivalent for $\mathbb{G}(n; \boldsymbol{\mu}, \boldsymbol{\Theta}_n)$, similarly with some well-known random graph models; e.g., inhomogeneous random key graphs [6], ER graphs [9], and (homogeneous) random key graphs [14].

Conditions (8) and (9) are enforced mainly for technical reasons and they are only needed in the proof of the one-law of Theorem 3.2. These conditions are likely to hold in real-world WSN implementations. In particular, (8) should hold in practice to ensure the resiliency of the WSN against node capture attacks [17], while (9) is needed as otherwise the network would be trivially disconnected [6, Section 3.2].

B. Comparison with related work

Our main results extend the work by Yağan [6] who established zero-one laws for the connectivity of inhomogeneous random key graph $\mathbb{G}(n, \boldsymbol{\mu}, \boldsymbol{K}, P)$ without employing a link-failure model. It is clear that, although a crucial first step in the study of heterogeneous key predistribution schemes, the assumption that all links are operational, i.e., *reliable*, is not likely to hold in most practical settings. In this regard, our work extends the results by Yağan [6] to more practical WSN scenarios where the unreliability of links are taken into account. In fact, by setting $\alpha_n = 1$ for each $n = 1, 2, \dots$ (i.e., by assuming that all links are *reliable*), our results reduce to those given in [6].

The reliability of secure WSNs was also studied in [13], but under the Eschenauer-Gligor scheme [2] where all sensors receive the same number of keys. However, when the network consists of sensors with varying level of resources (e.g., computational, memory, power) and/or with varying level of security and connectivity requirements, it may no longer be sensible to assign the same number of keys to all sensors. Our work addresses this issue by generalizing [13] to the cases where nodes can be assigned different number of keys. When $r = 1$, i.e., when all nodes belong to the same class and receive the same number of keys, our result recovers the main result in [13].

Another notable work that is related to ours is by Zhao et al. [12], who studied the k -connectivity and k -robustness in the inhomogeneous random key graph. A graph is said to be k -connected if it remains connected after removal (i.e., *failure*) of any $k-1$ nodes. Thus, the results obtained in [12] ensure the reliability of the network against the failure of $k-1$ nodes, for some integer constant k . Our work complements these results by considering the case when *each and every* edge fails with probability $1 - \alpha$, so that the total number of failed links is possibly infinite; e.g., as many as $O(n^2)$ links may fail.

C. Significance of the results

We present novel results on the network reliability problem of secure, large-scale WSNs. Our results are helpful in ensuring network reliability in multitude of applications where inhomogeneous random key graphs are utilized. For instance, reliability against the failure of wireless links is important in WSN applications where sensors are deployed in hostile environments (e.g., battlefield surveillance), or, are unattended

for long periods of time (e.g., environmental monitoring), or, are used in life-critical applications (e.g., patient monitoring).

Considering the asymptotic regime, a key question in network reliability analysis is whether or not there exists a threshold $\alpha_n^* \in (0, 1)$ such that if α_n is slightly smaller than (resp. slightly larger than) α_n^* then the probability that $\mathbb{G}(n; \boldsymbol{\mu}, \mathbf{K}, P, \alpha)$ is connected is close to zero (resp. close to one); e.g., see [9, Section 7.5]. Our results constitute an asymptotic solution of the network reliability problem for inhomogeneous random key graphs. More specifically, we show that α_n exhibits a threshold behavior as given at (6). Although asymptotic in nature, these results can still provide useful insights about the reliability of heterogeneous WSNs with number of sensors n being on the order of hundreds; see Section IV for numerical experiments.

IV. NUMERICAL RESULTS

We present numerical results that support Theorem 3.1 and Theorem 3.2 in the finite node regime. In all experiments, we fix the number of nodes at $n = 500$ and the size of the key pool at $P = 10^4$. In Figure 1, we consider the link-failure parameters $\alpha = 0.2$, $\alpha = 0.4$, $\alpha = 0.6$, and $\alpha = 0.8$, while varying the parameter K_1 (i.e., the smallest key ring size) from 5 to 35. The number of classes is fixed at 4 with $\boldsymbol{\mu} = \{0.25, 0.25, 0.25, 0.25\}$ and we set $K_2 = K_1 + 5$, $K_3 = K_1 + 10$, and $K_4 = K_1 + 15$. For each parameter pair (\mathbf{K}, α) , we generate 200 independent samples of the graph $\mathbb{G}(n; \boldsymbol{\mu}, \boldsymbol{\Theta})$ and count the number of times (out of a possible 200) that the obtained graphs i) have no isolated nodes and ii) are connected. Dividing the counts by 200, we obtain the (empirical) probabilities for the events of interest. We observed that $\mathbb{G}(n; \boldsymbol{\mu}, \boldsymbol{\Theta})$ is connected whenever it has no isolated nodes yielding the same empirical probability for both events. This is in parallel with the asymptotic equivalence of the two properties as implied by Theorems 3.1 and 3.2.

In Figure 1 we show the *critical* threshold of connectivity “predicted” by Theorem 3.2 by a vertical dashed line. More specifically, the vertical dashed lines stand for the minimum integer K_1 such that

$$\lambda_1(n) = \sum_{j=1}^4 \mu_j \left(1 - \frac{\binom{P-K_j}{K_1}}{\binom{P}{K_1}} \right) > \frac{1}{\alpha} \frac{\log n}{n}. \quad (10)$$

We see that the probability of connectivity transitions from zero to one within relatively small variation of K_1 , with critical values of K_1 from (10) lying within this transition interval.

Figure 2 is generated in a similar manner with Figure 1, this time with an eye towards understanding the impact of the minimum key ring size K_1 on network reliability. To that, we fix the number of classes at 2 with $\boldsymbol{\mu} = \{0.5, 0.5\}$ and consider three different key ring sizes \mathbf{K} each with mean 30; namely, we consider $\mathbf{K} = \{10, 50\}$, $\mathbf{K} = \{20, 40\}$, and $\mathbf{K} = \{30, 30\}$. As we compare the probability of connectivity in the resulting networks with link failure probability ranging from zero to one, we see that network reliability improves dramatically as the minimum key ring size K_1 increases.

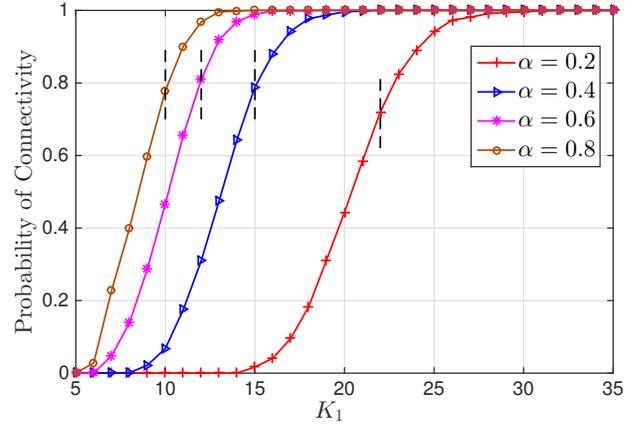


Fig. 1. Empirical probability that $\mathbb{G}(n; \boldsymbol{\mu}, \mathbf{K}, P, \alpha)$ is connected with $n = 500$, $\boldsymbol{\mu} = (1/4, 1/4, 1/4, 1/4)$, $\mathbf{K} = (K_1, K_1 + 5, K_1 + 10, K_1 + 15)$, and $P = 10^4$. Vertical dashed lines give the minimum K_1 for which (10) holds.

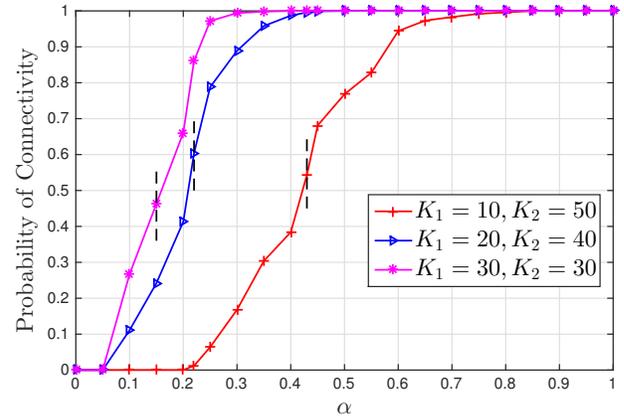


Fig. 2. Empirical probability that $\mathbb{G}(n; \boldsymbol{\mu}, \mathbf{K}, P, \alpha)$ is connected with $n = 500$, $\boldsymbol{\mu} = (1/2, 1/2)$, and $P = 10^4$; we consider three choices of $\mathbf{K} = (K_1, K_2)$ each with the same mean. Vertical dashed lines give the minimum α for which (10) holds.

V. PROOF OF THEOREM 3.1

The proof of Theorem 3.1 relies on the method of first and second moments applied to the number of isolated nodes in $\mathbb{G}(n; \boldsymbol{\mu}, \boldsymbol{\Theta}_n)$. Let $I_n(\boldsymbol{\mu}, \boldsymbol{\Theta}_n)$ denote the total number of isolated nodes in $\mathbb{G}(n; \boldsymbol{\mu}, \boldsymbol{\Theta}_n)$, namely,

$$I_n(\boldsymbol{\mu}, \boldsymbol{\Theta}_n) = \sum_{\ell=1}^n \mathbf{1}[v_\ell \text{ is isolated in } \mathbb{G}(n; \boldsymbol{\mu}, \boldsymbol{\Theta}_n)] \quad (11)$$

The method of first moment [18, Eqn. (3.10), p. 55] gives

$$1 - \mathbb{E}[I_n(\boldsymbol{\mu}, \boldsymbol{\Theta}_n)] \leq \mathbb{P}[I_n(\boldsymbol{\mu}, \boldsymbol{\Theta}_n) = 0]$$

A. Preliminaries

Several technical results are collected here for convenience. The first result is established in [6, Proposition 4.1] and follows easily from the scaling condition (5): For any scaling $K_1, K_2, \dots, K_r, P : \mathbb{N}_0 \rightarrow \mathbb{N}_0^{r+1}$, we have

$$\lambda_1(n) \leq \lambda_2(n) \leq \dots \leq \lambda_r(n) \quad (12)$$

for each $n = 2, 3, \dots$. Next, it was shown in [6, Proposition 4.4] that with positive integers K_1, \dots, K_r, P and $a \geq 1$,

$$\frac{\binom{P - \lceil aK_i \rceil}{K_j}}{\binom{P}{K_j}} \leq \left(\frac{\binom{P - K_i}{K_j}}{\binom{P}{K_j}} \right)^a, \quad i, j = 1, \dots, r \quad (13)$$

Moreover, we note from [6, Lemma 4.3] that for any scaling $\Theta : \mathbb{N}_0 \rightarrow \mathbb{N}_0^{r+1} \times (0, 1)$ such that (7) holds, we have

$$c_n \frac{\log n}{n\alpha_n} \leq p_{1r}(n) \leq \frac{c_n \log n}{\mu_r n\alpha_n} \quad (14)$$

Another useful bound that will be used throughout is

$$(1 \pm x) \leq e^{\pm x}, \quad x \in (0, 1) \quad (15)$$

Finally, we find it useful to write

$$\log(1 - x) = -x - \Psi(x) \quad (16)$$

where $\Psi(x) = \int_0^x \frac{t}{1-t} dt$. From L'Hôpital's Rule, we have

$$\lim_{x \rightarrow 0} \frac{\Psi(x)}{x^2} = \frac{-x - \log(1-x)}{x^2} = \frac{1}{2}. \quad (17)$$

B. Establishing the one-law

It is clear that in order to establish the one-law, namely that $\lim_{n \rightarrow \infty} \mathbb{P}[I_n(\boldsymbol{\mu}, \Theta_n) = 0]$, we need to show that $\lim_{n \rightarrow \infty} \mathbb{E}[I_n(\boldsymbol{\mu}, \Theta_n)] = 0$. Recalling (11), we have

$$\begin{aligned} & \mathbb{E}[I_n(\boldsymbol{\mu}, \Theta_n)] \\ &= n \sum_{i=1}^r \mu_i \mathbb{P}[v_1 \text{ is isolated in } \mathbb{G}(n; \boldsymbol{\mu}, \Theta_n) \mid t_1 = i] \\ &= n \sum_{i=1}^r \mu_i \mathbb{P}[\cap_{j=2}^r [v_j \approx v_1] \mid t_1 = i] \\ &= n \sum_{i=1}^r \mu_i (\mathbb{P}[v_2 \approx v_1 \mid t_1 = i])^{n-1} \end{aligned} \quad (18)$$

where (18) follows by the independence of the rvs $\{v_j \approx v_1\}_{j=1}^r$ given Σ_1 . By conditioning on the class of v_2 , we find

$$\begin{aligned} \mathbb{P}[v_2 \approx v_1 \mid t_1 = i] &= \sum_{j=1}^r \mu_j \mathbb{P}[v_2 \approx v_1 \mid t_1 = i, t_2 = j] \\ &= \sum_{j=1}^r \mu_j (1 - \alpha p_{ij}) = 1 - \Lambda_i(n). \end{aligned} \quad (19)$$

Using (19) in (18), and recalling (12), (15) we obtain

$$\begin{aligned} \mathbb{E}[I_n(\boldsymbol{\mu}, \Theta_n)] &= n \sum_{i=1}^r \mu_i (1 - \Lambda_i(n))^{n-1} \\ &\leq n (1 - \Lambda_1(n))^{n-1} \\ &= n \left(1 - c_n \frac{\log n}{n} \right)^{n-1} \\ &\leq e^{\log n (1 - c_n \frac{n-1}{n})} \end{aligned}$$

Taking the limit as n goes to infinity, we immediately get $\lim_{n \rightarrow \infty} \mathbb{E}[I_n(\boldsymbol{\mu}, \Theta_n)] = 0$ since $\lim_{n \rightarrow \infty} (1 - c_n \frac{n-1}{n}) = 1 - c < 0$ under the enforced assumptions (with $c > 1$) and the one-law is established.

C. Establishing the zero-law

Our approach in establishing the zero-law relies on the method of second moment applied to a variable that counts the number of nodes that are class-1 and isolated. Clearly if we can show that whp there exists at least one class-1 node that is isolated under the enforced assumptions (with $c < 1$) then the zero-law would immediately follow.

Let $Y_n(\boldsymbol{\mu}, \Theta_n)$ denote the number of nodes that are class-1 and isolated in $\mathbb{G}(n; \boldsymbol{\mu}, \Theta_n)$, and let

$$x_{n,i}(\boldsymbol{\mu}, \Theta_n) = \mathbf{1}[t_i = 1 \cap v_i \text{ is isolated in } \mathbb{G}(n; \boldsymbol{\mu}, \Theta_n)],$$

then we have $Y_n(\boldsymbol{\mu}, \Theta_n) = \sum_{i=1}^n x_{n,i}(\boldsymbol{\mu}, \Theta_n)$. By applying the method of second moments [18, Remark 3.1, p. 55] on $Y_n(\boldsymbol{\mu}, \Theta_n)$, we get

$$\mathbb{P}[Y_n(\boldsymbol{\mu}, \Theta_n) = 0] \leq 1 - \frac{\mathbb{E}[Y_n(\boldsymbol{\mu}, \Theta_n)]^2}{\mathbb{E}[Y_n(\boldsymbol{\mu}, \Theta_n)]^2} \quad (20)$$

where

$$\mathbb{E}[Y_n(\boldsymbol{\mu}, \Theta_n)] = n \mathbb{E}[x_{n,1}(\boldsymbol{\mu}, \Theta_n)] \quad (21)$$

and

$$\begin{aligned} \mathbb{E}[Y_n(\boldsymbol{\mu}, \Theta_n)^2] &= n \mathbb{E}[x_{n,1}(\boldsymbol{\mu}, \Theta_n)] \\ &\quad + n(n-1) \mathbb{E}[x_{n,1}(\boldsymbol{\mu}, \Theta_n) x_{n,2}(\boldsymbol{\mu}, \Theta_n)] \end{aligned} \quad (22)$$

by exchangeability and the binary nature of the rvs $\{x_{n,i}(\boldsymbol{\mu}, \Theta_n)\}_{i=1}^n$. Using (21) and (22), we get

$$\begin{aligned} \frac{\mathbb{E}[Y_n(\boldsymbol{\mu}, \Theta_n)^2]}{\mathbb{E}[Y_n(\boldsymbol{\mu}, \Theta_n)]^2} &= \frac{1}{n \mathbb{E}[x_{n,1}(\boldsymbol{\mu}, \Theta_n)]} \\ &\quad + \frac{n-1}{n} \frac{\mathbb{E}[x_{n,1}(\boldsymbol{\mu}, \Theta_n) x_{n,2}(\boldsymbol{\mu}, \Theta_n)]}{\mathbb{E}[x_{n,1}(\boldsymbol{\mu}, \Theta_n)]^2} \end{aligned}$$

In order to establish the zero-law, we need to show that

$$\lim_{n \rightarrow \infty} n \mathbb{E}[x_{n,1}(\boldsymbol{\mu}, \Theta_n)] = \infty,$$

and

$$\limsup_{n \rightarrow \infty} \left(\frac{\mathbb{E}[x_{n,1}(\boldsymbol{\mu}, \Theta_n) x_{n,2}(\boldsymbol{\mu}, \Theta_n)]}{\mathbb{E}[x_{n,1}(\boldsymbol{\mu}, \Theta_n)]^2} \right) \leq 1. \quad (23)$$

Proposition 5.1: Consider a scaling $K_1, \dots, K_r, P : \mathbb{N}_0 \rightarrow \mathbb{N}_0^{r+1}$ and a scaling $\alpha : \mathbb{N}_0 \rightarrow (0, 1)$ such that (6) holds with $\lim_{n \rightarrow \infty} c_n = c > 0$. Then, we have

$$\lim_{n \rightarrow \infty} n \mathbb{E}[x_{n,1}(\boldsymbol{\mu}, \Theta_n)] = \infty, \quad \text{if } c < 1$$

Proof. We have

$$\begin{aligned} & n \mathbb{E}[x_{n,1}(\boldsymbol{\mu}, \Theta_n)] \\ &= n \mathbb{E}[\mathbf{1}[t_1 = 1 \cap v_1 \text{ is isolated in } \mathbb{G}(n; \boldsymbol{\mu}, \Theta_n)]] \\ &= n \mu_1 \mathbb{P}[v_1 \text{ is isolated in } \mathbb{G}(n; \boldsymbol{\mu}, \Theta_n) \mid t_1 = 1] \\ &= n \mu_1 \mathbb{P}[\cap_{j=2}^r [v_j \approx v_1] \mid t_1 = 1] \\ &= n \mu_1 \mathbb{P}[v_2 \approx v_1 \mid t_1 = 1]^{n-1} \\ &= n \mu_1 \left(\sum_{j=1}^r \mu_j \mathbb{P}[v_2 \approx v_1 \mid t_1 = 1, t_2 = j] \right)^{n-1} \end{aligned}$$

$$= n\mu_1 \left(\sum_{j=1}^r \mu_j (1 - \alpha_n p_{1j}) \right)^{n-1} \quad (24)$$

$$= n\mu_1 (1 - \Lambda_1(n))^{n-1} = \mu_1 e^{\beta_n} \quad (25)$$

where

$$\beta_n = \log n + (n-1) \log(1 - \Lambda_1(n)).$$

Recalling (16), we get

$$\begin{aligned} \beta_n &= \log n - (n-1) (\Lambda_1(n) + \Psi(\Lambda_1(n))) \\ &= \log n - (n-1) \left(c_n \frac{\log n}{n} + \Psi \left(c_n \frac{\log n}{n} \right) \right) \\ &= \log n \left(1 - c_n \frac{n-1}{n} \right) \\ &\quad - (n-1) \left(c_n \frac{\log n}{n} \right)^2 \frac{\Psi \left(c_n \frac{\log n}{n} \right)}{\left(c_n \frac{\log n}{n} \right)^2} \end{aligned} \quad (26)$$

Recalling (17), we have

$$\lim_{n \rightarrow \infty} \frac{\Psi \left(c_n \frac{\log n}{n} \right)}{\left(c_n \frac{\log n}{n} \right)^2} = \frac{1}{2} \quad (27)$$

since $c_n \frac{\log n}{n} = o(1)$. Thus, $\beta_n = \log n (1 - c_n \frac{n-1}{n}) - o(1)$. Using (25), (26), (27), and letting n go to infinity, we get

$$\lim_{n \rightarrow \infty} n \mathbb{E}[x_{n,1}(\boldsymbol{\mu}, \boldsymbol{\Theta}_n)] = \infty$$

whenever $\lim_{n \rightarrow \infty} c_n = c < 1$. \blacksquare

Proposition 5.2: Consider a scaling $K_1, \dots, K_r, P : \mathbb{N}_0 \rightarrow \mathbb{N}_0^{r+1}$ and a scaling $\alpha : \mathbb{N}_0 \rightarrow (0, 1)$ such that (6) holds with $\lim_{n \rightarrow \infty} c_n = c > 0$. Then, we have (23) if $c < 1$.

Proof. Consider fixed $\boldsymbol{\Theta}$. We have

$$\begin{aligned} &\mathbb{E}[x_{n,1}(\boldsymbol{\mu}, \boldsymbol{\Theta}) x_{n,2}(\boldsymbol{\mu}, \boldsymbol{\Theta})] \\ &= \mathbb{E}[\mathbf{1}[v_1 \text{ is isolated}, v_2 \text{ is isolated} \cap t_1 = 1, t_2 = 1]] \\ &= \mu_1^2 \mathbb{E}[\mathbf{1}[v_1 \text{ is isolated}, v_2 \text{ is isolated}] \mid t_1 = 1, t_2 = 1] \\ &= \mu_1^2 \mathbb{E} \left[\mathbf{1}[v_1 \approx v_2] \prod_{m=3}^n \mathbf{1}[v_m \approx v_1, v_m \approx v_2] \mid t_1 = t_2 = 1 \right] \end{aligned}$$

Now we condition on Σ_1 and Σ_2 and note that i) Σ_1 and Σ_2 determine t_1 and t_2 ; and ii) the events $[v_1 \approx v_2], \{[v_m \approx v_1 \cap v_m \approx v_2]\}_{m=3}^n$ are mutually independent given Σ_1 and Σ_2 . Thus, we have

$$\begin{aligned} &\mathbb{E}[x_{n,1}(\boldsymbol{\mu}, \boldsymbol{\Theta}) x_{n,2}(\boldsymbol{\mu}, \boldsymbol{\Theta})] \\ &= \mu_1^2 \mathbb{E} \left[\mathbb{P} \left[v_1 \approx v_2 \mid \Sigma_1, \Sigma_2 \right] \times \right. \\ &\quad \left. \prod_{m=3}^n \mathbb{P} \left[v_m \approx v_1 \cap v_m \approx v_2 \mid \Sigma_1, \Sigma_2 \right] \mid t_1 = t_2 = 1 \right] \end{aligned} \quad (28)$$

Define the $\{0, 1\}$ -valued rv $u(\boldsymbol{\theta})$ by

$$u(\boldsymbol{\theta}) := \mathbf{1}[\Sigma_1 \cap \Sigma_2 \neq \emptyset]. \quad (29)$$

Next, with $\ell = 1, 2, \dots, n-1$, define $\nu_{\ell,j}(\alpha)$ by

$$\nu_{\ell,j}(\alpha) := \{i = 1, 2, \dots, \ell : B_{ij}(\alpha) = 1\} \quad (30)$$

for each $j = \ell+1, \dots, n$. Namely, $\nu_{\ell,j}(\alpha)$ is the set of nodes in $\{v_1, \dots, v_\ell\}$ whose *potential* link with node v_j will not fail. With these definitions in mind, (28) gives

$$\begin{aligned} &\mathbb{E}[x_{n,1}(\boldsymbol{\mu}, \boldsymbol{\Theta}) x_{n,2}(\boldsymbol{\mu}, \boldsymbol{\Theta})] \\ &= \mu_1^2 \mathbb{E} \left[(1 - \alpha)^{u(\boldsymbol{\theta})} \prod_{m=3}^n \frac{\binom{P - |\cup_{i \in \nu_{2,m}(\alpha)} \Sigma_i|}{|\Sigma_m|}}{\binom{P}{|\Sigma_m|}} \mid t_1 = t_2 = 1 \right] \end{aligned}$$

Conditioned on $u(\boldsymbol{\theta}) = 0$ and v_1, v_2 being class-1, we have

$$|\cup_{i \in \nu_{2,m}(\alpha)} \Sigma_i| = |\nu_{2,m}(\alpha)| K_1.$$

Also, we have

$$\mathbb{P}[u(\boldsymbol{\theta}_n) = 0 \mid t_1 = t_2 = 1] = 1 - p_{11}.$$

Thus, we get

$$\begin{aligned} &\mathbb{E}[x_{n,1}(\boldsymbol{\mu}, \boldsymbol{\Theta}) x_{n,2}(\boldsymbol{\mu}, \boldsymbol{\Theta}) \mathbf{1}[u(\boldsymbol{\theta}) = 0]] \\ &= \mu_1^2 (1 - p_{11}) \mathbb{E} \left[\prod_{m=3}^n \frac{\binom{P - |\nu_{2,m}(\alpha) K_1|}{|\Sigma_m|}}{\binom{P}{|\Sigma_m|}} \right] \\ &= \mu_1^2 (1 - p_{11}) \mathbb{E} \left[\frac{\binom{P - |\nu_{2,3}(\alpha) K_1|}{|\Sigma_3|}}{\binom{P}{|\Sigma_3|}} \right]^{n-2} \\ &= \mu_1^2 (1 - p_{11}) \left(\sum_{j=1}^r \mu_j \mathbb{E} \left[\frac{\binom{P - |\nu_{2,3}(\alpha) K_1|}{|\Sigma_3|}}{\binom{P}{|\Sigma_3|}} \mid t_3 = j \right] \right)^{n-2} \\ &= \mu_1^2 (1 - p_{11}) \left(\sum_{j=1}^r \mu_j \mathbb{E} \left[\frac{\binom{P - |\nu_{2,3}(\alpha) K_j|}{K_j}}{\binom{P}{K_j}} \right] \right)^{n-2} \\ &\leq \mu_1^2 (1 - p_{11}) \mathbb{E} \left[\sum_{j=1}^r \mu_j \left(\frac{\binom{P - K_1}{K_j}}{\binom{P}{K_j}} \right)^{|\nu_{2,3}(\alpha)|} \right]^{n-2} \end{aligned}$$

where we use (13) in the last step.

Now, let $Z(\boldsymbol{\theta})$ denote a rv that takes the value

$$\frac{\binom{P - K_1}{K_j}}{\binom{P}{K_j}} \quad \text{with probability} \quad \mu_j, \quad j = 1, \dots, r$$

In other words, $Z(\boldsymbol{\theta}) = 1 - p_{1j}$ with probability μ_j so that $\mathbb{E}[Z(\boldsymbol{\theta})] = 1 - \lambda_1$. Then,

$$\begin{aligned} &\mathbb{E}[x_{n,1}(\boldsymbol{\mu}, \boldsymbol{\Theta}) x_{n,2}(\boldsymbol{\mu}, \boldsymbol{\Theta}) \mathbf{1}[u(\boldsymbol{\theta}) = 0]] \\ &\leq \mu_1^2 (1 - p_{11}) \mathbb{E} \left[Z(\boldsymbol{\theta})^{|\nu_{2,3}(\alpha)|} \right]^{n-2} \end{aligned} \quad (31)$$

Under the independent Bernoulli link-failure model, we have that $|\nu_{2,3}(\alpha)|$ is a Binomial rv, i.e., $|\nu_{2,3}(\alpha)| =_{st} \text{Bin}(2, \alpha)$. Hence,

$$\mu_1^2 (1 - p_{11}) \mathbb{E} \left[Z(\boldsymbol{\theta})^{|\nu_{2,3}(\alpha)|} \right]^{n-2} \quad (32)$$

$$\begin{aligned}
&= \mathbb{E} \left[\sum_{i=0}^2 \binom{2}{i} \alpha^2 (1-\alpha)^{2-i} Z(\boldsymbol{\theta})^i \right]^{n-2} \\
&= \mu_1^2 (1-p_{11}) \mathbb{E} \left[(1-\alpha)^2 + 2\alpha(1-\alpha)Z(\boldsymbol{\theta}) + \alpha^2 Z(\boldsymbol{\theta})^2 \right]^{n-2}
\end{aligned}$$

Conditioning on $u(\boldsymbol{\theta}) = 1$ and $t_1 = t_2 = 1$, we have

$$|\cup_{i \in \nu_{2,m}(\alpha)} \Sigma_i| = \begin{cases} 0 & \text{if } |\nu_{2,m}(\alpha)| = 0 \\ K_1 & \text{if } |\nu_{2,m}(\alpha)| = 1 \\ 2K_1 - |\Sigma_1 \cap \Sigma_2| & \text{if } |\nu_{2,m}(\alpha)| = 2 \end{cases}$$

and by a crude bounding argument, we have

$$|\cup_{i \in \nu_{2,m}(\alpha)} \Sigma_i| \geq K_1 \mathbf{1}[|\nu_{2,m}(\alpha)| > 0] \quad (33)$$

Using (33) and recalling the analysis for $\mathbb{E}[x_{n,1}(\boldsymbol{\mu}, \boldsymbol{\Theta}) x_{n,2}(\boldsymbol{\mu}, \boldsymbol{\Theta}) \mathbf{1}[u(\boldsymbol{\theta}) = 0]]$, we obtain

$$\begin{aligned}
&\mathbb{E}[x_{n,1}(\boldsymbol{\mu}, \boldsymbol{\Theta}) x_{n,2}(\boldsymbol{\mu}, \boldsymbol{\Theta}) \mathbf{1}[u(\boldsymbol{\theta}) = 1]] \\
&\leq \mu_1^2 p_{11} (1-\alpha) \mathbb{E} \left[Z(\boldsymbol{\theta})^{\mathbf{1}[|\nu_{2,3}(\alpha)| > 0]} \right]^{n-2} \\
&= \mu_1^2 p_{11} (1-\alpha) \mathbb{E} \left[(1-\alpha)^2 + (1-(1-\alpha)^2) Z_n \right]^{n-2} \quad (34)
\end{aligned}$$

Combining (31), (32), and (34), we get

$$\begin{aligned}
&\mathbb{E}[x_{n,1}(\boldsymbol{\mu}, \boldsymbol{\Theta}) x_{n,2}(\boldsymbol{\mu}, \boldsymbol{\Theta})] \\
&= \mathbb{E}[x_{n,1}(\boldsymbol{\mu}, \boldsymbol{\Theta}) x_{n,2}(\boldsymbol{\mu}, \boldsymbol{\Theta}) (\mathbf{1}[u(\boldsymbol{\theta}) = 0] + \mathbf{1}[u(\boldsymbol{\theta}) = 1])] \\
&\leq \mu_1^2 (1-p_{11}) \left((1-\alpha)^2 + 2\alpha(1-\alpha) \mathbb{E}[Z(\boldsymbol{\theta})] \right. \\
&\quad \left. + \alpha^2 \mathbb{E}[Z(\boldsymbol{\theta})^2] \right)^{n-2} + \mu_1^2 p_{11} (1-\alpha) \left((1-\alpha)^2 \right. \\
&\quad \left. + (1-(1-\alpha)^2) \mathbb{E}[Z(\boldsymbol{\theta})] \right)^{n-2} \quad (35)
\end{aligned}$$

It is clear from (24) and the definition of $Z(\boldsymbol{\theta})$ that

$$\begin{aligned}
\mathbb{E}[x_{n,1}(\boldsymbol{\mu}, \boldsymbol{\Theta})] &= \mu_1 \left(\sum_{j=1}^r \mu_j (1-\alpha p_{1j}) \right)^{n-1} \\
&= \mu_1 ((1-\alpha) + \alpha \mathbb{E}[Z(\boldsymbol{\theta})])^{n-1} \quad (36)
\end{aligned}$$

Combining (35) and (36), we get

$$\begin{aligned}
&\frac{\mathbb{E}[x_{n,1}(\boldsymbol{\mu}, \boldsymbol{\Theta}) x_{n,2}(\boldsymbol{\mu}, \boldsymbol{\Theta})]}{\mathbb{E}[x_{n,1}(\boldsymbol{\theta})]^2} \\
&\leq (1-p_{11}) \frac{((1-\alpha)^2 + 2\alpha(1-\alpha) \mathbb{E}[Z(\boldsymbol{\theta})] + \alpha^2 \mathbb{E}[Z(\boldsymbol{\theta})^2])^{n-2}}{(1-\alpha + \alpha \mathbb{E}[Z(\boldsymbol{\theta})])^{2(n-1)}} \\
&\quad + p_{11} \frac{((1-\alpha)^2 + (1-(1-\alpha)^2) \mathbb{E}[Z(\boldsymbol{\theta})])^{n-2}}{(1-\alpha + \alpha \mathbb{E}[Z(\boldsymbol{\theta})])^{2(n-1)}} \\
&:= A + B \quad (37)
\end{aligned}$$

where we use the fact that $1-\alpha \leq 1$.

We now consider a scaling $\boldsymbol{\Theta} : \mathbb{N}_0 \rightarrow \mathbb{N}_0^{r+1} \times (0, 1)$ as stated in Proposition 5.2 and bound the terms A and B in turn. Our goal is to show that

$$\limsup_{n \rightarrow \infty} (A + B) \leq 1. \quad (38)$$

First, we write $\mathbb{E}[Z(\boldsymbol{\theta}_n)^2] = \mathbb{E}[Z(\boldsymbol{\theta}_n)]^2 + \text{var}[Z(\boldsymbol{\theta}_n)]$, where $\text{var}[Z(\boldsymbol{\theta}_n)]$ can be bounded by the Popoviciu's inequality [19, p. 9] as follows

$$\begin{aligned}
\text{var}[Z(\boldsymbol{\theta}_n)] &\leq \frac{1}{4} (\max(Z(\boldsymbol{\theta}_n)) - \min(Z(\boldsymbol{\theta}_n)))^2 \\
&= \frac{1}{4} \left(\frac{\binom{P_n - K_{1,n}}{K_{1,n}}}{\binom{P_n}{K_{1,n}}} - \frac{\binom{P_n - K_{1,n}}{K_{r,n}}}{\binom{P_n}{K_{r,n}}} \right)^2 \\
&\leq \frac{1}{4} \left(1 - \frac{\binom{P_n - K_{1,n}}{K_{r,n}}}{\binom{P_n}{K_{r,n}}} \right)^2 = \frac{1}{4} (p_{1r}(n))^2.
\end{aligned}$$

Then, we get from the scaling condition (7) and the immediate bound (14) that

$$\mathbb{E}[Z(\boldsymbol{\theta}_n)^2] \leq \mathbb{E}[Z(\boldsymbol{\theta}_n)]^2 + \frac{1}{4} \left(\frac{c_n \log n}{\mu_r n \alpha_n} \right)^2$$

Reporting this into (37) we get

A

$$\begin{aligned}
&\leq (1-p_{11}) \frac{\left(((1-\alpha_n) + \alpha_n \mathbb{E}[Z(\boldsymbol{\theta}_n)])^2 + \left(\frac{c_n \log n}{2\mu_r n} \right)^2 \right)^{n-2}}{((1-\alpha_n) + \alpha_n \mathbb{E}[Z(\boldsymbol{\theta}_n)])^{2(n-1)}} \\
&= (1+o(1))(1-p_{11}) \left(1 + \left(\frac{\frac{c_n \log n}{2\mu_r n}}{1-\alpha_n + \alpha_n \mathbb{E}[Z(\boldsymbol{\theta}_n)]} \right)^2 \right)^{n-2}
\end{aligned}$$

where we used the fact that

$$((1-\alpha_n) + \alpha_n \mathbb{E}[Z(\boldsymbol{\theta}_n)])^2 = (1-\alpha_n \lambda_1(n))^2 = 1 - o(1) \quad (39)$$

since $\alpha_n \lambda_1(n) = c_n \log n / n$. Finally, we have

$$\begin{aligned}
&\left(1 + \left(\frac{\frac{c_n \log n}{2\mu_r n}}{1-\alpha_n + \alpha_n \mathbb{E}[Z(\boldsymbol{\theta}_n)]} \right)^2 \right)^{n-2} \\
&\leq \exp \left\{ n \left(\frac{\frac{c_n \log n}{2\mu_r n}}{1-c_n \frac{\log n}{n}} \right)^2 \right\} = o(1)
\end{aligned}$$

since $\lim_{n \rightarrow \infty} c_n = c > 0$ and $\mu_r > 0$. Thus, we obtain the bound

$$A \leq (1-p_{11})(1+o(1)). \quad (40)$$

We now consider the second term in (37). Recall (39) and that $\mathbb{E}[Z(\boldsymbol{\theta}_n)] = 1 - \lambda_1(n) = 1 - c_n \log n / n$. We have

$$\begin{aligned}
B &= \frac{p_{11}}{(1-\alpha_n + \alpha_n \mathbb{E}[Z(\boldsymbol{\theta}_n)])^2} \\
&\quad \times \left(1 + \frac{\alpha_n^2 \mathbb{E}[Z(\boldsymbol{\theta}_n)](1-\mathbb{E}[Z(\boldsymbol{\theta}_n)])}{(1-\alpha_n + \alpha_n \mathbb{E}[Z(\boldsymbol{\theta}_n)])^2} \right)^{n-2} \\
&\leq p_{11} (1+o(1)) \exp \left\{ n \frac{\alpha_n^2 c_n \frac{\log n}{n} (1-c_n \frac{\log n}{n})}{(1-c_n \frac{\log n}{n})^2} \right\} \\
&\leq p_{11} (1+o(1)) \exp \left\{ \frac{c_n \alpha_n \log n}{\left(1-c_n \frac{\log n}{n}\right)^2} \right\} \quad (41)
\end{aligned}$$

We will now establish the desired result (38) by using (40) and (41). Our approach is based on the subsubsequence principle [18, p. 12] and considering the cases $\lim_{n \rightarrow \infty} \alpha_n \log n = 0$ and $\lim_{n \rightarrow \infty} \alpha_n \log n \in (0, \infty]$ separately.

a) Assume that $\lim_{n \rightarrow \infty} \alpha_n \log n = 0$: From (41) we get $B \leq (1 + o(1))p_{11}$ and upon using (40) we see that $A + B \leq (1 + o(1))$ establishing (38) along subsequences with $\lim_{n \rightarrow \infty} \alpha_n \log n = 0$.

b) Assume that $\lim_{n \rightarrow \infty} \alpha_n \log n \in (0, \infty]$: Since p_{1j} is monotonically increasing in $j = 1, \dots, r$ (see (12)), we have

$$\lambda_1 = \sum_{j=1}^r \mu_j p_{1j} \geq p_{11} \sum_{j=1}^r \mu_j = p_{11}$$

Thus, $p_{11} \leq \lambda_1(n) = c_n \log n / (\alpha_n n)$. Then, (41) gives

$$\begin{aligned} B &\leq (1 + o(1)) \frac{c_n \log n}{\alpha_n n} \exp \left\{ \frac{c_n \alpha_n \log n}{(1 - c_n \log n/n)^2} \right\} \\ &= (1 + o(1)) \frac{c_n (\log n)^2}{\alpha_n \log n} n^{-1 + \frac{c_n \alpha_n}{(1 - c_n \log n/n)^2}} \\ &= o(1) \end{aligned}$$

since $\lim_{n \rightarrow \infty} \alpha_n \log n > 0$ along this subsequence and

$$\lim_{n \rightarrow \infty} -1 + \frac{c_n \alpha_n}{(1 - c_n \log n/n)^2} < 0$$

given that $\lim_{n \rightarrow \infty} c_n = c < 1$. From (40) and the fact that $p_{11} \leq 1$, we have $A \leq 1 + o(1)$, and (38) follows.

The two cases considered cover all the possibilities for the limit of $\alpha_n \log n$. By virtue of the subsubsequence principle [18, p. 12], we get (38) without any condition on the sequence $\alpha_n \log n$; i.e., we obtain (38) even when the sequence $\alpha_n \log n$ does not have a limit! ■

VI. PROOF OF THEOREM 3.2

The proof of Theorem 3.2 is lengthy and technically involved. Therefore, we omit most of the details in this conference version. All details can be found in [20]. In this section, we present an outline of our proof. Let $C_n(\boldsymbol{\mu}, \boldsymbol{\Theta}_n)$ denote the event that the graph $\mathbb{G}(n, \boldsymbol{\mu}, \boldsymbol{\Theta}_n)$ is connected, and with a slight abuse of notation, let $I_n(\boldsymbol{\mu}, \boldsymbol{\Theta}_n)$ denote the event that the graph $\mathbb{G}(n, \boldsymbol{\mu}, \boldsymbol{\Theta}_n)$ has no isolated nodes. Clearly, if a random graph is connected then it does not have any isolated node, hence $C_n(\boldsymbol{\mu}, \boldsymbol{\Theta}_n) \subseteq I_n(\boldsymbol{\mu}, \boldsymbol{\Theta}_n)$ and we get

$$\mathbb{P}[C_n(\boldsymbol{\mu}, \boldsymbol{\Theta}_n)] \leq \mathbb{P}[I_n(\boldsymbol{\mu}, \boldsymbol{\Theta}_n)] \quad (42)$$

and

$$\mathbb{P}[C_n(\boldsymbol{\mu}, \boldsymbol{\Theta}_n)^c] = \mathbb{P}[I_n(\boldsymbol{\mu}, \boldsymbol{\Theta}_n)^c] + \mathbb{P}[C_n(\boldsymbol{\mu}, \boldsymbol{\Theta}_n)^c \cap I_n(\boldsymbol{\mu}, \boldsymbol{\Theta}_n)]. \quad (43)$$

In view of (42), we obtain the zero-law for connectivity, i.e., that

$$\lim_{n \rightarrow \infty} \mathbb{P}[\mathbb{G}(n; \boldsymbol{\mu}, \boldsymbol{\Theta}_n) \text{ is connected}] = 0 \quad \text{if } c < 1,$$

immediately from the zero-law part of Theorem 3.1, i.e., from that $\lim_{n \rightarrow \infty} \mathbb{P}[I_n(\boldsymbol{\mu}, \boldsymbol{\Theta}_n)] = 0$ if $c < 1$. It remains to establish the one-law for connectivity. From Theorem 3.1 and (43), we see that the one-law for connectivity, i.e., that

$$\lim_{n \rightarrow \infty} \mathbb{P}[\mathbb{G}(n; \boldsymbol{\mu}, \boldsymbol{\Theta}_n) \text{ is connected}] = 1 \quad \text{if } c > 1,$$

will follow if we show that

$$\lim_{n \rightarrow \infty} \mathbb{P}[C_n(\boldsymbol{\mu}, \boldsymbol{\Theta}_n)^c \cap I_n(\boldsymbol{\mu}, \boldsymbol{\Theta}_n)] = 0. \quad (44)$$

The proof of the one-law passes through obtaining a proper upper bound for (44) and then showing that the bound goes to zero as n gets to infinity (with $c > 1$) under appropriate conditions of the parameter scalings. Due to space limitations, the details of this technically involved result are given in [20].

REFERENCES

- [1] I. Akyildiz, W. Su, Y. Sankarasubramanian, and E. Cayirci, "A survey on sensor networks," *IEEE Communications Magazine*, vol. 40, no. 8, pp. 102–114, Aug 2002.
- [2] L. Eschenauer and V. D. Gligor, "A key-management scheme for distributed sensor networks," in *Proceedings of ACM CCS*, 2002, pp. 41–47.
- [3] A. Perrig, J. Stankovic, and D. Wagner, "Security in wireless sensor networks," *Commun. ACM*, vol. 47, no. 6, pp. 53–57, Jun. 2004.
- [4] X. Du and H.-H. Chen, "Security in wireless sensor networks," *Wireless Communications, IEEE*, vol. 15, no. 4, pp. 60–66, Aug 2008.
- [5] S. A. Çamtepe and B. Yener, "Key distribution mechanisms for wireless sensor networks: a survey," *Rensselaer Polytechnic Institute, Troy, New York, Technical Report*, pp. 05–07, 2005.
- [6] O. Yağan, "Zero-one laws for connectivity in inhomogeneous random key graphs," *ArXiv e-prints*, Aug 2015, available online at <http://arxiv.org/abs/1508.02407>.
- [7] M. Yarvis, N. Kushalnagar, H. Singh, A. Rangarajan, Y. Liu, and S. Singh, "Exploiting heterogeneity in sensor networks," in *Proceedings IEEE INFOCOM 2005*, vol. 2, March 2005, pp. 878–890 vol. 2.
- [8] C. J. Colbourn, "Combinatorial aspects of network reliability," *Annals of Operations Research*, vol. 33, no. 1, pp. 1–15.
- [9] B. Bollobás, *Random graphs*. Cambridge university press, 2001, vol. 73.
- [10] L. Valiant, "The complexity of enumeration and reliability problems," *SIAM Journal on Computing*, vol. 8, no. 3, pp. 410–421, 1979.
- [11] J. Provan and M. Ball, "The complexity of counting cuts and of computing the probability that a graph is connected," *SIAM Journal on Computing*, vol. 12, no. 4, pp. 777–788, 1983.
- [12] J. Zhao, O. Yağan, and V. Gligor, "On the strengths of connectivity and robustness in general random intersection graphs," in *IEEE Annual Conference on Decision and Control (CDC)*, 2014, pp. 3661–3668.
- [13] O. Yağan, "Performance of the Eschenauer-Gligor key distribution scheme under an ON/OFF channel," *IEEE Transactions on Information Theory*, vol. 58, no. 6, pp. 3821–3835, June 2012.
- [14] O. Yağan and A. M. Makowski, "Zero-one laws for connectivity in random key graphs," *IEEE Transactions on Information Theory*, vol. 58, no. 5, pp. 2983–2999, 2012.
- [15] M. Bloznelis, J. Jaworski, and K. Rybarczyk, "Component evolution in a secure wireless sensor network," *Netw.*, vol. 53, pp. 19–26, Jan. 2009.
- [16] E. Godehardt and J. Jaworski, "Two models of random intersection graphs for classification," in *Exploratory data analysis in empirical research*. Springer, 2003, pp. 67–81.
- [17] R. Di Pietro, L. V. Mancini, A. Mei, A. Panconesi, and J. Radhakrishnan, "Redoubtable sensor networks," *ACM Trans. Inf. Syst. Secur.*, vol. 11, no. 3, pp. 13:1–13:22, Mar. 2008.
- [18] S. Janson, T. Łuczak, and A. Ruciński, *Random Graphs. 2000*. Wiley–Intersci. Ser. Discrete Math. Optim, 2000.
- [19] S. T. Jensen, "The laguerre-samuels inequality with extensions and applications in statistics and matrix theory," Ph.D. dissertation, Department of Mathematics and Statistics, McGill University, 1999.
- [20] R. Eletreby and O. Yağan, "Reliability of wireless sensor networks under a heterogeneous key predistribution scheme," available online at <http://www.andrew.cmu.edu/user/reletreby/papers/CDCLongVersion.pdf>.