SPIRAL: Tuning DSP Transforms to Computing Platforms

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Sponsor

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Moore's Law and High(est) Performance Scientific Computing

(single processor, off-the-shelf)

Moore's Law:

- processor-memory bottleneck
- short life cycles of computers
- very complex architectures
  - vendor specific
  - special instructions (MMX, SSE, FMA, …)
  - undocumented features

Effects on software/algorithms:

- arithmetic cost model not accurate for predicting runtime (one cache miss = 10 floating point ops)
- better performance models hard to get
- best code is machine dependent (registers/caches size, structure)
- hand-tuned code becomes obsolete as fast as it is written
- compiler limitations
- full performance requires (in part) assembly coding

➡ **Portable performance requires automation**

Automatic Performance Tuning: Research

Linear Algebra:

- ATLAS (J. Dongarra et al.)
- LAPACK
- PhiPACK (J. Demmel et al.)

Signal Processing:

- FFTW (M. Frigo and S. Johnson)
- SPIRAL

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Automates

Implementation
- cuts development costs
- code less error-prone

Optimization
- systematic exploration of alternatives both at algorithmic and code level

Platform-Adaptation
- takes advantage of architecture specific features
- porting without loss of performance

of DSP algorithms
- are performance critical

A library generator for highly optimized signal processing algorithms

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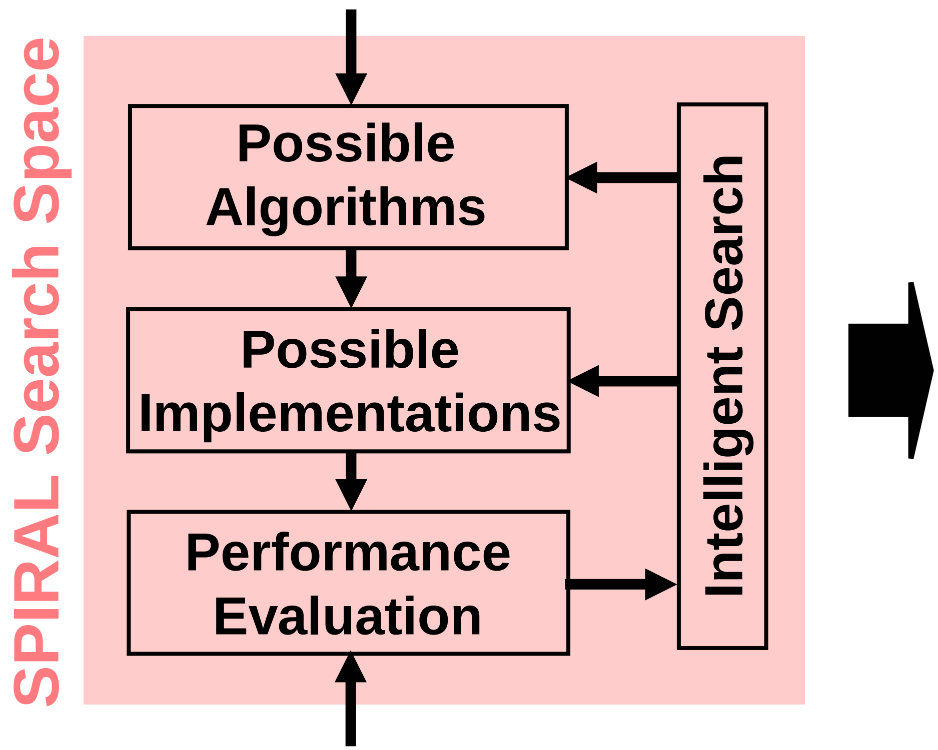
given → **DSP Transform**

(DFT, DCT, Wavelets etc.)



→ **adapted implementation**

given → **Computing Platform**

(Pentium III, Pentium 4, Athlon, SUN, PowerPC, Alpha, …)

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Organization

- **Mathematical Framework**

  Transforms, Rules, and Formulas
- Formula Generator

  Transform $\rightarrow$ Algorithm
- SPL and SPL Compiler

  Algorithm $\rightarrow$ Implementation
- Search Engine

  How to find the best implementation
- SPIRAL system

  Everything taken together
- Conclusions

DSP Algorithms: Example 4-point DFT

Cooley/Tukey FFT (size 4):

$$\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & i & -1 & -i\\ 1 & -1 & 1 & -1\\ 1 & -i & -1 & i\end{bmatrix} = \begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & -1 & 0\\ 0 & 1 & 0 & -1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & i\end{bmatrix}\begin{bmatrix}1 & 1 & 0 & 0\\ 1 & -1 & 0 & 0\\ 0 & 0 & 1 & 1\\ 0 & 0 & 1 & -1\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}$$

Fourier transform

Diagonal matrix (twiddles)

$$DFT_4 = (DFT_2 \otimes I_2) \cdot T_2^4 \cdot (I_2 \otimes DFT_2) \cdot L_2^4$$

Kronecker product

Identity

Permutation

- product of structured sparse matrices
- mathematical notation

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DSP Algorithms: Terminology

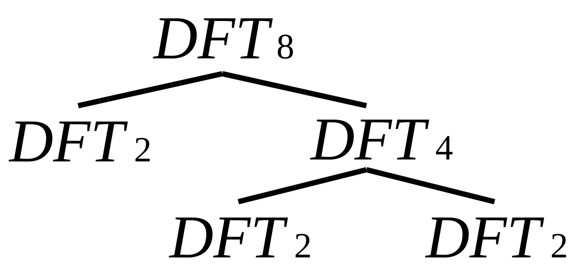
Transform DFT_n **parameterized matrix**

Rule

$$DFT_{nm} \rightarrow (DFT_n \otimes I_m) \cdot D \cdot (I_n \otimes DFT_m) \cdot P$$

- a breakdown strategy
- product of sparse matrices

Ruletree



- recursive application of rules
- uniquely defines an algorithm
- efficient representation
- easy manipulation

Formula

$$DFT_8 = (F_2 \otimes I_4) \cdot D \cdot (I_2 \otimes (I_2 \otimes F_2 \cdots)) \cdot P$$

- few constructs and primitives
- uniquely defines an algorithm
- can be translated into code

More Cooley-Tukey Rules

- **DFT is symmetric $\Rightarrow$ transpose the rule:**

$$F_{RS} = L_S^{RS}\left(I_R \otimes F_S\right) T_S^{RS}\left(F_R \otimes I_S\right)$$

CT rule transposed

- **Commuting tensor product factors**

$$B \otimes A = L_n^{mn}\left(A \otimes B\right) L_m^{mn}$$

A* and *B* square size *m* and *n

- **Commutation property $\Rightarrow$ further variations**

$$F_N = L_S^{RS}\left(I_S \otimes F_R\right) L_R^{RS} T_S^{RS}\left(I_R \otimes F_S\right) L_R^{RS}$$

$$F_N = \left(F_R \otimes I_S\right) T_S^{RS} L_S^{RS}\left(F_S \otimes I_R\right)$$

$$\left(F_2 \otimes I_2\right) x = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \hline x_2 \\ x_3 \end{bmatrix} \quad \left(I_2 \otimes F_2\right) = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ \hline x_2 \\ x_3 \end{bmatrix}$$

- **Different patterns for access, storage and flow of data**

DSP Transforms

discrete Fourier transform

$$DFT_n = \left[\exp\left(2kli\pi / n\right)\right]$$

Walsh-Hadamard transform

$$WHT_{2^k} = DFT_2 \otimes \cdots \otimes DFT_2$$

discrete cosine and sine Transforms (16 types)

$$DCT^{(II)}{}_n = \left[\cos\left(k\left(l + 1/2\right)\pi / n\right)\right]$$

$$DCT^{(IV)}{}_n = \left[\cos\left(\left(k + 1/2\right)\left(l + 1/2\right)\pi / n\right)\right]$$

$$DST^{(I)}{}_n = \left[\sin\left(kl\pi / n\right)\right]$$

modified discrete cosine transform

$$MDCT_{n \times 2n} = \left[\cos\left(\left(k + (n+1)/2\right)\left(l + 1/2\right)\pi / n\right)\right]$$

two-dimensional transform

$$T \otimes T$$

discrete wavelet transform

$$\mathrm{DTWT}_{2^n} = \left(\mathrm{DTWT}_{2^{n-1}} \oplus \mathrm{I}_{2^{n-1}}\right) \underbrace{\mathrm{L}^{2^n}_{2^{n-1}} \left(\mathrm{I}_{2^{n-1}} \otimes_{l-2} W\right)}_{H}$$

Others: filtering, Haar, Hartley, …

Rules = Breakdown Strategies

$$DCT_2^{(II)} \rightarrow diag\left(1, 1/\sqrt{2}\right) \cdot F_2$$
base case

$$DCT_n^{(II)} \rightarrow P \cdot \left(DCT_{n/2}^{(II)} \oplus DCT_{n/2}^{(IV)}\right) \cdot \left(I_{n/2} \otimes F_2\right)^Q$$
recursive

$$DCT_n^{(IV)} \rightarrow S \cdot DCT_n^{(II)} \cdot D$$
translation

$$DCT_n^{(IV)} \rightarrow M_1 \cdots M_r$$
iterative

$$DFT_n \rightarrow CosDFT_n + j \cdot SinDFT_n$$
recursive

$$DFT_n \rightarrow B \cdot (DCT_{n/2}^{(I)} \oplus DST_{n/2}^{(I)}) \cdot C$$
recursive

$$DFT_{nm} \rightarrow (DFT_n \otimes I_m) \cdot D \cdot (I_n \otimes DFT_m) \cdot P$$
recursive

$$CosDFT_n \rightarrow \cdots CosDFT_{n/2} \cdots DCT_{n/4}^{(II)} \cdots$$
recursive

$$SinDFT_n \rightarrow \cdots SinDFT_{n/2} \cdots DCT_{n/4}^{(II)} \cdots$$
recursive

$$WHT_{2^n} \rightarrow \prod_{i=1}^{n} (I_{2^{n_1+\ldots+n_{i-1}}} \otimes WHT_{2^{n_i}} \otimes I_{2^{n_{i+1}+\ldots+n_t}})$$
iterative/ recursive

$$MDCT_{n \times 2n} \rightarrow S \cdot DCT_n^{(IV)} \cdot P$$
translation

$$\mathrm{DTWT}_{2^n} = \left(\mathrm{DTWT}_{2^{n-1}} \oplus \mathrm{I}_{2^{n-1}}\right) \underbrace{\mathrm{L}_{2^{n-1}}^{2^n}\left(\mathrm{I}_{2^{n-1}} \otimes_{l-2} W\right)}_{H}$$

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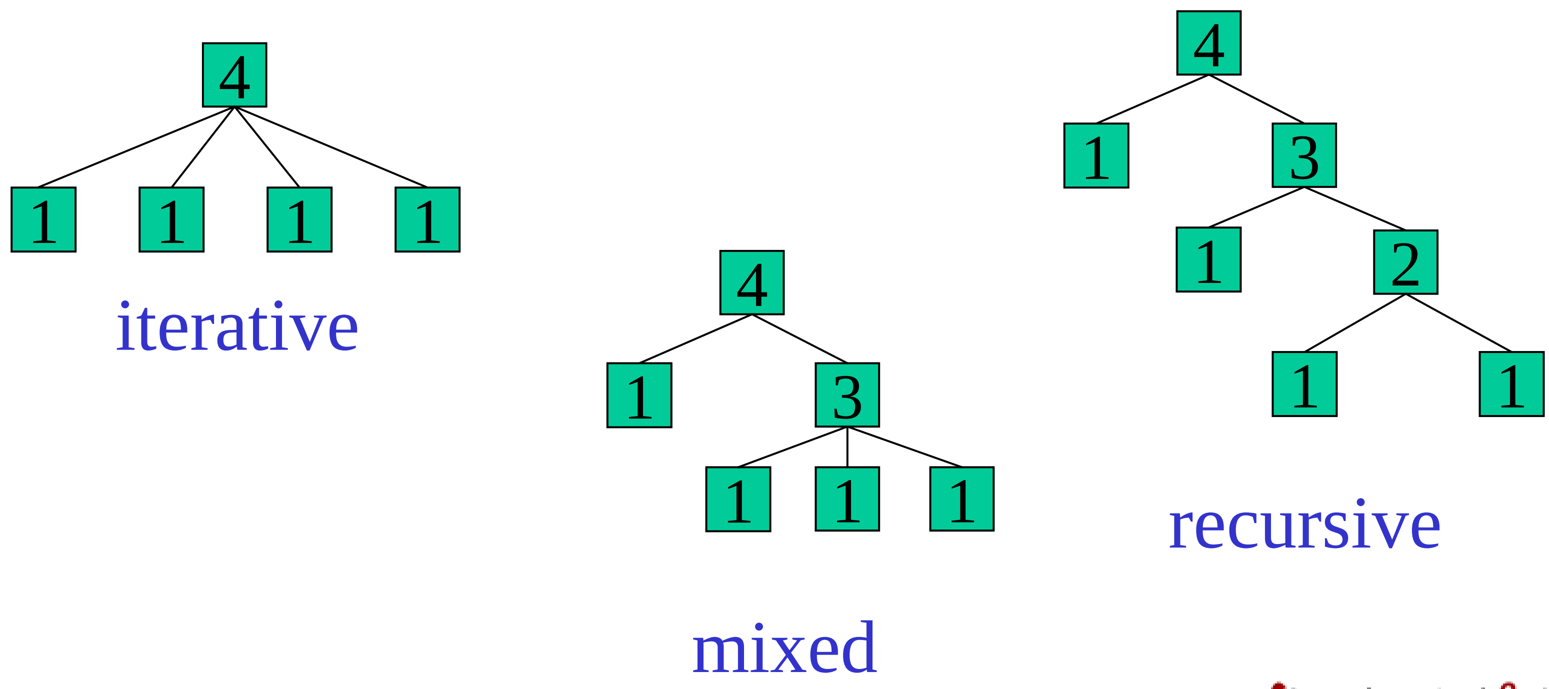
Partition Trees

Each WHT algorithm can be represented by a tree, where a node labeled by n corresponds to WHT_{2^n}



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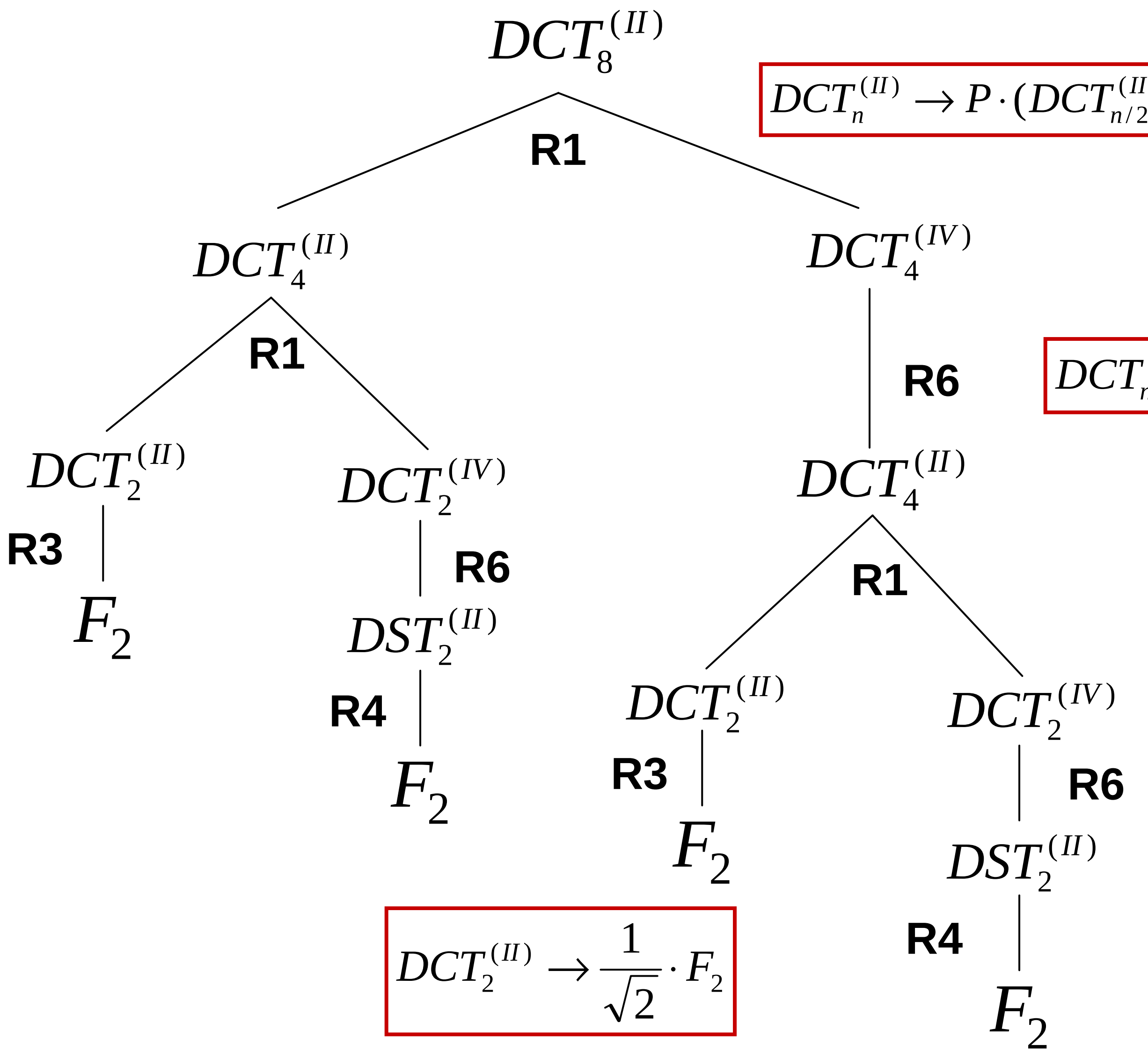
Algorithms = Ruletrees = Formulas



$$\boxed{DCT_n^{(II)} \rightarrow P \cdot (DCT_{n/2}^{(II)} \oplus DCT_{n/2}^{(IV)}) \cdot (F_2 \otimes I_{n/2})}$$

$$\boxed{DCT_n^{(IV)} \rightarrow P \cdot DCT_n^{(II)} \cdot S}$$

$$\boxed{DCT_2^{(II)} \rightarrow \frac{1}{\sqrt{2}} \cdot F_2}$$

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Mathematical Framework for Filters and Discrete-Time Wavelet Transforms

- **Overlapped direct sum of matrices A (*m* x *n*) and B (*r* x *s*)**

column overlap $$A \oplus_v B = \begin{bmatrix} \boxed{A} & \\ & \overset{\overbrace{\ }^{v}}{\boxed{B}} \end{bmatrix} = (A \oplus B)(I_n \oplus_v I_s)$$

row overlap $$A \oplus^v B = \begin{bmatrix} \boxed{A} & \}^{v}\,\boxed{B} \end{bmatrix} = (I_m \oplus^v I_r)(A \oplus B)$$

- **Overlapped tensor product**

column overlap $$I_k \otimes_v B = \underbrace{B \oplus_v B \oplus_v \cdots \oplus_v B}_{k-fold}$$

row overlap $$I_k \otimes^v B = \underbrace{B \oplus^v B \oplus^v \cdots \oplus^v B}_{k-fold}$$

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Properties of Overlapped Direct Sums and Tensor Products

- **Column and row overlap direct sum**

$$A \oplus^{v} B = \begin{bmatrix} \boxed{A} & \\ & \boxed{B} \end{bmatrix} = \begin{bmatrix} \boxed{A^{T}} & \\ & \boxed{B^{T}} \end{bmatrix}^{T} = \left(A^{T} \oplus_{v} B^{T}\right)^{T}$$

$$\mathbf{I}_n \oplus^{v} \mathbf{I}_s = \left(\mathbf{I}_n \oplus_{v} \mathbf{I}_s\right)^{T}$$

- **Column and row overlapped tensor product**

$$I_k \otimes_v B_{m \times n} = \left(\bigoplus_{l=1}^{k} B_{m \times n}\right) \bullet \left(\bigoplus_{l=1}^{k}{}_{v}\, I_n\right) = \left(I_k \otimes B_{m \times n}\right) \bullet \left(I_k \otimes_v I_n\right)$$

$$I_k \otimes^{v} B_{m \times n} = \left(\bigoplus_{l=1}^{k}{}_{v}\, I_m\right)^{T} \bullet \left(\bigoplus_{l=1}^{k} \left(B_{m \times n}\right)^{T}\right)^{T} = \left(I_k \otimes_v I_m\right)^{T} \bullet \bigoplus_{l=1}^{k} B_{m \times n}$$

$$= \left(I_k \otimes^{v} I_m\right) \bullet \left(I_k \otimes B_{m \times n}\right)$$

Filters

- **Linear convolution in matrix form**

$$C_n(\underline{h}) = \mathrm{I}_n \otimes^{l-1} \begin{bmatrix} h_0 & h_1 & \cdots & h_{l-1} \end{bmatrix}^{\mathrm{T}}$$

- **Block convolutions**
  - **Overlap-save rule**

$$C_n(\underline{h}) = \left(\mathrm{I}_{n/b} \otimes^{l-1} \mathrm{I}_{b+l-1}\right)\left(\mathrm{I}_{n/b} \otimes C_b(\underline{h})\right)$$

  - **Overlap-add rule**

$$C_n(\underline{h}) = \left(\mathrm{I}_{m/b} \otimes C_b^{\mathrm{T}}(\underline{h})\right)\left(\mathrm{I}_{m/b} \otimes_{l-1} \mathrm{I}_{b+l-1}\right)E_{l-1,l-1} \qquad m=n+l-1$$

- **Circular convolution rule for circulant *K***

$$K(\underline{h}) = \mathrm{DFT}_n^{-1} \cdot \mathrm{diag}\left(\mathrm{DFT}_n \cdot \underline{h}\right) \cdot \mathrm{DFT}_n$$

$\underline{h}$-first column of K

n-length of $\underline{h}$

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Discrete-Time Wavelet Transform

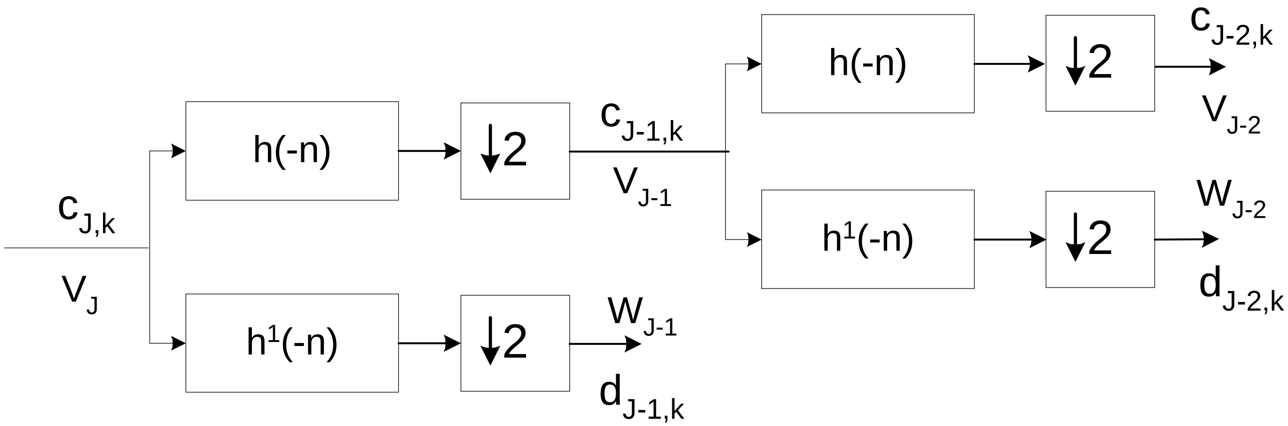
- **DTWT scaling and wavelet expansion coefficients**

$$c_{0,k} = \sum_n x_n \, h_J\left(2^J k - n\right) \qquad \text{scaling coefficients}$$

$$d_{j,k} = \sum_n x_n \, h^1_{J-j}\left(2^{J-j} k - n\right) \quad j = 0, \ldots, J-1 \qquad \text{wavelet coefficients}$$

- **Recursive algorithm (Mallat)**

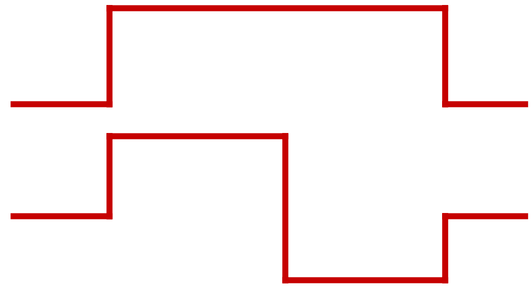
$$c_{j,k} = \sum_m h_{m-2k} c_{j+1,m} \qquad d_{j,k} = \sum_m h^1_{m-2k} c_{j+1,m}$$



Haar Wavelets – Example

- **Haar wavelets = square waves**

$$h = \frac{1}{\sqrt{2}}[\,1,\ 1],\quad h^1 = \frac{1}{\sqrt{2}}[\,1, -1]$$



- **First stage: $V_2 \Rightarrow V_1 \oplus W_1$**

$$H = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} = L^4_2 \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} = L^4_2\left(I_2 \otimes F_2\right) \qquad \begin{bmatrix} \underline{c}_1 \\ \underline{d}_1 \end{bmatrix} = H\,\underline{c}_2$$

- **The process is repeated for the upper half of the output**

$$HT_{2^n} = \left(HT_{2^{n-1}} \oplus I_{2^{n-1}}\right) \underbrace{L^{2^n}_{2^{n-1}}\left(I_{2^{n-1}} \otimes F_2\right)}_{H},\quad HT_2 = F_2$$

Discrete-Time Wavelet Transform

- **Discrete-Time Wavelet Transform (DTWT) rule**

$$\mathrm{DTWT}_{2^n} = \left(\mathrm{DTWT}_{2^{n-1}} \oplus \mathrm{I}_{2^{n-1}}\right) \underbrace{\mathrm{L}^{2^n}_{2^{n-1}}\left(\mathrm{I}_{2^{n-1}} \otimes_{l-2} W\right)}_{H}$$

- **Scaling (lowpass) and wavelet (highpass) filter coefficients**

$$W = \begin{bmatrix} h_0 & h_1 & \cdots & h_{l-1} \\ h'_0 & h'_1 & \cdots & h'_{l-1} \end{bmatrix}$$

- **DTWT - convolution rule**

$$H = \Big(\big([1 \quad 1] \otimes \mathrm{I}_{m/2} \big) \cdot \big(C^{\mathrm{T}}_{m/2}(\underline{lo}) \oplus C^{\mathrm{T}}_{m/2}(\underline{le}) \big) \cdot \mathrm{L}^n_2 \oplus$$
$$\big([1 \quad 1] \otimes \mathrm{I}_{m/2} \big) \cdot \big(C^{\mathrm{T}}_{m/2}(\underline{ho}) \oplus C^{\mathrm{T}}_{m/2}(\underline{he}) \big) \cdot \mathrm{L}^n_2 \Big) \cdot \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} \otimes \mathrm{I}_n \right)$$

$\underline{lo}$-lowpass odd coeffs., $\underline{le}$-lowpass even coeffs.

$\underline{ho}$-highpass odd coeffs., $\underline{he}$-highpass even coeffs.

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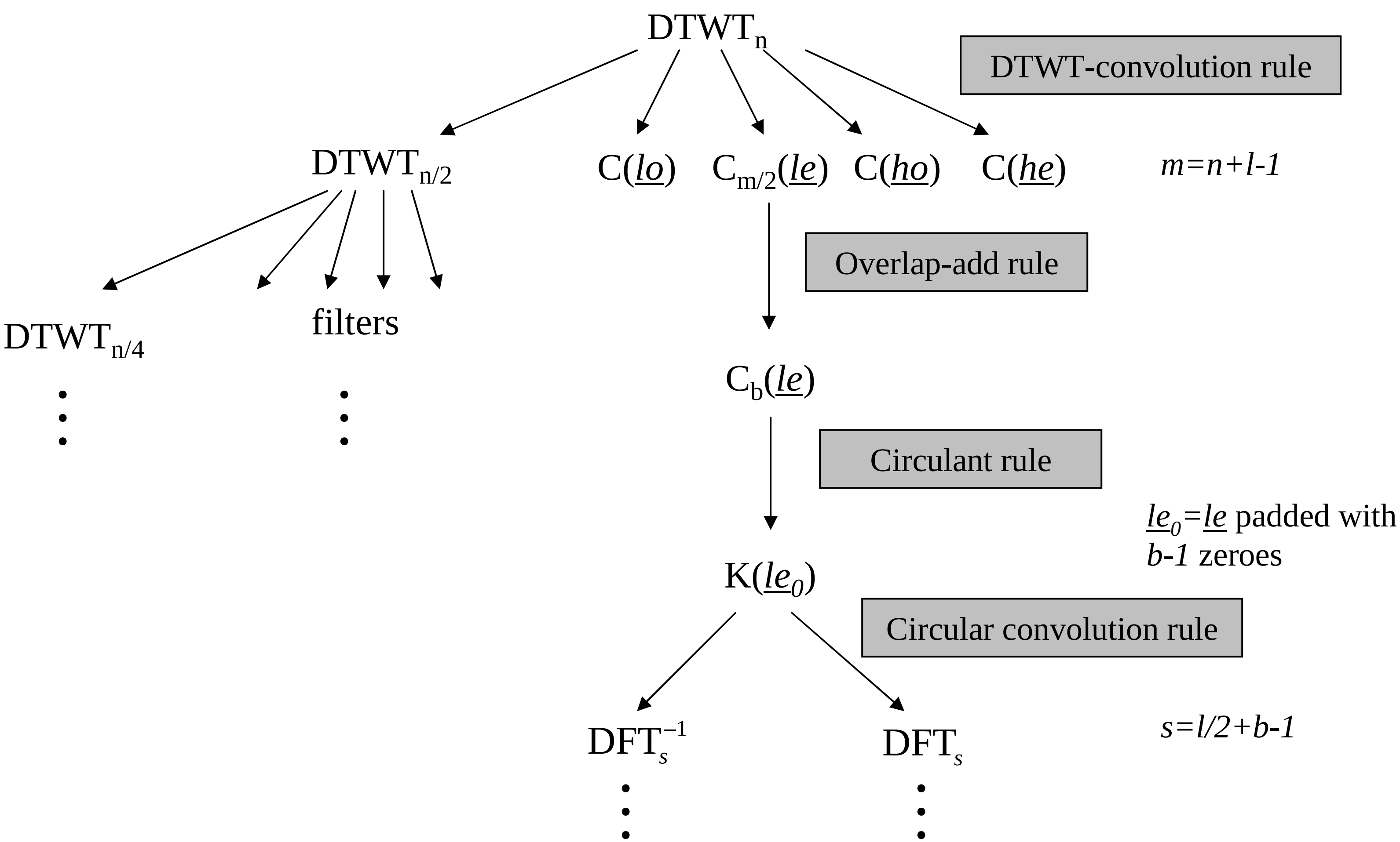
DTWT Ruletree – Example



Formula for a DCT, size 16

$$
\begin{aligned}
&[(2,16,9,5,3)(4,15,8,13,7)(6,14,10,12,11),16]\cdot\\
&(([(2,8,5,3)(4,7),8]\cdot(([(2,4,3),4]\cdot((\mathrm{diag}(1,\sqrt{\tfrac{1}{2}})\cdot\mathrm{DFT}_2)\oplus([(1,2),2]\cdot\mathrm{R}_{\frac{13}{8}\pi})^{[(1,2),2]})\cdot\\
&(\mathbf{1}_2\otimes\mathrm{DFT}_2)^{[(2,4,3),4]})\oplus(\mathrm{diag}(\tfrac{1}{2\cos(\frac{1}{16}\pi)},\tfrac{1}{2\cos(\frac{3}{16}\pi)},\tfrac{1}{2\cos(\frac{5}{16}\pi)},\tfrac{1}{2\cos(\frac{7}{16}\pi)})\cdot(\mathbf{1}_2\otimes\mathrm{DFT}_2)^{[(2,4,3),4]}\cdot\\
&((\mathrm{DFT}_2\cdot\mathrm{diag}(1,\sqrt{\tfrac{1}{2}}))\oplus([(1,2),2]\cdot\mathrm{R}_{\frac{13}{8}\pi})^{[(1,2),2]})\cdot[(2,3,4),4]\cdot\begin{bmatrix}1&0&0&0\\1&1&0&0\\0&1&1&0\\0&0&1&1\end{bmatrix})^{[(1,4)(2,3),4]})\cdot\\
&(\mathbf{1}_4\otimes\mathrm{DFT}_2)^{[(2,8,5,3)(4,7),8]})\oplus([(2,5,4,3,7,6,8),8]\cdot(\mathbf{1}_2\otimes(\mathbf{1}_2\oplus([(1,2),2]\cdot\mathrm{R}_{\frac{7}{4}\pi})))\cdot\\
&(\mathbf{1}_2\otimes\mathrm{DFT}_2\otimes\mathbf{1}_2)\cdot(\mathbf{1}_4\oplus([(1,2),2]\cdot\mathrm{R}_{\frac{13}{8}\pi})\oplus([(1,2),2]\cdot\mathrm{R}_{\frac{1}{8}\pi}))\cdot(\mathbf{1}_1\otimes\mathrm{DFT}_2\otimes\\
&\mathbf{1}_4)\cdot(([(1,2),2]\cdot\mathrm{R}_{\frac{49}{32}\pi})\oplus([(1,2),2]\cdot\mathrm{R}_{\frac{53}{32}\pi})\oplus([(1,2),2]\cdot\mathrm{R}_{\frac{57}{32}\pi})\oplus([(1,2),2]\cdot\\
&\mathrm{R}_{\frac{61}{32}\pi}))\cdot[(2,8)(4,6),8])^{[(1,8)(2,7)(3,6)(4,5),8]})\cdot\\
&(\mathbf{1}_8\otimes\mathrm{DFT}_2)^{[(2,16,9,5,3)(4,15,8,13,7)(6,14,10,12,11),16]}
\end{aligned}
$$

Helpful Concept

DSP Transforms		Formal Languages
DSP transform (of size)	⟷	Non-terminal symbol (with attribute)
Rule	⟷	Rule (production)
Formula/Algorithm	⟷	Element in Language (only terminals)

Mathematical Framework: Summary

- fast algorithms represented as ruletrees (easy generation/manipulation) and as formulas (can be translated into code)
- formulas built from few constructs and primitives
- many different algorithms/formulas generated from few rules (combinatorial explosion)
- these algorithms are (essentially) equal in arithmetic cost, but differ in data flow

Organization

- Mathematical Framework

  Transforms, Rules, and Formulas

- **Formula Generator**

  Transform $\rightarrow$ Algorithm

- SPL and SPL Compiler

  Algorithm $\rightarrow$ Implementation

- Search Engine

  How to find the best implementation

- SPIRAL system

  Everything taken together

- Conclusions

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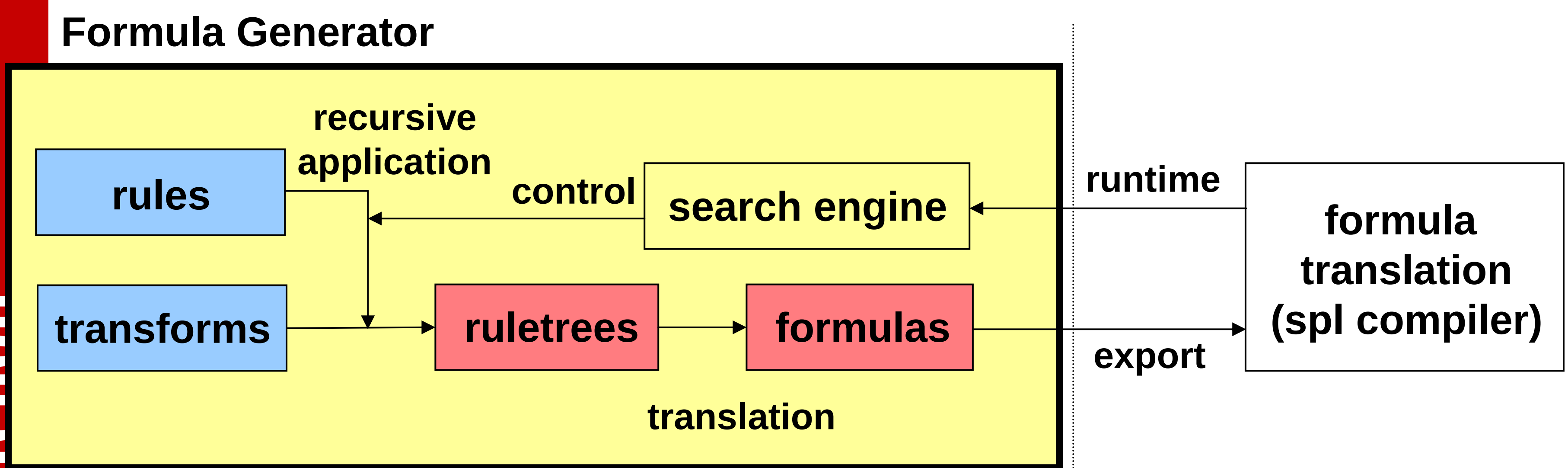
Formula Generation

data base (extensible!)
data type



cut here for other optimization problems

- written in GAP/AREP (computer algebra system)
- all computation/manipulation is symbolic
- exact arithmetic
- easy extensible rule and transform data base
- verification of rules and formulas

Formula Generator: Summary

Concept:

- easy extensible
- formulas as ruletrees (efficient representation, easy manipulation)
- interface to spl
- allows implementation of search methods

Implementation:

- efficient implementation moves bottleneck to spl compiler
- fully symbolic derivation/manipulation of formulas
- verification
- tools to analyze structure, arithmetic cost of formulas
- interfaces with search engine

Organization

- Mathematical Framework

  Transforms, Rules, and Formulas

- Formula Generator

  Transform $\rightarrow$ Algorithm

- SPL and SPL Compiler

  Algorithm $\rightarrow$ Implementation

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  How to find the best implementation

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Formulas in SPL

....

```
( compose
     ( diagonal ( 2*cos(1/16*pi) 2*cos(3/16*pi) 2*cos(5/16*pi) 2*cos(7/16*pi) ) )
     ( permutation ( 1 3 4 2 ) )
     ( tensor
       ( I 2 )
       ( F 2 )
     )
     ( permutation ( 1 4 2 3 ) )
     ( direct_sum
       ( compose
         ( F 2 )
         ( diagonal ( 1 sqrt(1/2) ) )
       )
       ( compose
         ( matrix
           ( 1 1 0 )
           ( 0 (-1) 1 )
         )
         ( diagonal ( cos(13/8*pi)-sin(13/8*pi) sin(13/8*pi) cos(13/8*pi)+sin(13/8*pi) ) )
         ( matrix
           ( 1 0 )
           ( 1 1 )
           ( 0 1 )
         )
         ( permutation ( 2 1 ) )
```

....

SPL Syntax (Subset)

- matrix operations:
```
(compose formula formula ...)
(tensor formula formula ...)
(direct_sum formula formula ...)
```
- direct matrix description:
```
(matrix (a11 a12 ...) (a21 a22 ...) ...)
(diagonal (d1 d2 ...))
(permutation (p1 p2 ...))
```
- parameterized matrices:
```
(I n)
(F n)
```
- scalars:
```
1.5, 2/7, cos(..), w(3), pi, 1.2e-04
```
- definition of new symbols: ← **allows extension of SPL**
```
(define name formula)
(template formula (i-code-list)
```
- directives for code generation
```
#codetype real/complex
#unroll on/off
```
← **controls loop unrolling**

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SPL Compiler, 4-point FFT

```
(compose (tensor (F 2) (I 2)) (T 4 2)
  (tensor (I 2) (F 2)) (L 4 2))
```

fast algorithm as **formula** as **SPL program**

#codetype

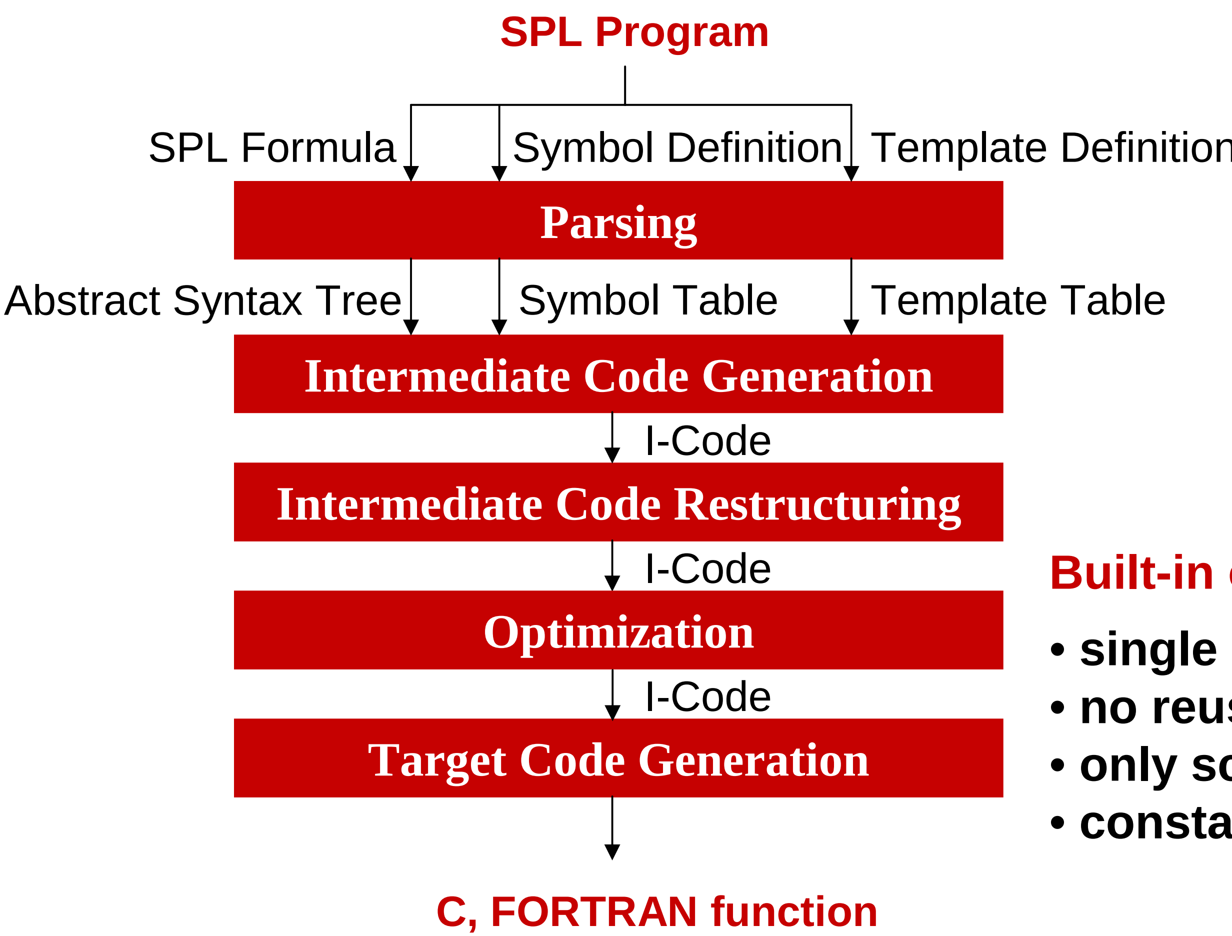
complex

real

```
f0   = x(1) + x(3)
f1   = x(1) - x(3)
f2   = x(2) + x(4)
f3   = x(2) - x(4)
f4   = (0.00d0,-1.00d0)*f(3)
y(1) = f0 + f2
y(2) = f0 - f2
y(3) = f1 + f4
y(4) = f1 - f4
```

```
r0   = x(1) + x(5)
r1   = x(1) - x(5)
r2   = x(2) + x(6)
r3   = x(2) - x(6)
r4   = x(3) + x(7)
r5   = x(3) - x(7)
r6   = x(4) + x(8)
r7   = x(4) - x(8)
y(1) = r0 + r4
y(2) = r1 + r5
y(3) = r0 - r4
y(4) = r1 - r5
y(5) = r2 + r7
y(6) = r3 - r6
y(7) = r2 - r7
y(8) = r3 + r6
```

SPL Compiler: Summary



Built-in optimizations:

- **single static assignment code**
- **no reuse of temporary vars**
- **only scalar temporary vars**
- **constants precomputed**

Extensible through templates

Templates

```
(template
  (F n)[ n >= 1 ]
  ( do i=0,n-1
      y(i)=0
      do j=0,n-1
        y(i)=y(i)+W(n,i*j)*x(j)
      end
    end ))
```

Pattern

Condition

I-code

Code Generation and Template Matching

(F 2) matches pattern **(F n)** and assigns 2 to n.
Because n=2 satisfies the condition n>=1,
the following i-code is generated from the template:

```
do i = 0,1
  y(i) = 0
  do j = 0,1
    y(i) = y(i)+W(2,i*j)*x(j)
  end
end
```

Unrolling & Optimization

```
Y(0)=x(0)+x(1)
y(1)=x(0)-x(1)
```

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SIMD Short Vector Extensions

- Extension to instruction set architecture
- Available on most current architectures
  (SSE on Pentium, AltiVec on Motorola G4)
- Originally for multimedia (like MMX for integers)
- Requires fine grain parallelism
- Large potential speed-up

Problems:

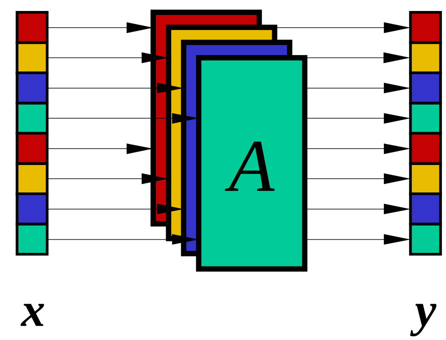
- SIMD instructions are architecture specific
- No common API (usually assembly hand coding)
- Performance very sensitive to memory access
- Automatic vectorization very limited

Vector Code Generation from SPL Formulas

Naturally vectorizable construct

$$A \otimes I_4$$

vector length



(Current) generic construct completely vectorizable:

$$\prod_{i=1}^{k} P_i D_i (A_i \otimes I_\nu) E_i Q_i$$

P_i, Q_i	**permutations**
D_i, E_i	**diagonals**
A_i	**arbitrary formulas**
ν	**SIMD vector length**

Vectorization in two steps:

1. **Formula manipulation using manipulation rules**
2. **Code generation (vector code + C code)**

SPL Compiler: Vector Code Generation

$$DFT_{16} = \left(DFT_4 \otimes I_4\right) \cdot T_4^{16} \cdot \left(I_4 \otimes DFT_4\right) \cdot L_4^{16}$$

Symbolic vectorization (automatic formula manipulation)

$$\overline{DFT}_{16} = \left(\left(I_4 \otimes L_4^8\right) \cdot \left(\overline{DFT}_4 \otimes I_4\right) \cdot \overline{T'}_4^{16}\right) \cdot \left(\left(I_4 \otimes L_2^8\right)\left(L_4^{16} \otimes I_2\right)\left(I_4 \otimes L_4^8\right) \cdot \left(\overline{DFT}_4 \otimes I_4\right) \cdot \left(I_4 \otimes L_2^8\right)\right)$$

Mapping to C code + vector API

```
LOAD_VECT(xl0, x + 0);
LOAD_VECT(xl4, x + 16);
f0 = SIMD_SUB(xl0, xl4);
LOAD_VECT(xl1, x + 4);
LOAD_VECT(xl5, x + 20);
f1 = SIMD_SUB(xl1, xl5);
...
yl7 = SIMD_SUB(f1, f4);
STORE_L_8_4(yl6, yl7, y + 24);
yl2 = SIMD_SUB(f0, f5);
yl3 = SIMD_ADD(f1, f4);
STORE_L_8_4(yl2, yl3, y + 8);
```

SSE
SSE2
AltiVec
...

Organization

- Mathematical Framework

  Transforms, Rules, and Formulas
- Formula Generator

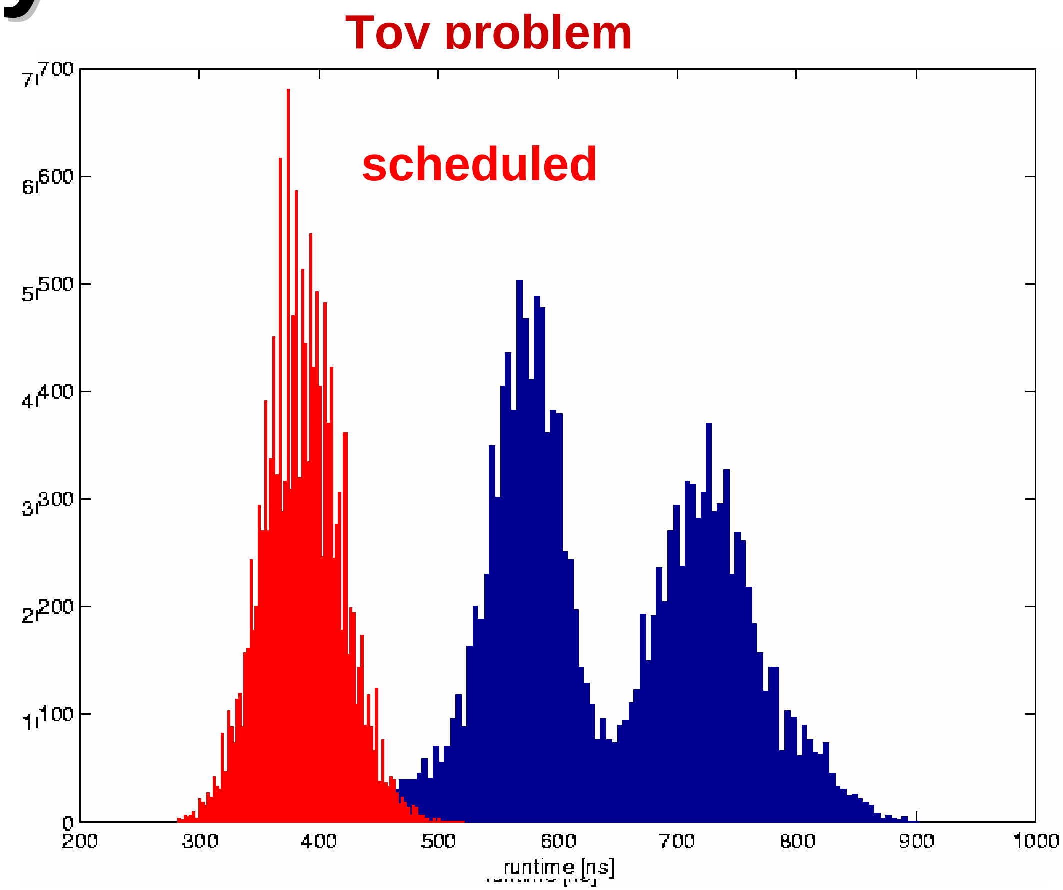
  Transform $\rightarrow$ Algorithm
- SPL and SPL Compiler

  Algorithm $\rightarrow$ Implementation
- **Search Engine**

  How to find the best implementation
- SPIRAL system

  Everything taken together
- Conclusions

Why Search?

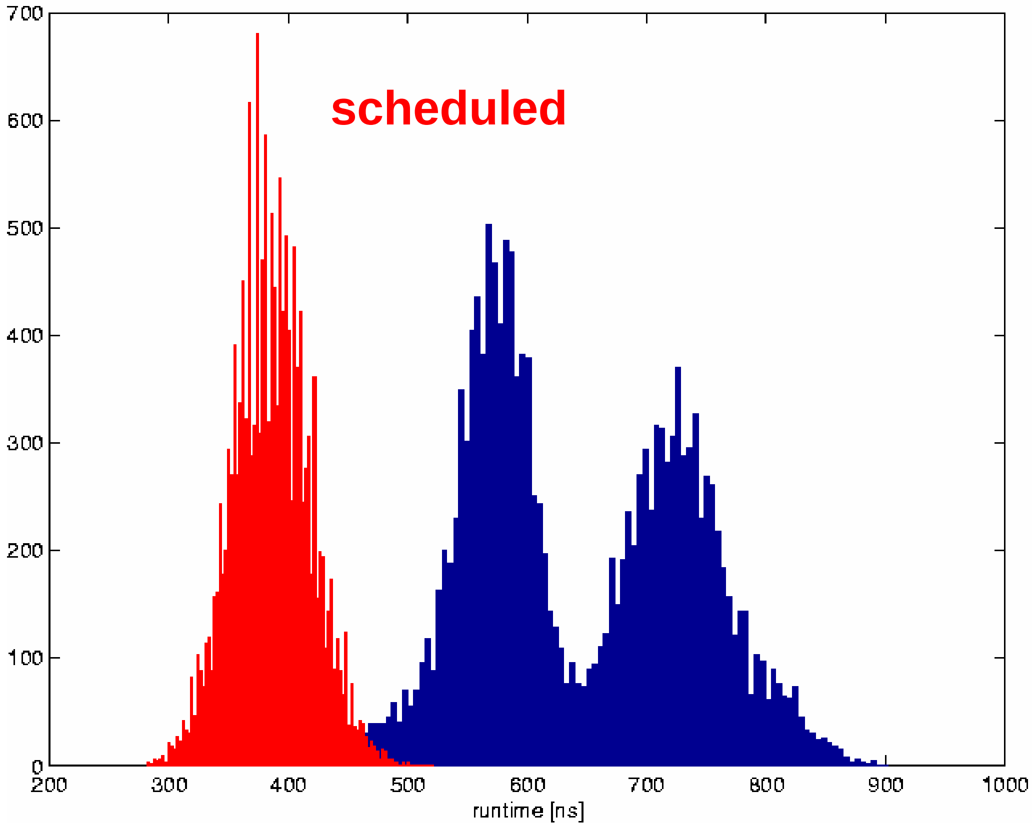


DCT IV- size 2^4

~31000 formulas

Search in algorithm space in SPIRAL:
Exhaustive & Random Search, DP, Hill climbing, Genetic Algorithms
Beyond search: design of optimal tree

Why Search?



Toy problem:

DCT, type IV, size 16

~31000 formulas

- **maaaany different formulas**
- **large spread in runtimes, even for modest size**
- **precisely equal arithmetic cost**
- **best formula is platform-dependent**

Number of Formulas/Algorithms

k	# DFT, size 2^k	# DCT-IV, size 2^k
1	1	1
2	6	10
3	40	126
4	296	31242
5	27744	1924443362
6	162570361280	7343815121631354242
7	~1.01 • 10^27	~1.07 • 10^38
8	~2.31 • 10^61	~2.30 • 10^76
9	~2.86 • 10^133	~1.06 • 10^153

- differ in data flow not in arithmetic cost
- exponential search space

Search Methods Available in SPIRAL

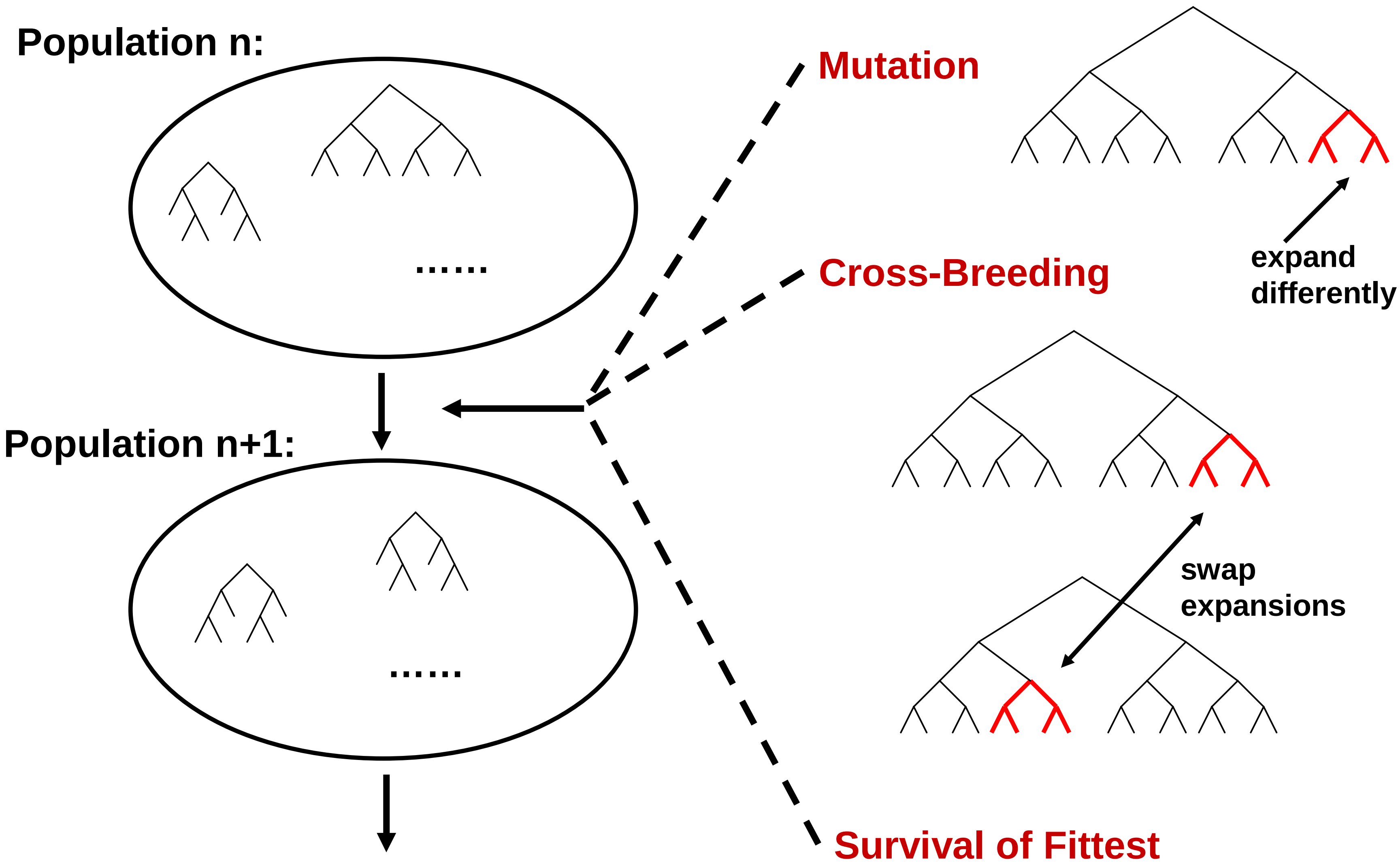
- Exhaustive Search
- Dynamic Programming (DP)
- Random Search
- Hill Climbing
- STEER (similar to a genetic algorithm)

	Possible Sizes	Formulas Timed	Results
Exhaust	Very small	All	Best
DP	All	10s-100s	(very) good
Random	All	User decided	fair/good
Hill Climbing	All	100s-1000s	Good
STEER	All	100s-1000s	(very) good

Search over

- algorithm space and
- implementation options (degree of unrolling)

STEER


Population n:
......
Population n+1:
......
Mutation
expand differently
Cross-Breeding
swap expansions
Survival of Fittest

Learning to Generate Fast Algorithms

- Learns from given dataset (formulas+runtimes) how to design a fast algorithm (breakdown strategy)
- Learns from a transform of one size, generates the best algorithm for many sizes
- Tested for DFT and WHT

Fast Formula Generation Results

Size	Number of Formulas Generated	Generated Included the Fastest Known	Top N Fastest Known Formulas in Generated
2^{12}	101	yes	77
2^{13}	86	yes	4
2^{14}	101	yes	70
2^{15}	86	yes	11
2^{16}	101	yes	68
2^{17}	86	yes	15
2^{18}	101	yes	25
2^{19}	86	yes	16
2^{20}	101	yes	16

Some Experimental Results

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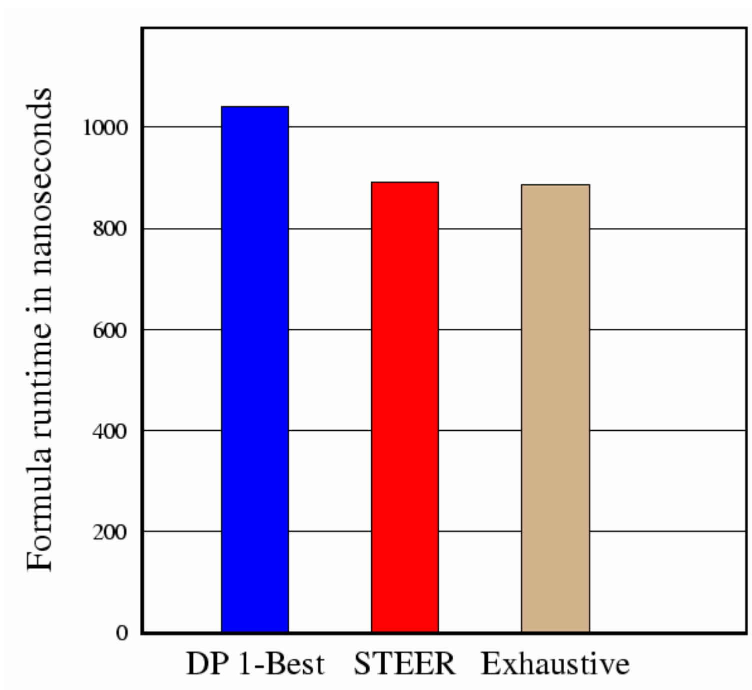
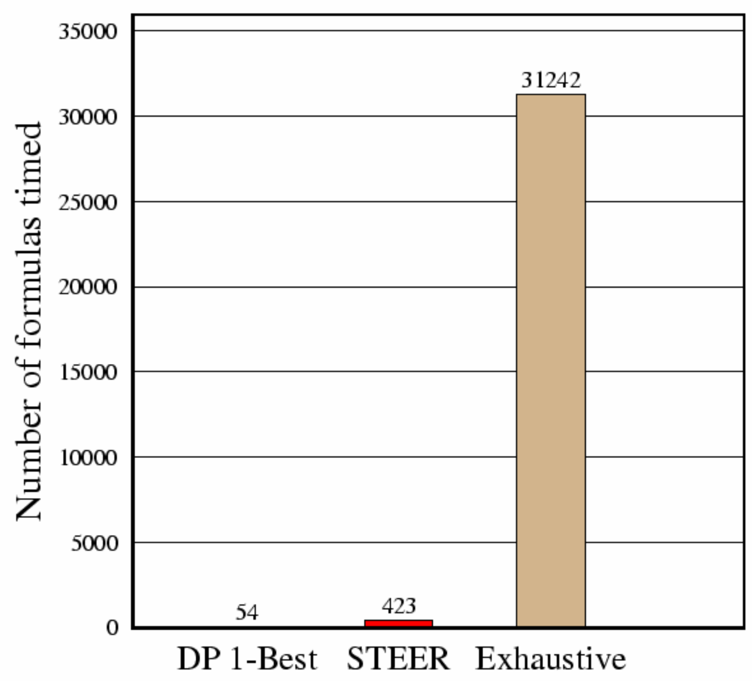
DCT Type IV Size 16

Fastest Found Formulas



Number of Formulas Timed



Experimental Results

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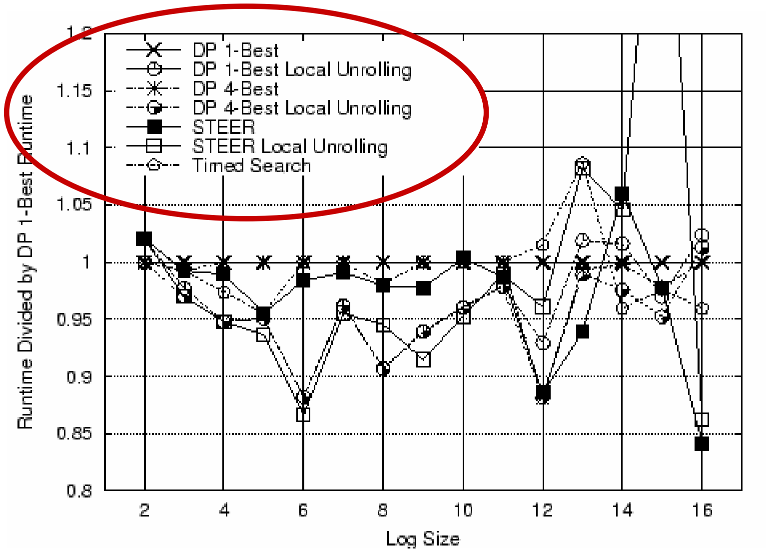
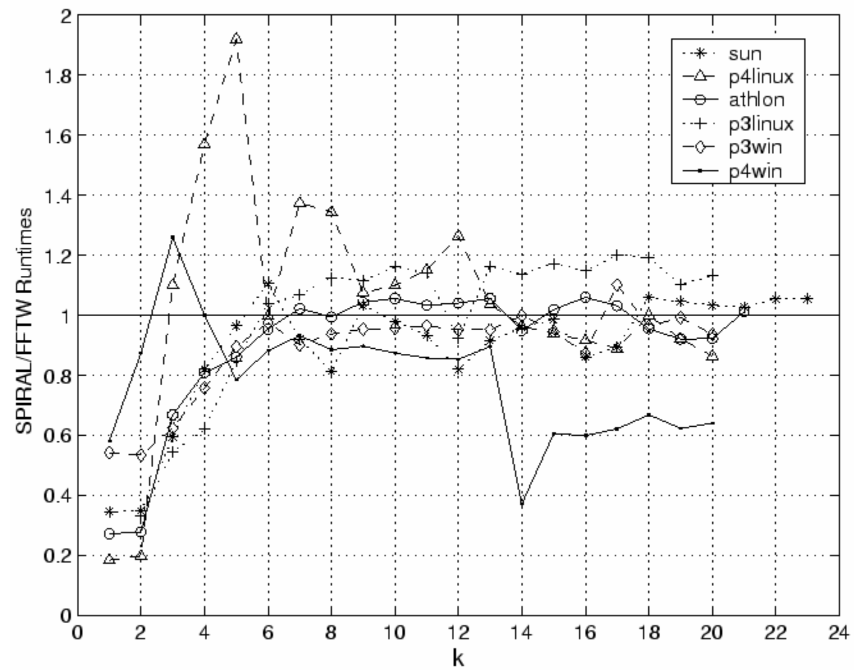
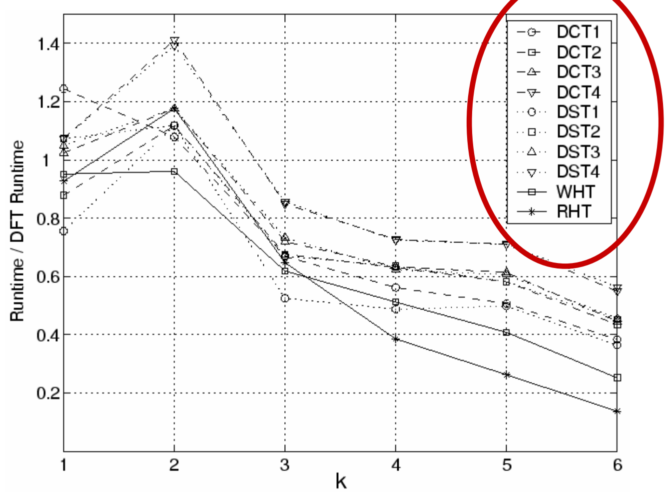
search methods (applicable to all transforms)



high performance code (compared with FFTW)





different transforms

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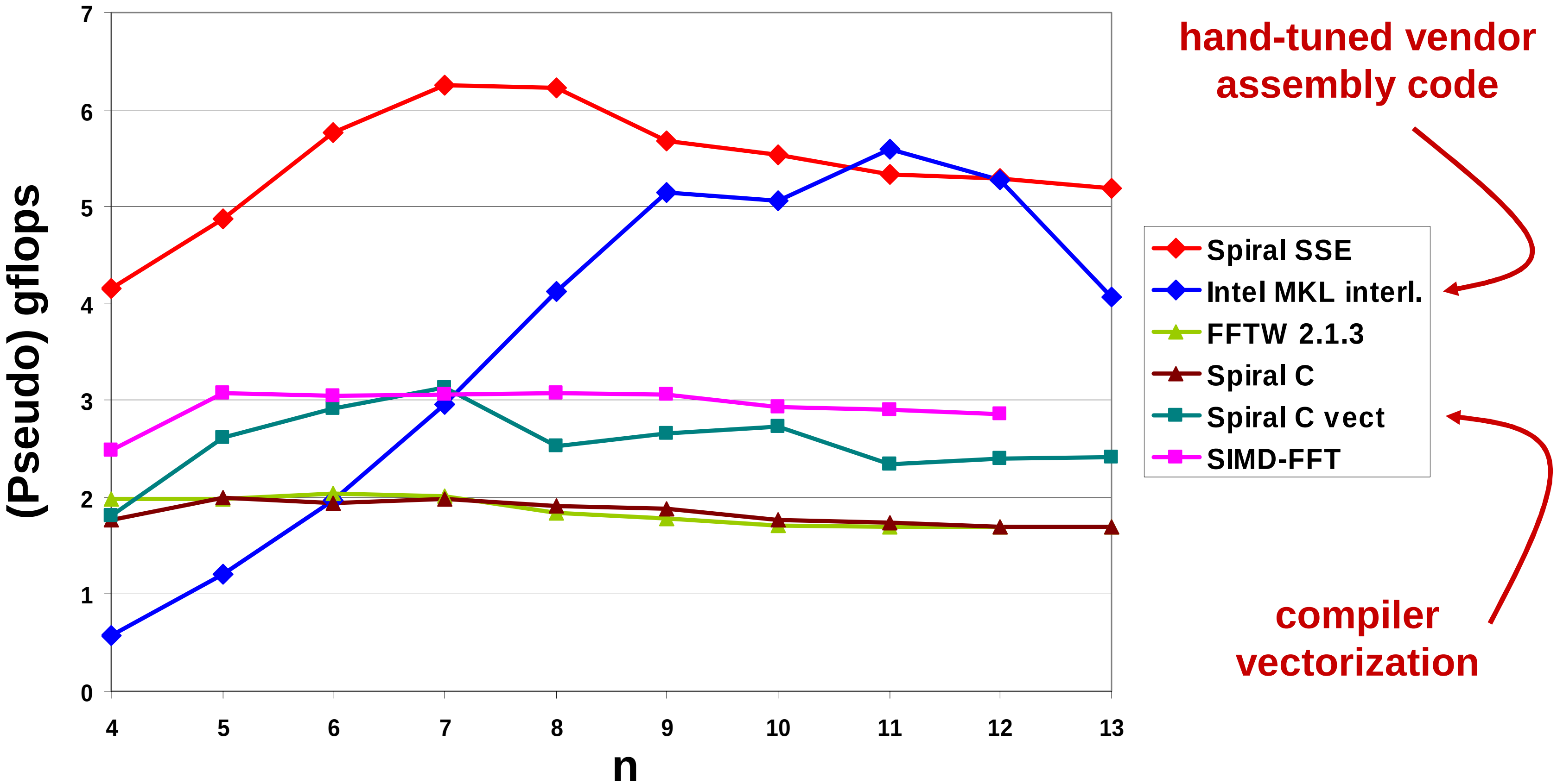
Some Experimental Results

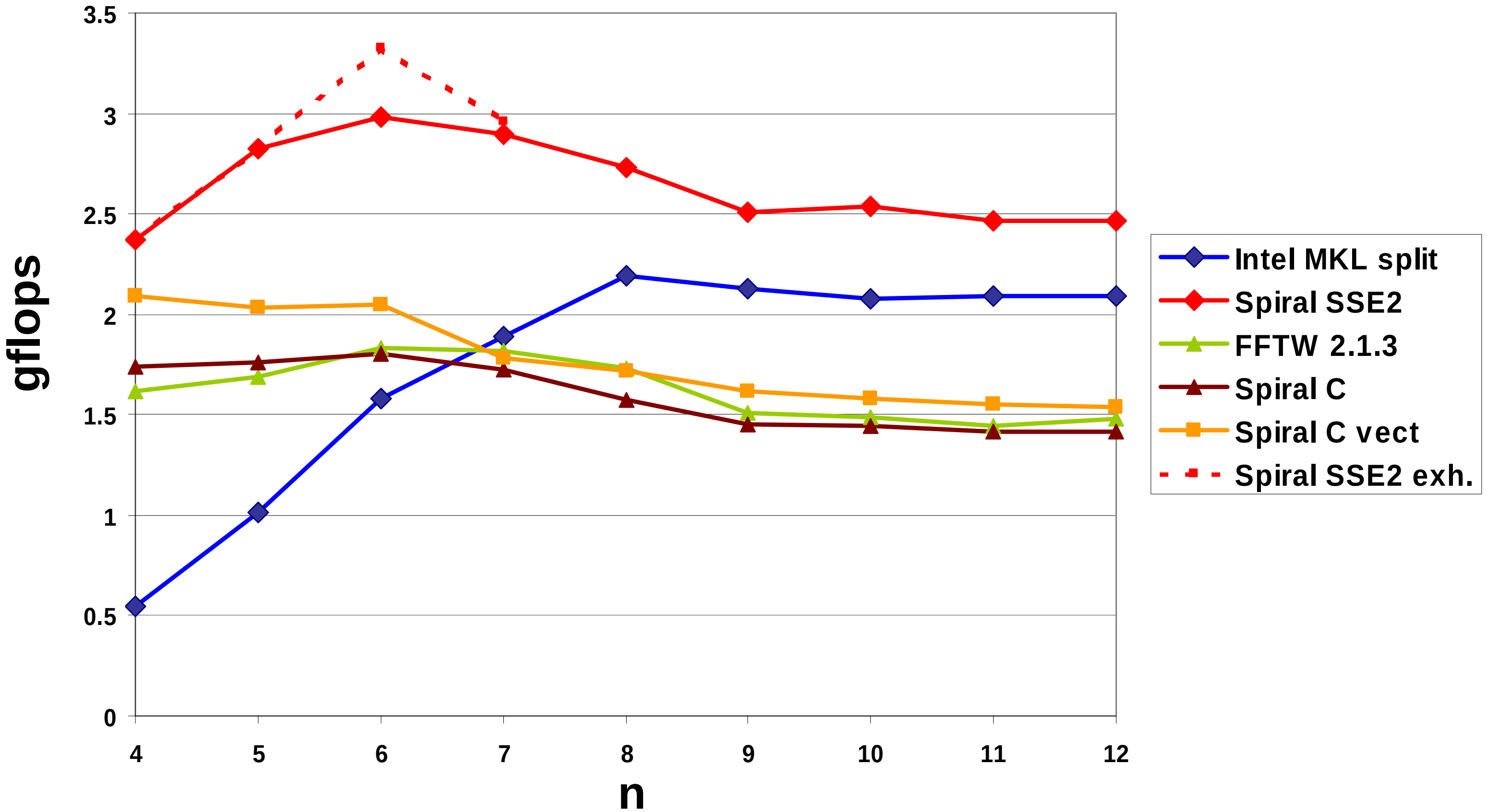
Generated DFT Code: Pentium 4, SSE



DFT 2^n *single precision*, Pentium 4, 2.53 GHz, using Intel C compiler 6.0

➡ **speedups (to C code) up to factor of 3.1**

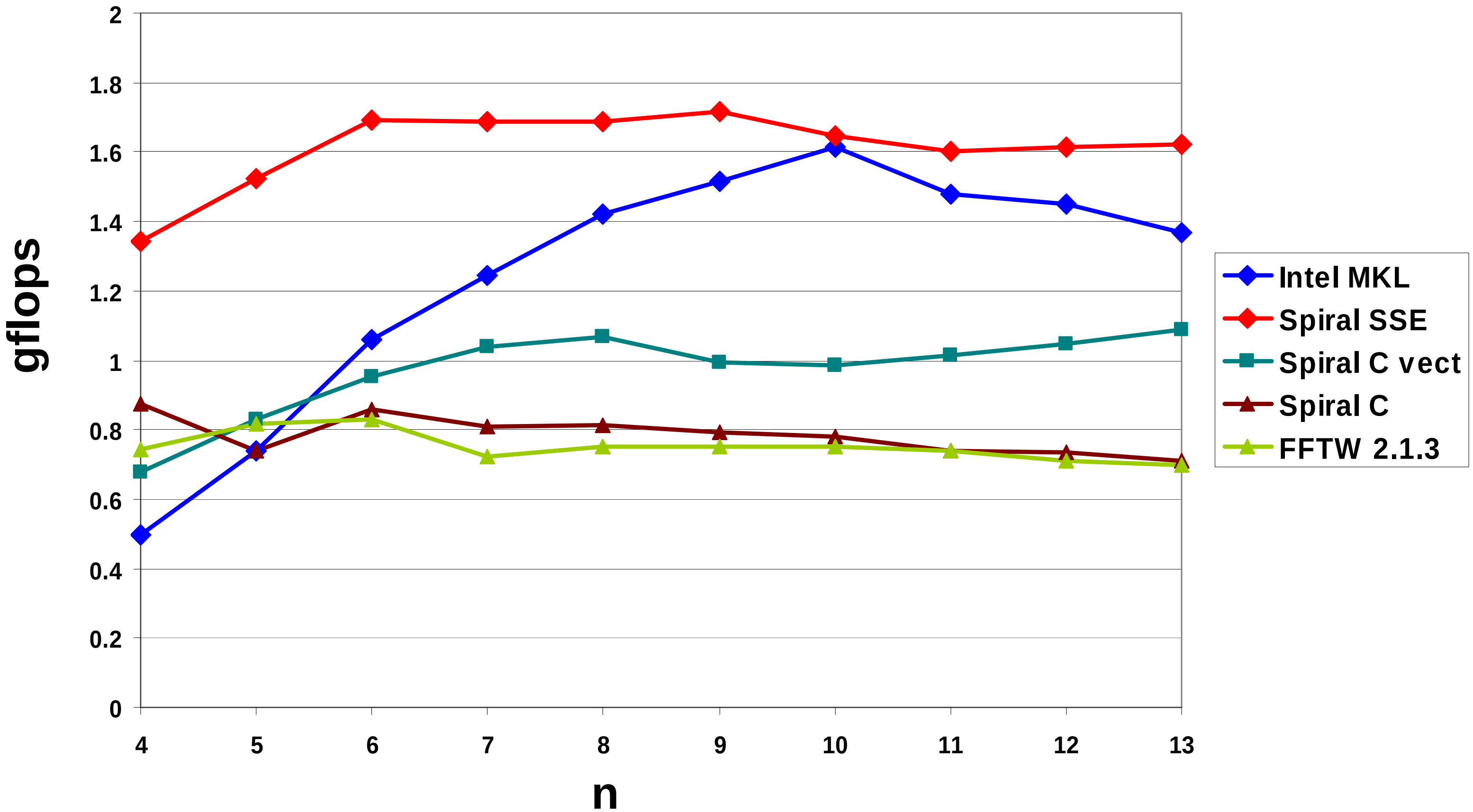
Generated DFT Code: Pentium 4, SSE2



DFT 2^n *double precision*, Pentium 4, 2.53 GHz, using Intel C compiler 6.0

speedups (to C code) up to factor of 1.8

Generated DFT Code: Pentium III, SSE



DFT 2^n *single precision*, Pentium III, 1 GHz, using Intel C compiler 6.0

speedups (to C code) up to factor of 2.1

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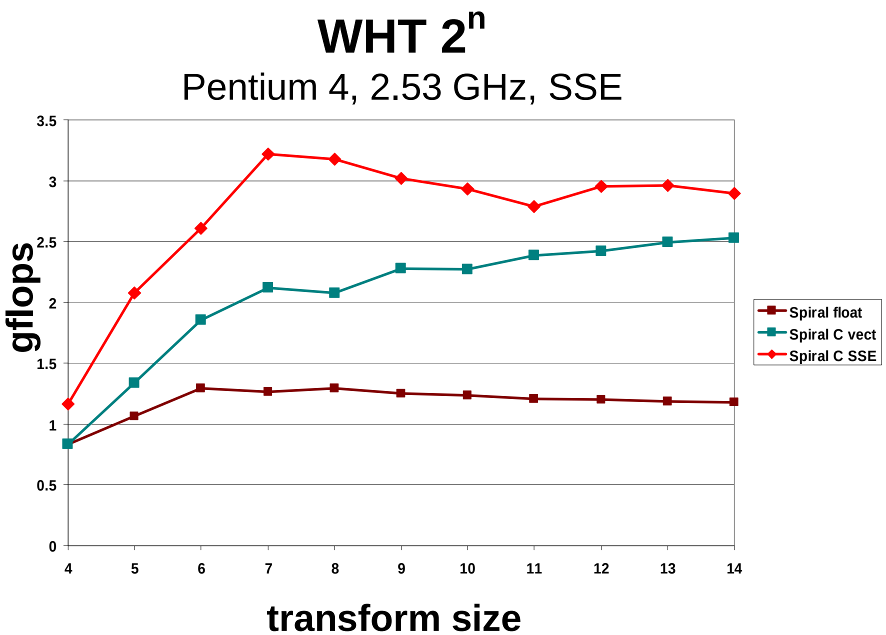
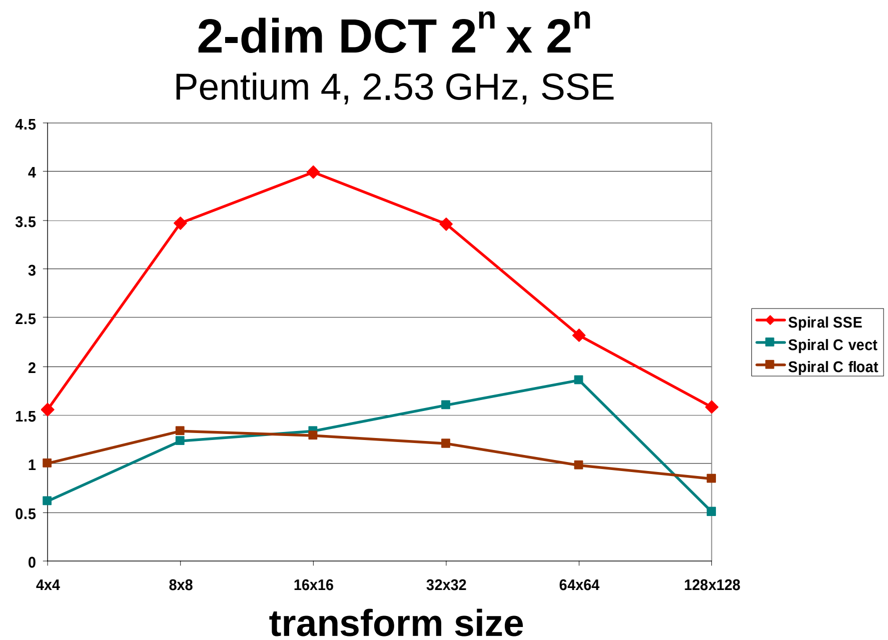
Other transforms





- WHT has only additions
- very simple transform

speedups (to C code) up to factor of 3

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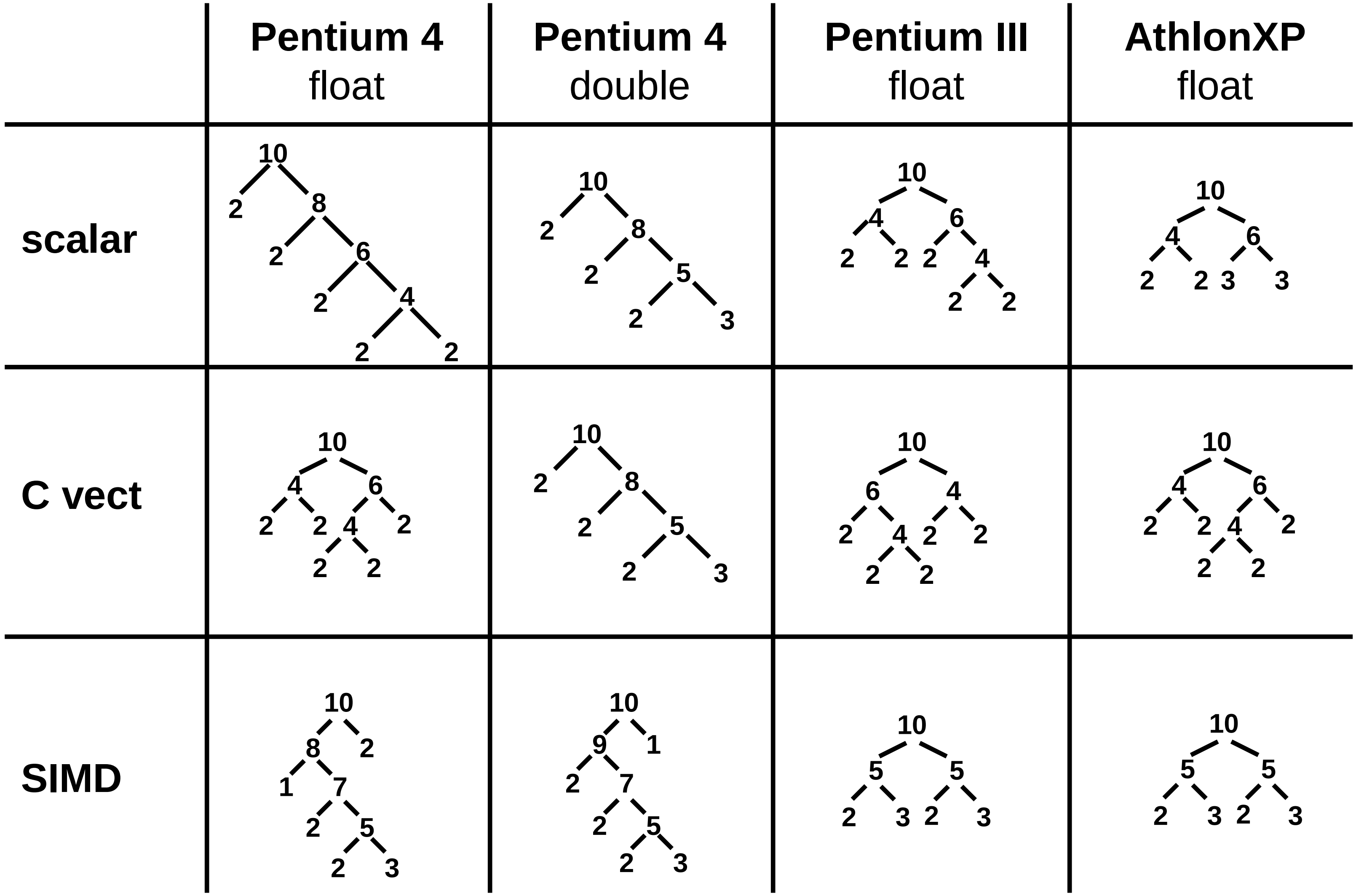
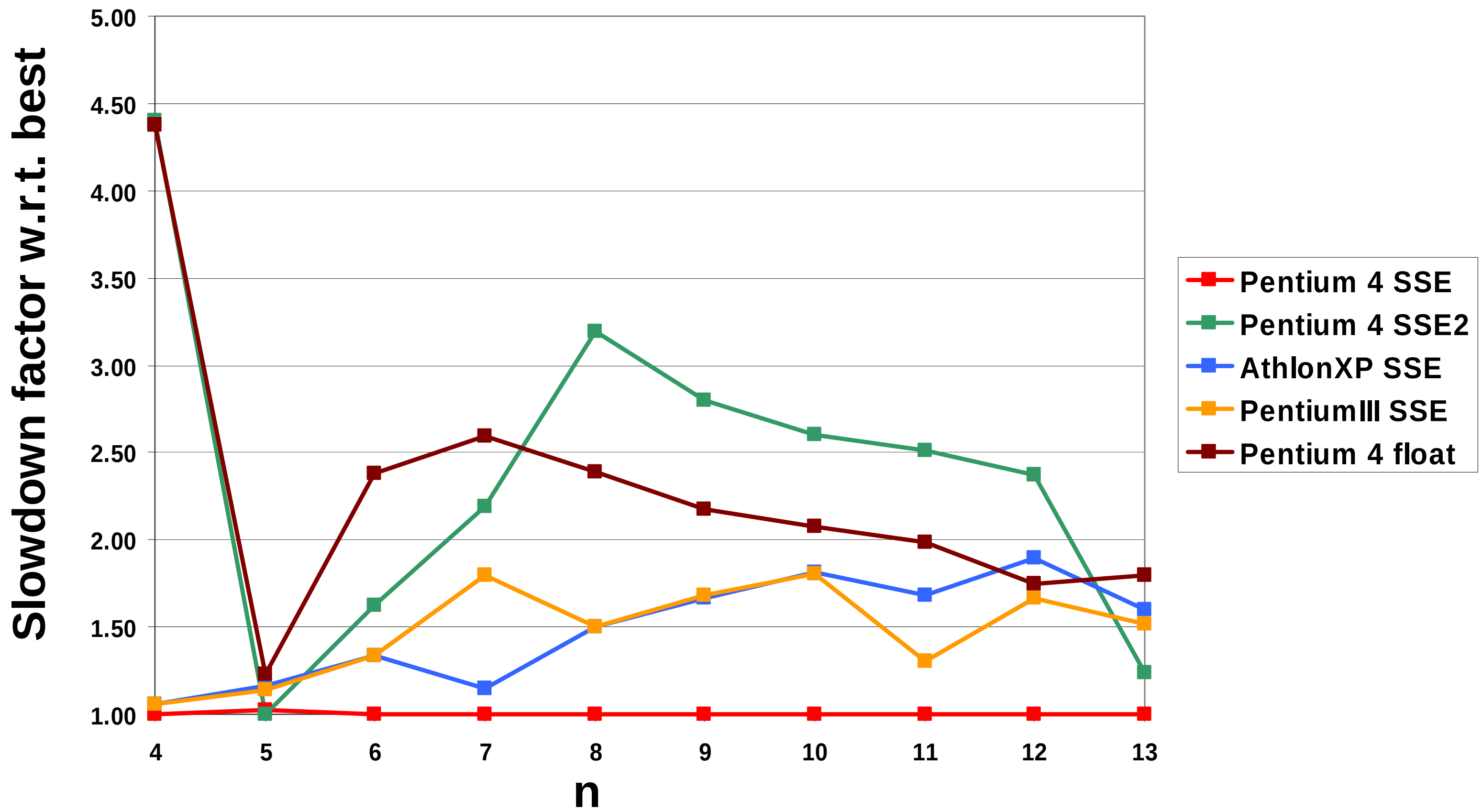
Best DFT Trees, size 2^{10} = 1024



trees platform/datatype dependent

Crosstiming of best trees on Pentium 4



DFT 2^n *single precision*, runtime of best found of other platforms

software adaptation is necessary

Organization

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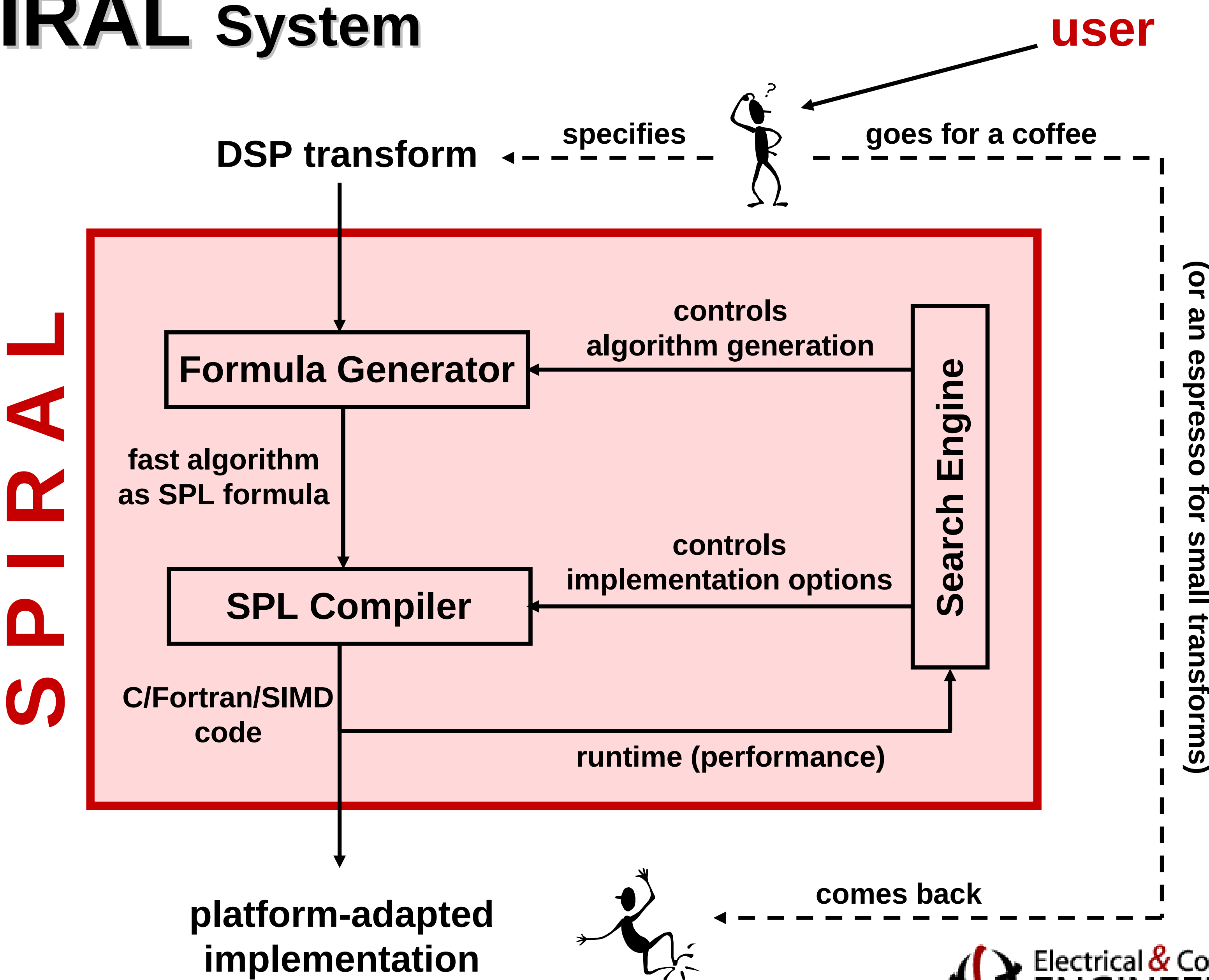
  - **Algorithm $\rightarrow$ Implementation**
- Search Engine
  - **How to find the best implementation**
- **SPIRAL system**
  - **Everything taken together**
- Conclusions

SPIRAL System



SPIRAL System

- Available for download (v3.1): `www.ece.cmu.edu/~spiral`
- Easy installation (Unix: configure/make; Windows: install shield)
- Unix/Linux and Windows 98/ME/NT/2000/XP
- Current transforms: DFT, DHT, WHT, RHT, DCT/DST type I – IV, MDCT, Filters, Wavelets, Toeplitz, Circulants
- Extensible
- New version (4.0) in preparation

- ~ 30 Publications in the areas of Signal Processing, High Performance Computing, Compilers, Machine Learning, Mathematics

SPIRAL System: Summary

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Organization

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  Transforms, Rules, and Formulas

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  Transform $\rightarrow$ Algorithm

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  Algorithm $\rightarrow$ Implementation

- Search Engine

  How to find the best implementation

- SPIRAL system

  Everything taken together

- **Conclusions**

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Conclusions

SPIRAL closes the gap between math domain (algorithms) and implementation domain (programs)

- **Mathematical computer representation of algorithms**
- **Automatic translation of algorithms into code**

SPIRAL does automatic optimization by intelligent search/learning in the space of alternatives

- **High level: Mathematical manipulation of algorithms**
- **Low level: Coding degrees of freedom**

➡ **a new paradigm of software development**