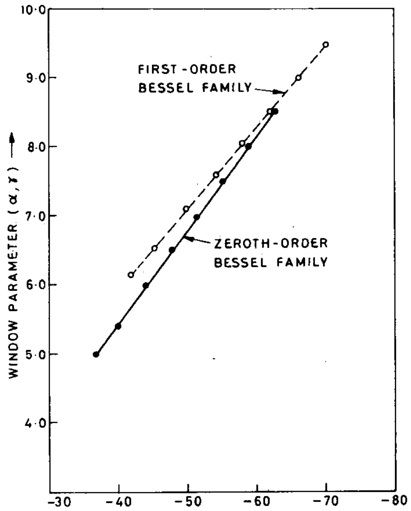

Fig. 4. Window parameters versus maximum sidelobe level for I_0-sinh and I_1-cosh window families.

rameters (α, γ) and the maximum sidelobe level (in decibels) are shown in Fig. 4.

IV. Conclusions

The desired design equations for digital spectrum analysis using I_1-cosh window and raised-cosine family have been derived along the lines of Kaiser and Schafer [15]. Because of the variable parameter, these window functions are more flexible in digital spectrum analysis than the fixed windows, such as Hanning, Hamming, etc. The I_1-cosh window has the advantage of lower first sidelobe level compared to the I_0-sinh window or prolate-spheroidal wave functions.

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Alternate Bounds on the Resolvability Constraints of Spatial Smoothing

M. J. D. Rendas and J. M. F. Moura

***Abstract*—In this correspondence, bounds for the number of subarrays required by the spatial smoothing technique for direction finding are discussed. It is proved that for a source matrix of rank r, q directions can be resolved with $M > q - r$ subarrays. It is also shown that when the source matrix is similar to a block-diagonal matrix through a permutation matrix, this bound can be further reduced to the largest rank deficiency presented by the diagonal blocks: $M > \max(n_i - r_i)$, where n_i and r_i are, respectively, the dimension and the rank of the ith diagonal block. Another bound for M is derived, which relates to the number of nonzero components in the eigenvectors of the source covariance matrix.**

I. Introduction

For linear uniform arrays, the spatial smoothing technique [1], [6] provides a solution to the problem of rank deficiency that affects the narrow-band high-resolution direction finding algorithms when the signal components of the observations are perfectly correlated. This method is based on averaging the sample covariance matrix over contiguous subarrays, resulting in a "smoothed" covariance matrix

$$\bar{S} = \sum_{k=1}^{M} D^{k-1} S (D^{+})^{k-1} \tag{1}$$

where M is the number of subarrays, D is a diagonal matrix

$$D = \operatorname{diag}\{e^{j\omega_0\tau_1} \cdots e^{j\omega_0\tau_q}\}, \tag{2}$$

q is the number of impinging replicas, and S is the $(q \times q)$ covariance matrix of the source vector

$$S = E[s(t)\, s^{+}(t)]. \tag{3}$$

We will denote the rank of S by r.

The goal of the spatial smoothing technique is to replace S, that for perfectly correlated sources is a singular matrix, with $\bar{S}$, which, for conveniently large M, will have rank equal to the dimension of S independently of its rank.

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The number of subarrays M necessary to generate a nonsingular $\bar{S}$ determines the resolvability constraint of the array, i.e., the maximum number of sources that can be detected using high-resolution algorithms. This constraint is obtained combining the bound on M together with the condition that each individual subarray must have a number of sensors greater than the number of sources. For an array of size K, the length of each subarray is given by $K - M + 1$. The two conditions

$$K - M + 1 > q$$
$$M \geq M_{\min} \tag{4}$$

imply

$$K \geq q + M_{\min}. \tag{5}$$

The constraint $K \geq 2q$ presented in [6] corresponds to the case when the q impinging replicas are perfectly correlated, for which $M_{\min} = q$. In fact, the maximum number of sources that can be detected depends on the rank of the source covariance matrix, as it has already been shown in [5] and [4].

In this correspondence, we assess the general case and the special case of block-diagonal covariance matrices, for which bounds on the number of subarrays were presented in [5] and [4]. We also improve the bounds in the case of block-diagonal matrices and present a new bound in terms of the number of null components in the eigenvectors of the source covariance matrix. Namely, we prove that the following are sufficient conditions for the spatial smoothing technique to yield a matrix with rank equal to the number of impinging replicas, q:

i) in the general case, the minimum number of subarrays required is $q - r + 1$ where r is the rank of the source covariance matrix;

ii) when the source covariance matrix S is similar, through a permutation matrix, to a block-diagonal matrix, the previous bound can be reduced to $M > \max (n_i - r_i)$, where n_i is the dimension of the ith diagonal block, r_i is its rank, and the maximum is taken over all diagonal blocks;

iii) when the null space of the source covariance matrix admits a basis formed by vectors with at most w nonzero elements, the minimum number of subarrays can be reduced from $q - r + 1$ to w.

The analysis in [5] and [4] is based on properties of polynomials. Here, we use the special structure of the invariant subspaces of diagonal matrices to prove the bounds i)–iii).

II. General Case

We establish in this section the bound for the general case i). We need the concept of invariant subspace for a matrix [3].

Let A be the matrix that represents in a given basis the linear transformation $\mathcal{A}$

$$\mathcal{A}: \boldsymbol{K}^q \rightarrow \boldsymbol{K}^q \tag{6}$$

where the elements of the matrix A take values in the ground field $\boldsymbol{K}$ (in our case, $\boldsymbol{K} = \boldsymbol{C}$, the field of complex numbers).

Definition 1: A subspace $\mathcal{U} \subseteq \boldsymbol{K}^q$ is said to be an invariant subspace of A iff

$$A\mathcal{U} \subset \mathcal{U}, \tag{7}$$

i.e.,

$$Av \in \mathcal{U}, \qquad \forall v \in \mathcal{U}. \tag{8}$$

□

In the sequel, $\mathrm{Sp}\{\cdot\}$ stands for the linear span of the enclosed vectors, while $\rho(A)$ stands for the rank of the matrix A.

To motivate our proof, let us consider briefly the special case when S has rank one: $S = vv^+$. Then, $\bar{S}$ admits the following factorization:

$$\bar{S} = [v | Dv | \cdots | D^{M-1}v] \begin{bmatrix} v^+ \\ v^+D^+ \\ \vdots \\ \vdots \\ v^+D^{+M-1} \end{bmatrix}. \tag{9}$$

For $\bar{S}$ to have rank q, the left factor of $\bar{S}$ in (9) must contain exactly q linearly independent vectors. Define the sets C_n of the first n columns of the left factor of $\bar{S}$, for $n = 1, \cdots, M$. Assume that for a given $n < q$, the vectors in C_n were all contained in an invariant subspace of D of dimension $n^* < q$, that we denote by $\mathcal{U}$. If this would be the case, all the subsequent remaining column vectors in the left factor of $\bar{S}$ would all also belong to $\mathcal{U}$, and the rank of $\bar{S}$ would be at most n^*.

The structure of the set of invariant subspaces of diagonal matrices with distinct entries, like D, is described in Lemma 1 below. As we shall see, this structure imposes $n^* = q$. So, when $\rho(S) = 1$, as long as $M \geq q$, the first q columns of the left factor in (9) are in fact linearly independent, and $\bar{S}$ has full rank q.

Lemma 1 [3, p. 51]: Let D be a diagonal matrix with distinct nonzero diagonal entries: $D_{ii} \neq D_{jj}$, $i \neq j$. Any invariant subspace of D is of the following form:

$$\mathcal{V} = \mathrm{Sp}\{e_{\sigma_1}, \cdots, e_{\sigma_n}\} \tag{10}$$

where $(\sigma_1 \cdots, \sigma_n)$ is a partition of length $1 \leq n \leq q$ of the integers $\{1, \cdots, q\}$ and e_i is the q-dimensional vector with all entries equal to zero, except the ith:

$$e_i \triangleq [\underbrace{0 \cdots 0}_{i-1} \;\; 1 \;\; 0 \cdots 0]^T. \tag{11}$$

□

An immediate consequence of this lemma is the following: if v belongs to an n-dimensional invariant subspace of a $(q \times q)$ matrix D in the conditions of the lemma, then at least $q - n$ of its elements are zero.

Lemma 2: Let S be a Hermitian matrix with nonzero diagonal entries of rank r, D a matrix in the conditions of Lemma 1, $\{u_i\}_{i=1}^r$ the eigenvectors of S corresponding to the positive eigenvalues λ_i. Then, the only invariant subspace of D that contains $\{u_i\}_{i=1}^r$ is the whole space $\boldsymbol{K}^q$. □

What Lemma 2 is saying is that $\boldsymbol{K}^q$ is the minimal D-invariant subspace over $\mathrm{Sp}\{u_i\}_{i=1}^r$, i.e., there is no invariant subspace of D of dimension smaller than q that contains all the eigenvectors of S corresponding to nonzero eigenvalues.

Proof (by contradiction): If $r = q$, there is nothing to prove. Assume $r < q$. Consider the spectral representation of the matrix S [2]:

$$S = \sum_{i=1}^{r} \lambda_i u_i u_i^+ = U\Lambda U^+. \tag{12}$$

Assume there is an invariant subspace $\mathcal{V}$ of D of dimension n, $r \leq n < q$, which contains all the columns of the matrix U, i.e., the eigenvectors $\{u_i\}_{i=1}^r$. By Lemma 1, $\mathcal{V} = \mathrm{Sp}\{e_{\sigma_1}, \cdots, e_{\sigma_n}\}$ for some partition σ. Define the $(q \times n)$ matrix

$$E_n \triangleq [e_{\sigma_1} \cdots e_{\sigma_n}]. \tag{13}$$

If $\{u_i\}_{i=1}^r \in \mathcal{V}$ then there is an $(n \times r)$ matrix T such that

$$U = E_n T. \tag{14}$$

Using (14) in (12),

$$S = E_n T \Lambda T^+ E_n^+, \tag{15}$$

which implies

$$S_{ii} = 0, \qquad i \notin \{\sigma_1, \cdots, \sigma_n\} \tag{16}$$

contradicting the assumption that the diagonal elements of S are all different from zero. □

We can now state result i) above.

Fact 1: Let S and D be matrices in the conditions of Lemma 2 above. For $M > q - r$, the matrix $\bar{S}$ defined in (1) has full rank q. □

Proof: If $r = q$, S is itself full rank and the fact is trivially verified. Assume $r < q$. As in Lemma 2, let $\{u_i\}_{i=1}^r$ be the eigenvectors of S that do not belong to its kernel. Define

$$u_i^k \triangleq D^{k-1} u_i \qquad k = 1, \cdots, M. \tag{17}$$

Then, from (1),

$$\bar{S} = \sum_{i=1}^{r} \sum_{k=1}^{M} u_i^k \lambda_i u_i^{k+}. \tag{18}$$

We prove that $\rho(\bar{S}) = q$ by showing that

$$\mathrm{Sp}\{u_i^k\}_{\substack{i=1,\cdots,r\\k=1,\cdots,M}} = K^q. \tag{19}$$

Let

$$\mathcal{U}^1 = \mathrm{Sp}\{u_i^1\}_{i=1,\cdots,r}. \tag{20}$$

Note that

$$\mathrm{Sp}\{u_i^k\}_{\substack{i=1,\cdots,r\\k=1,\cdots,M}} = \mathcal{U}^1 + D\mathcal{U}^1 + \cdots + D^{M-1}\mathcal{U}^1. \tag{21}$$

Since the vectors $\{u_i\}_{i=1}^r$ are linearly independent, $\mathcal{U}^1$ has dimension r. By Lemma 2, we know that $\mathcal{U}^1$ cannot be an invariant subspace of D, i.e.,

$$D\mathcal{U}^1 \not\subset \mathcal{U}^1. \tag{22}$$

So, there is at least a vector $v_1 \in \mathcal{U}^1$ such that

$$Dv_1 \notin \mathcal{U}^1. \tag{23}$$

Using v_1, define

$$\mathcal{U}^2 = \mathrm{Sp}\{u_1^1, \cdots, u_r^1, Dv_1\} \subseteq \mathcal{U}^1 + D\mathcal{U}^1. \tag{24}$$

Again, unless $r = q - 1$, $\mathcal{U}^2$ cannot be an invariant subspace of D. So, we can find a vector $v_2 \in \mathcal{U}^2$ whose image by D is not in $\mathcal{U}^2$, and add the new direction Dv_2 to $\mathcal{U}^2$ to yield a subspace $\mathcal{U}^3$ of dimension $r + 2$. This process can be repeated t times until the whole space K^q is reached, i.e.,

$$\mathcal{U}^t = K^q. \tag{25}$$

But,

$$\begin{aligned}\mathcal{U}^t &= \mathrm{Sp}\{u_1^1, \cdots, u_r^1, Dv_1, \cdots, Dv_{t-1}\}\\ &\subseteq \mathcal{U}^1 + D\mathcal{U}^1 + \cdots + D^{t-1}\mathcal{U}^1\end{aligned} \tag{26}$$

which, together with (25), implies

$$\mathcal{U}^1 + D\mathcal{U}^1 + \cdots + D^{t-1}\mathcal{U}^1 = K^q. \tag{27}$$

Since $\dim\{\mathcal{U}^t\} = r + t - 1$, the whole space is generated for $r + t - 1 = q$, i.e., $t = q - r + 1$, and the proof is completed. □

III. Special Cases

A particular case for which a tighter bound can be found is motivated by the situation in which the matrix S is block diagonal:

$$S = \begin{bmatrix} W_1 & & & \\ & W_2 & & \underline{0} \\ & & \ddots & \\ \underline{0} & & & W_l \end{bmatrix} \tag{28}$$

where W_i are $(n_i \times n_i)$ matrices of rank r_i,

$$\sum_{i=1}^{l} n_i = q \qquad \sum_{i=1}^{l} r_i = r. \tag{29}$$

This corresponds, for example, to the existence of l physically distinct (uncorrelated) sources, each one propagating over n_i paths to the receiving structure. As in the previous case, q denotes the total number of replicas received. Generally, the components of the vector $s(t)$ are not arranged by correlated groups. In this case, even though the matrix S is not block diagonal, it is related by a similarity transformation to a block-diagonal matrix S^*

$$S^* = PSP^{-1} \tag{30}$$

where P is a permutation matrix.

Lemma 3: Let S be a $(q \times q)$ Hermitian matrix with nonzero diagonal entries which is block-diagonalizable by a permutation matrix [(30) and (28)], and D a diagonal matrix with distinct nonzero diagonal entries. The eigenvectors of S that do not belong to its null subspace can be divided into l disjoint groups, of r_i vectors each, such that each group is contained in an invariant subspace $\mathcal{Y}^i$ of D of dimension n_i

$$u_{\sigma_1^i}, \cdots, u_{\sigma_{r_i}^i} \in \mathcal{Y}^i, \; i = 1, \cdots, l. \tag{31}$$

Furthermore, the subspaces $\mathcal{Y}^i$ satisfy

$$\mathcal{Y}^i \cap \mathcal{Y}^j = \{0\}, \; i \neq j \tag{32}$$

and

$$K^q = \mathcal{Y}^1 + \mathcal{Y}^2 + \cdots + \mathcal{Y}^l. \tag{33}$$

Proof: Partition D as in (28) and apply Lemma 2 to each submatrix in the block-diagonal matrix S^*. □

For this particular case, the following holds.

Fact 2: Let S and D be matrices in the conditions of Lemma 3 above. For

$$M > \max_{i=1,\cdots,l} (n_i - r_i), \tag{34}$$

the averaged matrix given by (1) has full rank q. □

Proof: It suffices to show that

$$\mathcal{V}^i + D\mathcal{V}^i + \cdots + D^{M-1}\mathcal{V}^i = \mathcal{Y}^i, \qquad i = 1, \cdots, l, \tag{35}$$

and then use (33). Applying Fact 1 to the restriction of D to each $\mathcal{Y}^i$, it is seen that (35) will be verified for $M > n_i - r_i$. Since this must hold for all values of i, bound (34) is obtained. □

A different bound can be found that relates to the number of nonzero elements in the eigenvectors of S that do not belong to its null space.

Fact 3: Let S, D, and $\{u_i\}_{i=1}^r$ have the same meaning as in Fact 1. Let w_i be the number of nonzero elements in u_i. Then, for

$$M \geq \max_{i=1,\cdots,r} w_i, \tag{36}$$

$\bar{S}$, given by (1), has full rank q. □

Proof: Let M^* be in the conditions of Fact 1: $M^* > q - r$. Define

$$\mathcal{K}^i \triangleq \mathrm{Sp}\{u_i, Du_i, \cdots, D^{M^*-1}u_i\}. \tag{37}$$

By Fact 1, we know that

$$\mathrm{Sp}\{u_i^k\}_{\substack{i=1,\cdots,r\\k=1,\cdots,M^*}} = K^q, \tag{38}$$

where u_i^k have been defined in (17). Also, in terms of the subspaces $\mathcal{K}^i$,

$$\mathrm{Sp}\{u_i^k\}_{\substack{i=1,\cdots,r\\k=1,\cdots,M^*}} = \mathcal{K}^1 + \cdots + \mathcal{K}^r. \tag{39}$$

If u_i has only w_i nonzero elements, then u_i belongs to an invariant subspace of D of dimension w_i, and consequently,

$$\mathrm{Sp}\{u_i, Du_i, \cdots, D^{M^*-1}u_i\} = \mathrm{Sp}\{u_i, Du_i, \cdots, D^{w_i-1}u_i\}. \tag{40}$$

So, for any $M \geq \max_{i=1,\cdots,r} w_i$,

$$\mathcal{K}^i = \mathrm{Sp}\{u_i, Du_i, \cdots, D^{M-1}u_i\}, \quad i = 1, \cdots r, \tag{41}$$

and we get

$$\mathrm{Sp}\{u_i^k\}_{\substack{i=1,\cdots,r\\k=1,\cdots,M}} = \boldsymbol{K}^q \tag{42}$$

which concludes the proof. □

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Asymptotic Second-Order Properties of Sample Partial Correlations

PETRE STOICA

Abstract—**The asymptotic behavior of the sample partial correlations (PARCOR's) is studied in the case of a general stationary process. An explicit formula for PARCOR's second-order moments is provided. The analysis presented in this correspondence extends to more general processes the results on the statistics of sample partial correlations of autoregressive (AR) and mixed autoregressive moving-average (ARMA) processes reported in [1], [2], [10], and [11]. This extension should be useful as the processes encountered in applications are not exactly AR or ARMA.**

I. Introduction

Partial correlations (also known as reflection coefficients) are of considerable interest in a number of signal processing applications. Their estimation from the data may be done by a variety of methods. The difference between the various estimators currently in use is, however, negligible for large data samples (see [1]). In other words, all of the commonly used estimators of partial correlations (abbreviated "PARCOR's" in the following) have the same asymptotic properties. In this correspondence, we assume that the Yule-Walker method is used to estimate the PARCOR sequence.

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The interest in the statistical properties of sample PARCOR estimates has various roots. For example, one may be interested in the performance of some technique which uses sample PARCOR's, such as spectral estimation and linear prediction. In other cases, such as fault detection, speech recognition, or AR order estimation, the statistical properties of the sample PARCOR's are used to devise the technique itself.

Early studies of the sample fluctuations of PARCOR estimators appeared in [3]-[5]. More recent studies can be found in [1], [2], [10]-[12].

In this correspondence, we establish the large sample properties of PARCOR's, under more general conditions than those assumed in earlier works. For regular processes, the PARCOR coefficients are uniquely determined from the serial covariances. The distribution of the latter, and the so-called Bartlett formula for the covariance matrix of that distribution in particular, are well known [6]-[12]. A similar formula for sample PARCOR's, which in view of the relationship between PARCOR and serial covariances should hold under similar conditions to the Bartlett formula, does not seem to be available in the literature. Our aim here is to provide such a formula for sample PARCOR's of a general stationary process. This is done in Section II. In Section III we present a compact version of our formula, which holds for linear processes. Note that we derive our formula by first linearizing the dependence PARCOR-serial covariances and then using the Bartlett formula. This procedure has the virtue of clearly showing that the Bartlett formula for serial covariances and our formula for PARCOR's hold under analogous conditions. Also, the linearized dependence PARCOR-serial covariances obtained as a byproduct may be interesting in its own right. An alternative way to derive our results would be to use the analysis technique of [14].

II. The General Result

Let $y(t)$, $t = 1, 2, \cdots$, denote a zero-mean stationary process. The (serial) covariance of $y(t)$ at lag k is denoted by

$$\rho_k \triangleq Ey(t)y(t+k) \qquad k = 0, \pm 1, \pm 2, \cdots \tag{2.1}$$

and the kth PARCOR coefficient by

$$\phi_k \triangleq u_k^T R_k^{-1} r_k \qquad k = 1, 2, \cdots \tag{2.2}$$

where

$$u_k = [0 \ \cdots \ 0 \ \ 1]^T$$

$$R_k = \begin{bmatrix} \rho_0 & \rho_1 & \cdots & \rho_{k-1} \\ \vdots & & & \vdots \\ & & & \rho_1 \\ \rho_{k-1} & & \cdots & \rho_0 \end{bmatrix}$$

$$r_k = [\rho_1 \ \cdots \ \rho_k]^T.$$

Let us introduce the following assumptions, for later use.

A1: The covariance matrices R_k, $k = 1, 2, \cdots$, are positive definite.

A2: The process $y(t)$ is Gaussian and its covariance sequence $\{\rho_k\}$ is absolutely summable.

Assumption A1 guarantees that the PARCOR sequence is uniquely derived from $\{\rho_k\}$, see (2.2). Assumption A2 essentially contains the conditions under which the Bartlett formula is known to hold (see below). In particular, A2 is satisfied if $y(t)$ is a linear (Gaussian) process, i.e., if there exists a stable linear filter $H(q^{-1}) = \Sigma_{k=-\infty}^{\infty} h_k q^{-k}$ such that

$$y(t) = H(q^{-1})e(t) \tag{2.3}$$

where q^{-1} denotes the backward shift operator and $e(t)$ is a zero-mean (Gaussian) white noise. We will denote the variance of $e(t)$