Cramér–Rao Bound for Location Systems in Multipath Environments

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Abstract—**We study the Cramér–Rao lower bound for source localization in the context of multiple stochastic sources, multipath propagation, and observations in an array of sensors. We derive a general expression which is then specialized to simpler configurations and related to results previously reported in the literature. The special case of a single stochastic source in a multipath environment is treated in detail. We assess the relative importance for source localization of the temporal (multipath) and spatial (array baseline) structures of the incoming wavefield. We show that for an array of K sensors the multipath contribution to the Fisher information matrix can be interpreted as the contribution of K independent arrays whose size depends on the number of spatially resolved replicas. We analyze the degradation due to unknown source spectra. When the source spectrum is completely arbitrary, source location is not possible with a single sensor. If a parametric form of the source spectrum is available, we show that the multipath structure can be used to locate the source. In particular, localization with a single sensor may then be possible.**

I. Introduction

The paper considers the localization of acoustic sources in the ocean. The source location is estimated from the directions of arrival (DOA's) of the wavefronts present. The current discussion focuses on passive systems, but many of the concepts and results are relevant to active systems as well.

In the past, we have considered ranging by exploring the spatial curvature of the incoming wavefront, either with an extended array of sensors [12] or with an array synthetically generated by the relative motions [8], [9]. This procedure was carried out in the context of homogeneous medium, where the effects of media boundaries were neglected. We showed that the relevant information is contained in the direct source/receiver path.

The presence of boundaries or inhomogeneities in the propagating channel resulting, for example, from a nonzero velocity gradient, gives rise to a complex multipath structure. Multipath is usually ignored or taken as a degradation effect that deteriorates the performance of systems based on direct-path-only propagation. Here, we take a different route and address the question of what can be gained if the location systems explore to their advantage the additional information encoded in the temporal structure of the observed wavefield, i.e., in the set of interpath delays. This approach is becoming increasingly relevant due to the progressive silencing of underwater platforms. The paper studies the local performance of location systems that explore in a coherent way the information in the intersensor and interpath delays. We derive the Cramér–Rao bound for the errors in the location estimates. This paper considers the following:

1) *multipath:* Each source is received by the array as the superposition of several attenuated and delayed replicas;
2) *stochastic multiple sources:* The signals corresponding to different sources are stationary wide-band Gaussian signals, not completely correlated (coherent) with each other;
3) *time invariant delays:* Source and receiver have no relative motion;
4) *known array:* The observations are obtained from a multisensor array of known geometry and location.

Going beyond the single path/direct propagation framework, the above assumptions may still be unrealistic in many practical problems. The context of Gaussian signals is fairly common in analytical performance studies. The effects of sensor uncertainty and of array calibration have been the object of recent studies. Motion and tracking considered previously in [10], [11] are ignored here. In the sense that we do not address many of these issues, our results correspond to a "best" scenario.

Some authors have studied the Cramér–Rao bound for wide-band stochastic source signals of known spectral density function. In [2], the Cramér–Rao bound is studied, considering an array of two sensors and two propagation paths (direct path and surface or bottom reflected path). In [3], [4], the importance of the multipath contribution for the range-only estimation problem is studied for two coherent paths. In [6], [7] the closely related problem of delay estimation for wide-band source signals with observations in a single sensor is addressed where the number of paths is confined to three. In [19], the Cramér–Rao bound for the estimation of the location param-

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eters of an arbitrary number of partially correlated narrow-band sources is presented. This work uses an unknown deterministic model for the source signals.

Here, we consider the case of an arbitrary number of coherent replicas of an arbitrary number of partially correlated wideband stationary sources received over an arbitrary number of sensors. We address both the case of known and of unknown source spectra. In doing so, we consider a model for the signals different from the one used by [19] and go beyond the simpler contexts of [2], [3], [4], [6], [7].

In focusing on the multipath/stochastic multisource problem, the paper generalizes work previously reported in the literature. Briefly, the major points are as follows.

1) Presentation in Section III of a general expression, (3.11), for the Cramér–Rao bound (CRB) on the error of the location estimates. This expression is valid for multiple stochastic sources of known spectral density, multipath propagation, and multisensor arrays. The expression is highly compact. To provide insight into the result, we interpret it in the context of high signal-to-noise ratio (SNR)/uncorrelated sources and in the case of multiple sources/large uniform array/single propagation path.

2) Interpretation of the inverse of the CRB, the Fisher information matrix (FIM), as the sum of two terms, one related to the spatial structure of the incoming wavefields (array effect) and another to its temporal structure (multipath effect). We show in Section III that these two terms have remarkably distinct asymptotic behavior with SNR. While the first grows without limit with SNR, the second attains a limit that is dependent on the geometry of the problem. Intuitively, the residual error associated with localization using the temporal structure has to do with the error incurred in estimating the source signals.

3) Consideration of the special case of a single source in a multipath environment that leads, in Section IV, to (4.4)–(4.7). These equations describe the CRB in terms of an orthogonal projection matrix and they provide further insight into the different behavior of the spatial and temporal effects. By restricting further the problem, we show how previously reported results are recovered for the single source/single path [18], single source/single sensor [14], single source/single sensor/two paths [5] problems.

4) Quantitative assessment of the multipath contribution to the CRB for the case of a single source, see Section V, in particular (5.18). This result generalizes to a different context the bound for a two path configuration contained in [4]. The multipath structure is interpreted in terms of a virtual array. First, we show that the additional virtual sensors arising from the secondary paths contribute significantly only if they are spatially separated. In fact, the number of "virtual" sensors that have a positive impact on the system performance equals the number of groups of resolved paths. This means for example that with a single (spatial) sensor, the technique of "multipath ranging," e.g., see [17], cannot have a performance which is comparable to the "virtual" sensor technique used for triangulation. Second, we prove that the performance of a system that uses the multipath structure is not worse than that of a system accounting solely for the spatial array effects. In fact, for the case where P paths are resolved, we show that the resulting gain has the form of the CRB for an array of P sensors receiving a single path, where both spatial and temporal effects are combined.

5) Study in Section VI, see (6.45), of the problem of unknown spectra for a single source/multipath context and its impact on the multipath effects contribution. We demonstrate that, when the spectra is completely unknown, the information corresponding to the temporal structure is lost. This result means that a single sensor cannot locate the source. It is also proved that lacking knowledge of the signal only affects the multipath contribution to the CRB, not the term corresponding to the spatial contribution. The analysis suggests that the method of "multipath ranging" may be severely adversely affected when the source spectrum is unknown. When the spectrum is parameterized by unknown parameters, we show that the information contained in the multipath structure can be used to advantage. In particular, single sensor localization is possible. We quantify the loss term for this case with respect to the case of known spectrum. This loss depends on the specific spectral parameterization.

The paper does not assume any particular propagation model. All the results remain valid as long as the observed signal is the superposition of attenuated and delayed replicas of the source signal, corrupted by additive noise. In a companion paper [16], we conduct an extensive numerical study of the Cramér–Rao bound, using a bilinear approximation to the velocity profile. In the next section, we describe the model and notation used.

II. Model and Notation

A. Problem Formulation

Consider the general case of several radiating sources, each propagating to the receiving array through multiple paths (Fig. 1.) Each source signal is a zero-mean wideband Gaussian signal, which can be partially correlated with the signals emitted by the other sources. Coherent signals are assumed to come from the same source.

Let α_s be the vector of parameters that describes the location of source s. For this general configuration, the signal received at sensor k in the array is given by

$$r_k(t) = \sum_{s=1}^{S} \sum_{p=1}^{P_s} a_{ksp} s_s(t - \tau_{ksp}(\alpha_s)) + w_k(t)$$

$$k = 1, \cdots, K; \; t \in T \qquad (2.1)$$

where

- S number of sources,
- P_s number of paths for source s,
- K number of sensors,
- a_{ksp} attenuation from source s to sensor k through path p,
- τ_{ksp} delay from source s to sensor k through path p,

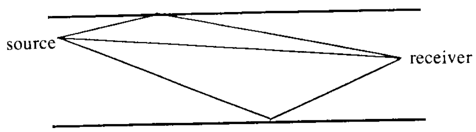

Fig. 1. Geometry.

$s_s(t)$ signal emitted by source s,
$w_k(t)$ observation noise at sensor k, statistically independent from the source signals.

Since the present paper is concerned with the Cramér–Rao bound, we assume that S and P are known. In (2.1) all the propagation parameters that define the medium's response (P_s, a_{ksp}, τ_{ksp}) depend on the source locations α_s. However, we assume throughout this study that P_s and a_{ksp} do not vary significantly with α_s, and consider only the dependency of the travel times on α_s, as it is indicated in (2.1). Accordingly, we make the following assumption:

Assumption 2.1 (Invariance of P_s and a_{ksp}): In a neighborhood $\mathfrak{N}(\alpha_s)$ of each source location

$$P_s(\alpha) = P_s(\alpha_s), \qquad \alpha \in \mathfrak{N}(\alpha_s), \qquad s = 1, \cdots, S \tag{2.2}$$

$$\left.\frac{\partial a_{ksp}}{\partial \alpha}\right|_{\alpha_s} = 0, \qquad \alpha \in \mathfrak{N}(\alpha_s); \qquad s = 1, \cdots, S; \qquad k = 1, \cdots, K; \qquad p = 1, \cdots, P_s. \tag{2.3}$$

Consider that the observation interval is large compared to the signals' correlation time (large time-bandwidth approximation) and compared to the interpaths delays. The Fourier components of the received signal become then uncorrelated and given by

$$r_k(\omega) = \sum_{s=1}^{S} \sum_{p=1}^{P_s} a_{ksp} e^{j\omega\tau_{ksp}} s_s(\omega) + w_k(\omega) \qquad k = 1, \cdots, K. \tag{2.4}$$

For each source, define the K-dimensional vector h_s with kth component:

$$[h_s(\omega)]_k = \sum_{p=1}^{P_s} a_{ksp} e^{j\omega\tau_{ksp}} \qquad k = 1, \cdots, K; \qquad s = 1, \cdots, S. \tag{2.5}$$

Let $r(\omega)$ be the vector that collects the K observations (2.4):

$$r(\omega) = \sum_{s=1}^{S} s_s(\omega) h_s(\omega) + w(\omega). \tag{2.6}$$

For each frequency bin, h_s is the vector that combines the coherent replicas received from source s. It will be referred to as the resultant vector. Being colinear with h_s, the combination of the replicas corresponding to source s is thus confined to a one-dimensional subspace of $\boldsymbol{C}^K$, the K-dimensional complex vector space where the observation vector $r(\omega)$ takes value. For a single propagation path, h_s reduces to the steering vector for source s which is usually parametrized by the direction of arrival (DOA). For multipath situations, the resultant vector h_s plays an analogous role, defining the direction of the K-dimensional space $\boldsymbol{C}^K$ that corresponds to the position of the source s. For a given source location, the resultant vector is itself a fixed linear combination of P_s steering vectors, each corresponding to a single path.

Finally, let H be the $(K \times S)$ matrix with columns h_s:

$$H(\omega) = [h_1(\omega) h_2(\omega) \cdots h_S(\omega)]. \tag{2.7}$$

The covariance matrix of the vector $r(\omega)$ can then be written as

$$S_r(\omega) = H(\omega) S_s(\omega) H^H(\omega) + \Sigma(\omega) \tag{2.8}$$

where $S_s(\omega)$ denotes the covariance matrix of the source vector[1]:

$$S_s(\omega) = E\{s(\omega) s^H(\omega)\} \tag{2.9}$$

and $\Sigma(\omega)$ is the covariance matrix of the observation noise. For simplicity, we assume that the noise is spatially incoherent:

$$\Sigma(\omega) = S_n(\omega) I_K \tag{2.10}$$

where $S_n(\omega)$ is the power spectral density of the noise.

B. Nomenclature

We present here some definitions that are used throughout the paper.

One-Form: The "one-form" $\mathbf{1}_n$ is the n-dimensional vector with all its elements equal to one.

Derivatives: The following two notations for the first-order partial derivative are used indistinctively: $\partial x / \partial \alpha = \dot{x}_\alpha$. When α is an n-dimensional vector and x is a scalar, $\dot{x}_\alpha$ is the n-dimensional row vector: $\dot{x}_\alpha = [\dot{x}_{\alpha_1} \cdots \dot{x}_{\alpha_n}]$. When x and α are vectors of dimensions m and n, respectively, $\dot{x}_\alpha$ is the $(m \times n)$ matrix of generic element: $[\dot{x}_\alpha]_{ij} = \dot{x}_{i\alpha_j}$.

vec*(A):* Let a_i be the ith column of the $(n \times m)$ matrix A. Then vec (A) is the nm-dimensional column vector: vec $(A)^T = [a_1^T \cdots a_m^T]$.

e^A is the exponential matrix of matrix A.

δ_{ij} is the Kronecker symbol.

I_K denotes the identity matrix of order K.

$\mathfrak{F}(\alpha)$ denotes the Fisher information matrix for the vector of parameters α.

A^H: $[A^H]_{ij} = [A]_{ji}^*$.

diag (x): Let x be an n-dimensional vector. Then diag (x) denotes the $(n \times n)$ diagonal matrix, with generic element: $[\text{diag}\,(x)]_{ij} = x_i \delta_{ij}$.

$A \geq B$ means that $A - B$ is a positive semidefinite matrix.

[1] E denotes the mean value operator.

Vector notation: When we drop the subindex in a sequence, we mean the column vector that stacks the indexed variables, i.e., for the sequence $\{r_k\}_{k=1}^K$, r denotes the K-dimensional vector with generic component r_k.

$\langle x, y\rangle$: Let x and y be elements of an Hilbert space H. Then $\langle x, y\rangle$ denotes the inner product of x and y. When H is finite dimensional $\langle x, y\rangle = x^H y$. For the infinite dimensional space $\mathcal{L}^2$ of square integrable functions, we use $\langle x, y\rangle = \int x(t)^* y(t)\, dt$.

III. Cramér–Rao Bound Multipath/Multiple Source

In this section, we derive the Cramér-Rao bound (CRB) for the general multiple source/multipath/multiple sensor problem with wide-band source signals of known spectra, and interpret the resulting expression in terms of physically meaningful parameters. Finally, we consider limiting behaviors of the CRB.

A. Derivation of the General Expression

Based on the model presented in the previous section, we establish in the sequel the CRB for any unbiased estimate of the location of the S sources. Throughout this section, we assume that the source spectrum is known to the receiver so that the only unknown entities in the received spectral density matrix are the locations of the sources.

Assumption 3.1 (Known Spectra): The spectral density matrix S_s of the sources is known.

Definition 3.1 (α): Define the 2S-dimensional vector

$$\alpha^T = [\alpha_1^T \cdots \alpha_S^T] \tag{3.1}$$

where each α_s is the vector of range (R_s) and depth (Y_s) for source s: $\alpha_s^T = [R_s Y_s]$.

It is well known that the CRB is given by the inverse of the Fisher information matrix (FIM):

$$\text{CRB}(\alpha) = \mathfrak{F}(\alpha)^{-1}. \tag{3.2}$$

In the Gaussian stationary case under consideration, the generic element of this matrix is given asymptotically (in the large sample limit) by Whittle's formula [2], [21]

$$[\mathfrak{F}(\alpha)]_{ij} = \frac{N}{4\pi}\int \operatorname{tr}\left\{\frac{\partial S_r(\omega)}{\partial \alpha_s} S_r^{-1}(\omega)\frac{\partial S_r(\omega)}{\partial \alpha_j} S_r^{-1}(\omega)\right\} d\omega. \tag{3.3}$$

In this equation N is the number of (temporal) samples, which, assuming sampling at the Nyquist rate, equals

$$N = 2BT \tag{3.4}$$

where B is the bandwidth and T is the observation interval. Approximation (3.3) gives the FIM as the integral of the FIM for the same problem at each single frequency. This is equivalent to saying that the processing is done independently at each frequency (in a statistical sense), which is valid in the large BT (i.e., large N) limit.

Partition (3.1) of the vector α implies the following partition for the FIM:

$$\mathfrak{F}(\alpha) = \begin{bmatrix} \mathfrak{F}(\alpha_1) & \mathfrak{F}(\alpha_1, \alpha_2) & \cdots & \mathfrak{F}(\alpha_1, \alpha_S) \\ & \mathfrak{F}(\alpha_2) & & \vdots \\ \vdots & & \ddots & \\ \mathfrak{F}(\alpha_S, \alpha_1) & \cdots & & \mathfrak{F}(\alpha_S)\end{bmatrix}. \tag{3.5}$$

In (3.5), $\mathfrak{F}(\alpha_i) = \mathfrak{F}(\alpha_i, \alpha_i)$, and each ($2 \times 2$) subblock is

$$\mathfrak{F}(\alpha_i, \alpha_j)_{nm} = \frac{N}{4\pi}\int \operatorname{tr}\left\{\frac{\partial S_r}{\partial n_i} S_r^{-1} \frac{\partial S_r}{\partial m_i} S_r^{-1}\right\} d\omega \tag{3.6}$$

and n and m take values on $\{R, Y\}$.

Using Woodbury's identity in (2.8)[2]

$$S_r^{-1} = \frac{1}{S_n}[I_K - H(S_n S_s^{-1} + H^H H)^{-1} H^H]. \tag{3.7}$$

Since each h_s depends only on the position of source s

$$\frac{\partial S_r}{\partial R_s} = \frac{\partial h_s}{\partial R_s} s_s^H H^H + H s_s \frac{\partial h_s^H}{\partial R_s} \tag{3.8}$$

$$\frac{\partial S_r}{\partial Y_s} = \frac{\partial h_s}{\partial Y_s} s_s^H H^H + H s_s \frac{\partial h_s^H}{\partial Y_s} \tag{3.9}$$

where s_s denotes the sth column of the source spectral density matrix:

$$S_s = [s_1 | \cdots | s_S]. \tag{3.10}$$

Using (3.7)–(3.9) in (3.3), we obtain (see Appendix A) the following result:

Fact 3.1 (Multiple Source, Multipath CRB): Let the observation model be described by (2.1) where the source signal covariance function is known. Then, the Cramér–Rao lower bound for the location parameters α is given by

$$\begin{aligned}\text{CRB}(\alpha)^{-1} = \frac{N}{2\pi}\int &\{\text{Re}\,\{[(S_s H^H H(S_n S_s^{-1} + H^H H)^{-1}) \\ &\otimes (\mathbf{1}_2 \mathbf{1}_2^T)] \odot D^H S_r^{-1} D\} \\ &+ \text{Re}\,\{D^H H(S_n S_s^{-1} + H^H H)^{-1} \\ &\odot (S_n S_s^{-1} + H^H H)^{-T} H^T D^*\}\}\, d\omega. \end{aligned} \tag{3.11}$$

In (3.11) $\odot$ denotes the Hadamard (element-by-element) product of matrices

$$[A \odot B]_{ij} = a_{ij} b_{ij} \tag{3.12}$$

and $\otimes$ denotes the Kronecker product

$$A \otimes B = \begin{bmatrix} a_{11}B & \cdots & a_{1n}B \\ \vdots & & \vdots \\ a_{m1}B & \cdots & a_{mn}B\end{bmatrix}. \tag{3.13}$$

[2]For simplicity, we drop the frequency dependency in the rest of the paper.

The $(K \times 2S)$ matrix D groups all the derivatives of the resultant vectors h_s with respect to the source location parameters R_s, Y_s:

$$D = \left[\frac{\partial h_1}{\partial R_1} \ \frac{\partial h_1}{\partial Y_1} \ \frac{\partial h_2}{\partial R_2} \ \frac{\partial h_2}{\partial Y_2} \cdots \frac{\partial h_S}{\partial R_S} \ \frac{\partial h_S}{\partial Y_S}\right]. \quad (3.14)$$

B. Interpretation of CRB

To clarify the meaning of (3.11), consider the following rewriting of (2.4) that emphasizes the spatial dependency of the received sequence. We exchange the subindex k with the argument ω and use the definition (2.5) of the resultant vector h_s to get

$$r^\omega(k) = \sum_{s=1}^{S} s_s^\omega h_s(k) + w^\omega(k), \qquad k = 1, \cdots, K. \quad (3.15)$$

The (super) index ω is held fixed. Recall that $w^\omega(k)$ is a (spatially) white noise process, with zero mean and covariance given by (2.10) and that s^ω is a zero mean random vector with covariance matrix S_s given by (2.9).

Consider now that $\{h_s(k)\}_{s=1,\cdots,S}$ are known functions of k. In the location problem, this assumption amounts to knowing the location of the sources. As in [13], the minimum mean square error estimate (MMSEE) of s^ω is

$$\hat{s}^\omega = (S_n S_s^{-1} + H^H H)^{-1} H^H r^\omega \quad (3.16)$$

the covariance matrix of the signal MMSEE is

$$P_s \triangleq S_s H^H H (S_n S_s^{-1} + H^H H)^{-1} \quad (3.17)$$

and the mean square error matrix

$$\Sigma_s \triangleq S_n (S_n S_s^{-1} + H^H H)^{-1}. \quad (3.18)$$

Using these definitions in (3.11):

$$\text{CRB}(\alpha)^{-1} = \frac{N}{2\pi} \int \text{Re}\left\{(P_s \otimes (\mathbf{1}_2 \mathbf{1}_2^T)) \odot (D^H S_r^{-1} D) + \left(D^H H\, \Sigma_s S_n^{-1}\right) \odot \left(D^H H\, \Sigma_s S_n^{-1}\right)\right\} d\omega. \quad (3.19)$$

Thus, we see that the first term of the CRB is proportional to the product of the signal's estimated covariance P_s and the inner product of the steering vector's derivatives D, taken in the norm of the observations' covariance matrix S_r.

We establish a parallel between (3.19) and [19, eq. (E.8g)] showing heuristically that the first term of (3.19) is the stochastic equivalent of the CRB for the unknown deterministic signal case. To do that, we relate some of the statistical quantities appearing in (3.19) with their deterministic equivalent. Take the source signal as deterministic, which is the context of [19]. Then i) there is no signal estimation error, so $\Sigma_s = 0$ and the second term of (3.19) vanishes; ii) P_s is reinterpreted as the temporal signal covariance matrix S_s; iii) finally, the observations' covariance matrix S_r is reduced to the noise covariance matrix S_n. Equation (3.19) is then rewritten as

$$\text{CRB}(\alpha)^{-1} = \frac{N}{2\pi} \int S_n^{-1} (S_s \otimes (\mathbf{1}_2 \mathbf{1}_2^T)) \odot D^H D \, d\omega. \quad (3.20)$$

Equation (3.20) is identical to [19, eq. (E.8g)]. This analogy suggests that the second term in (3.19) is related to the stochastic character of the source signals.

In the following subsections, we consider two limiting situations: large signal-to-noise ratio (SNR) and uncorrelated sources (Section III-C) and large array size (Section III-D). For each case, we present a simplified expression for the CRB, and relate it to expressions presented in the literature.

C. High SNR and Uncorrelated Sources

Consider the case of uncorrelated sources for which

$$S_s = \text{diag}\{S_{s_1}, \cdots, S_{sS}\}.$$

For very high signal-to-noise ratio

$$\frac{S_n}{S_{s_i}} \ll \|h_i\|^2, \qquad i = 1, \cdots, S$$

and (3.11) can be simplified, using the following relations:

$$(S_n S_s^{-1} + H^H H)^{-1} \rightarrow (H^H H)^{-1} \quad (3.21)$$

and

$$S_r^{-1} \rightarrow \frac{1}{S_n}(I - H(H^H H)^{-1} H^H) \triangleq \frac{1}{S_n} P_H^\perp \quad (3.22)$$

where $P_H^\perp$ denotes the orthogonal projection operator in the orthogonal complement of the space spanned by the columns of H. Using these relations, (3.11) simplifies to

$$\text{CRB}(\alpha)^{-1} = \frac{N}{2\pi} \int \text{Re}\left\{\frac{1}{S_n}[S_s \otimes (\mathbf{1}_2 \mathbf{1}_2^T)] \odot (D^H P_H^\perp D) + D^H H (H^H H)^{-1} \odot (D^H H (H^H H)^{-1})^T\right\} d\omega. \quad (3.23)$$

Large values of the signal's covariance mean that the uncertainty about the signal's distribution is large, similarly to what happens in the unknown deterministic model. Thus it is not surprising that (3.23) closely resembles the CRB for this problem derived using a deterministic signal model. In fact, the first term in this expression coincides with the CRB expression given by Stoica and Nehorai in [19], where a deterministic model is used for the source signals. The two expressions should not be the same since in our case the information with respect to the source location is in the covariance of the process, while in [19] it is in the mean value of the process. In the case of a stochastic source signal with known spectrum, as we assume here, the CRB must contain an additional term, reflecting the prior knowledge about the source covariance matrix. Note, however, that the additional term in (3.23) is independent of the signal-to-noise ratio and becomes negligible for high signal-to-noise ratios. In this situation, the impact of the prior knowledge is not important, since the signal is well estimated in either case, and the two expressions agree.

D. Multiple Source, Large Uniform Array, Single Path (P = 1)

For a single propagation path, $\{P_s = 1\}_{s=1,\cdots,S}$, H coincides with the steering matrix for the incoming incoherent S wavefronts. Instead of definition 3.1, consider (just for this subsection) the following alternative definition of α:

Definition 3.2: Consider the case of a uniform linear array with S impinging planar wavefronts. We define α as the S-dimensional vector of the directions of arrival (DOA's).

For the particular case of uniform linear array and planar wavefronts, let the intersensor spacing be constant and the number of sensors grow. Then the following relations hold (see [19, appendix G]):

$$\lim_{K\to\infty} \frac{1}{K} H^H H = I \tag{3.24}$$

$$\lim_{K\to\infty} \frac{2}{K^2} H^H D = jI \tag{3.25}$$

$$\lim_{K\to\infty} \frac{3}{K^3} D^H D = I. \tag{3.26}$$

Using these in the general equation (3.11) yields for high values of K

$$\text{CRB}(\alpha)^{-1} = \frac{N}{4\pi}\frac{K^4}{6}\int \frac{S_s}{S_n}(S_n S_s^{-1} + H^H H)^{-1}\, d\omega. \tag{3.27}$$

For the high signal-to-noise ratio situation

$$\text{CRB}(\alpha)^{-1} = \frac{N}{4\pi}\frac{K^3}{6}\int \frac{S_s}{S_n}\, d\omega. \tag{3.28}$$

This expression for a single path multiple source problem agrees with the large array size limit presented in [19] if we consider narrow-band signals. It shows that in the large signal-to-noise ratio limit, the estimates of the several source positions depend on the signal-to-noise ratio. This large array size limit is of questionable application since a large array is inconsistent with the planar wavefront assumption. From our point of view, the relevancy of the results in this and the previous subsection lies in the further insight they provide into the general CRB expression. For actual evaluations of the CRB, one should resort to the general expressions.

IV. Single Source ($S = 1$)

A. Mulitpath/Multiple Sensor

In the case of a single propagating source ($S = 1$), the general expression (3.11) simplifies considerably. In this case the source covariance S_s is a scalar and the resultant matrix H reduces to a single vector h. Then

$$(S_n S_s^{-1} + H^H H)^{-1} = \frac{S_s}{E} \tag{4.1}$$

where we defined

$$E \triangleq S_n + S_s\|h\|^2. \tag{4.2}$$

The inverse of S_r is obtained making $H = h$, i.e., a single vector, in (3.7)

$$S_r^{-1} = \frac{1}{S_n}\left(I_K - \frac{S_s}{E} hh^H\right). \tag{4.3}$$

Using these in (3.11) and after algebraic manipulations (see Appendix B) we obtain the following.

Fact 4.1 (Single Source, Multipath CRB): Consider the single source/multipath problem version of (2.1), where the spectral density function of the source is known. Then, the Cramér–Rao bound for the location parameters α is

$$\text{CRB}(\alpha)^{-1} = \frac{N}{2\pi}\int \left\{K_1 \,\text{Re}\left\{\frac{\partial h^H}{\partial\alpha} P_h^{\perp} \frac{\partial h}{\partial \alpha}\right\} + K_2 \,\text{Re}\left\{\frac{\partial h^H}{\partial \alpha} h\right\}\text{Re}\left\{h^H \frac{\partial h}{\partial\alpha}\right\}\right\} d\omega \tag{4.4}$$

where

$$K_1 = \frac{S_s^2}{S_n E}\|h\|^2 \tag{4.5}$$

$$K_2 = 2\frac{S_s^2}{E^2} \tag{4.6}$$

and $P_h^{\perp}$ denotes the orthogonal projection matrix in the orthogonal complement of the resultant vector h:

$$P_h^{\perp} = I_K - \frac{1}{\|h\|^2} hh^H. \tag{4.7}$$

□

Expression (4.4) is the CRB when a single source is received over several distinct paths and has been presented in [15]. We point out that the two terms in this expression do not have a one-to-one correspondence with the two terms in (3.11). There has been a rearrangement of the detailed structure of (3.11) to obtain (3.19).

1) Interpretation: Expression (4.4) has an interesting interpretation in terms of the dual temporal/spatial domains of variation of the observed wavefield. The first term is related to the information in the spatial domain, measuring how much the resultant direction vector h changes with source position. The second term is related to the temporal structure. As we will see in Section VI, it represents the additional information about the source location due to knowledge of the source covariance matrix.

To get further insight into the nature of the second term in (4.4), let us rewrite its (α, β) element as

$$\mathrm{Re}\,\{\dot{h}_\alpha^H h\}\ \mathrm{Re}\,\{h^H \dot{h}_\beta\} = \sum_{k=1}^{K} \mathrm{Re}\,\{\dot{h}_{\alpha_k}^* h_k\} \sum_{l=1}^{K} \mathrm{Re}\,\{h_l^* \dot{h}_{\beta_l}\}. \tag{4.8}$$

The component at sensor k of the resultant vector h_k is the sum of the P incoming replicas at sensor k, and is given by (2.5). In vector notation

$$h_k = \boldsymbol{a}_k^T \epsilon_k \tag{4.9}$$

where we defined

$$\epsilon_k \triangleq [e^{j\omega\tau_{k1}} \cdots e^{j\omega\tau_{kP}}] \tag{4.10}$$

and $\boldsymbol{a}_k$ groups all the attenuation coefficients for sensor k. Alternatively, we can write h_k as

$$h_k = \rho_k e^{j\delta_k} \tag{4.11}$$

where $\rho_k \in \boldsymbol{R}^+$ is the modulus of the complex number h_k, $\delta_k \in [-\pi, \pi]$ is its phase, and they both depend on the travel times $\{\tau_{kp}\}_{p=1}^{P}$. Using this, we can write

$$[\dot{h}_\alpha]_k = e^{j\delta_k}([\dot{\rho}_\alpha]_k + j\rho_k[\dot{\delta}_\alpha]_k) \tag{4.12}$$

resulting in

$$\mathrm{Re}\,\{[\dot{h}_\alpha^*]_k h_k\} = [\dot{\rho}_\alpha]_k \rho_k \tag{4.13}$$

which finally gives the following expression for the (α, β) element in the second term of CRB $(\alpha)^{-1}$:

$$\mathrm{Re}\,\{\dot{h}_\alpha^H h\}\ \mathrm{Re}\,\{h^H \dot{h}_\beta\} = \sum_{k=1}^{K} ([\dot{\rho}_\alpha]_k \rho_k) \sum_{l=1}^{K} ([\dot{\rho}_\beta]_l \rho_l). \tag{4.14}$$

Equation (4.14) expresses the second term as the product of two factors, each one being the inner product of the vector of derivatives of the amplitudes of the resultant vector $(\dot{\rho}_\alpha)$ by the vector of amplitudes itself (ρ):

$$\mathrm{Re}\,\{\dot{h}_\alpha^H h\}\ \mathrm{Re}\,\{h^H \dot{h}_\beta\} = \dot{\rho}_\alpha^T \rho \rho^T \dot{\rho}_\beta. \tag{4.15}$$

To have a more precise idea about the dependency of this term on the delays $\{\tau_{kp}\}$, we rewrite it in terms of the individual travel times. We begin by noting that

$$\rho_k^2 = \boldsymbol{a}_k^T \epsilon_k \epsilon_k^H \boldsymbol{a}_k = \boldsymbol{a}_k^T \Delta_k \boldsymbol{a}_k \tag{4.16}$$

where we have defined the $(P \times P)$ matrix Δ_k:

$$[\Delta_k]_{pq} = e^{j\omega(\tau_{kp} - \tau_{kq})}. \tag{4.17}$$

Using these equations, we obtain for the derivative $[\dot{\rho}_k]_\alpha$

$$[\dot{\rho}_k]_\alpha = \omega \frac{1}{\rho_k} (\boldsymbol{a}_k \odot \dot{\tau}_k)^T\, \mathrm{Im}\,\{\Delta_k\} \boldsymbol{a}_k \tag{4.18}$$

and

$$\sum_{k=1}^{K} [\dot{\rho}_k]_\alpha \rho_k = \beta \sum_{k=1}^{K} (\boldsymbol{a}_k \odot [\dot{\tau}_k]_\alpha)^T\, \mathrm{Im}\,\{\Delta_k\} \boldsymbol{a}_k. \tag{4.19}$$

We have used here the vector notation introduced in Section II-B where by τ_k we mean the vector with components τ_{kp}, $p = 1, \cdots, P$. We can finally write the right-hand side of (4.14) as

$$\sum_{k=1}^{K} [\dot{\rho}_k]_\alpha \rho_k \sum_{k=1}^{K} [\dot{\rho}_k]_\beta \rho_k = \omega^2 \sum_{k=1}^{K} (\boldsymbol{a}_k \odot [\dot{\tau}_k]_\alpha)^T\, \mathrm{Im}\,\{\Delta_k\} \boldsymbol{a}_k \cdot \sum_{k=1}^{K} (\boldsymbol{a}_k \odot [\dot{\tau}_k]_\beta)^T\, \mathrm{Im}\,\{\Delta_k\} \boldsymbol{a}_k. \tag{4.20}$$

This term is dependent on the sine of the time differences of arrival at sensor k:

$$[\mathrm{Im}\,\{\Delta_k\}]_{pq} = \sin \omega(\tau_{kp} - \tau_{kq}) \tag{4.21}$$

yielding for (4.14) an expression with the general form of the single sensor case, see [14, eq. (23)] and Section IV-C. Note that in (4.20) no intersensor delays appear. The expression is only dependent on the interpath delays. This shows that the second term of (4.4) is only dependent on the signals' temporal structure (multipath) and not on their spatial structure (intersensor delay for the same path).

We finally point out another important characteristic that distinguishes this term from the (spatially related) first term in (4.4): for each frequency component it has rank 1, while the first term has rank min $\{\dim(\alpha), (K - 1)\}$ (refer to (4.4)).

2) Behavior with Signal-to-Noise Ratio: The two factors K_1 and K_2 exhibit different variations with signal-to-noise ratio. While $K_1 \to \infty$, where $S_S/S_n \to \infty$, K_2 tends to

$$K_2 \to \frac{2}{\|h\|^4}. \tag{4.22}$$

As long as the received signal energy is different from zero, i.e., $\|h\| \neq 0$, this limit is finite. For this reason, we will often refer to the second term in (4.4) as the "saturating term" of the CRB. The same behavior has already been noted [5], [4] for the multipath contribution to the FIM of the range-only estimation problem with $P = 2$ (two paths).

In Section V we treat the case of a single source in full detail, analyzing the influence of the temporal and the spatial structure of the incoming wavefield in the location performance. Since particularly simple expressions are obtained when, in addition to single source $(S = 1)$, we assume single propagation path $(P = 1)$ or single sensor $(K = 1)$, we present here the CRB for these two situations.

B. Single Source, Single Path ($S = 1$, $P = 1$)

For this case, see (2.5)

$$h_k = \boldsymbol{a}_k e^{j\iota\omega\tau_k} \tag{4.23}$$

i.e.,

$$\rho_k = a_k \tag{4.24}$$

and according to assumption 2.1

$$\dot{\rho}_\alpha = 0 \tag{4.25}$$

and the "saturating term" of CRB(α) is zero, (4.15), leaving just the first term in (4.4).

Definition 4.1 (θ): Consider the single source/single path version of (2.1). Define the vector of delays relative to sensor 1:

$$\theta^T = [0 \quad \tau_2 - \tau_1 \quad \cdots \quad \tau_K - \tau_1]. \tag{4.26}$$

Definition 4.2 (a): Consider the single source/single path version of (2.1). Let a be the vector of attenuations from the source to the different sensors:

$$a^T = [a_1 \quad a_2 \quad \cdots \quad a_K]. \tag{4.27}$$

Fact 4.2 (Single Source/Single Path CRB): Consider the single source ($S = 1$), single path ($P_s = 1$) version of (2.1), where the source spectrum is known. Then the Cramér-Rao bound on the error of the source location is

$$\text{CRB}(\alpha)^{-1} = \frac{N}{2\pi} \int K_1 \omega^2 \frac{\partial \theta^H}{\partial \alpha} \Lambda_a \left(I_K - \frac{aa^T}{\|a\|^2} \right) \Lambda_a \frac{\partial \theta}{\partial \alpha} \, d\omega \tag{4.28}$$

where θ and a were defined in definitions 4.1 and 4.2, respectively, K_1 was defined in (4.5), and $\Lambda_a = \text{diag}\{a\}$.

Equation (4.28) follows immediately from the first term of (4.4), and coincides with the expression given for this simple case in [18]. It confirms our previous discussion about the "saturating term" in the CRB, that vanishes in this case since there are no interpath delays.

C. Single Source, Single Sensor ($S = 1$, $K = 1$)

The case of a single receiving sensor is particularly interesting, since in this case all the information about the source location is contained in the multipath delays. In this case, h is a complex scalar and (from (4.7)), $P_h^\perp = 0$, leaving just the second term in (4.4).

Fact 4.3 (Single Source/Single Sensor CRB): Consider the single source ($S = 1$), single sensor ($K = 1$) version of (2.1), where the source spectrum is known. Then the Cramér-Rao bound on the error of the source location is

$$\text{CRB}(\alpha)^{-1} = \frac{N}{2\pi} \int K_2 \,\text{Re}\left\{ \frac{\partial h^H}{\partial \alpha} h \right\} \text{Re}\left\{ h^T \frac{\partial h^*}{\partial \alpha} \right\} d\omega \tag{4.29}$$

where K_2 was defined in (4.6). □

Using (4.21) and the equations that precede it, we obtain the following expression:

$$\text{CRB}(\alpha)^{-1} = \frac{N}{2\pi} \int K_2 \omega^2 \sum_{p=1}^{P} a_p \dot{\tau}_{p_\alpha}^T \sum_{n=1}^{P} a_n \sin \omega(\tau_p - \tau_n) \cdot \sum_{q=1}^{P} \sum_{m=1}^{P} a_m \sin \omega(\tau_q - \tau_m) \dot{\tau}_{q_\alpha} a_q \, d\omega \tag{4.30}$$

where $\dot{\tau}_{p_\alpha}$ is the 2-dimensional row vector of derivatives of the delay τ_p with respect to the location of the source. This expression is given in [14] for the single sensor case. Using (4.14) with $K = 1$, and the definition of K_2 in (4.6), the integrand of (4.29) can be written as

$$K_2 \,\text{Re}\left\{ \frac{\partial h^H}{\partial \alpha} h \right\} \text{Re}\left\{ h^H \frac{\partial h}{\partial \alpha} \right\} = \frac{1}{2} \frac{S_s^2}{(S_n + S_s \rho^2)^2} \frac{\partial \rho^{2T}}{\partial \alpha} \frac{\partial \rho^2}{\partial \alpha} = \frac{1}{2} \left(\frac{\partial \ln E}{\partial \alpha} \right)^T \frac{\partial \ln E}{\partial \alpha}$$

where ρ is introduced in (4.11) (because $K = 1$ we omit the subindex) and the received energy E is in (4.2). This last equation shows that the contribution of each frequency to the FIM is proportional to the variation of ln E, the logarithm of the received energy. The frequencies at which this variation is more rapid are those that give a larger contribution to the FIM. For two propagation paths, it is shown in [5] that the integrand of (4.29) is maximized when the paths have equal factors which occurs in the vicinity of the frequency for which the received energy is zero.

Equation (4.30) shows that the CRB decreases with the temporal resolvability. It indicates that to obtain an acceptable performance, high signal bandwidths may be required. For each frequency, this term has rank one. Then, the estimation problem will be singular unless as a function of ω there is sufficient variability on the 2-dimensional vectors Re $\{(\partial h^H / \partial \alpha) h\}$. In particular, for narrow-band signals no estimation will be possible.

Fact 4.4 (Single Narrow-Band Source, Single Sensor): For a narrow-band source and a single sensor, the FIM matrix is singular.

The proof is trivial considering the single-frequency limit of fact 4.3, when the integral reduces to a single frequency contribution.

V. Influence of Spatial and Temporal Structure

In this section, we analyze the multipath contribution to the CRB on the location parameters of a single radiating source. We assume assumption 3.1 is valid.

Classical methods for source location with array observations are based on the estimation of wavefront curvature and orientation at the receiving array. In these systems, only the spatial structure of the wavefield is analyzed. With the use of more sophisticated propagation models, increasing interest in assessing the temporal structure of the observations arose [2], [4], [1], [20]. Here we evaluate, for a multipath model, the impact that temporal processing may have in the quality of the estimates.

To simplify the analysis, we consider in this section the case of a single propagating source. For this case, see fact 4.1, the CRB on the location of the source is given by (4.4). According to assumptions 2.1 and 3.1, the resultant vector h depends on the source location α through the set of travel times τ_{kp} (refer to (2.1)).

To make the study of the impact of the temporal structure of the observations clear, we decompose each individual τ_{kp} into a multipath (θ_{mp}) and an intersensor (θ_s) component:

$$\tau_{kp} = \tau_{11} + \theta_{mp_p} + \theta_{s_{kp}} \qquad k = 1, \cdots, K; \quad p = 1, \cdots, P. \tag{5.1}$$

This decomposition is illustrated in Fig. 2. We distinguish two kinds of delays: i) the intersensor delays $\theta_{s_{kp}}$, that are the delays for path p at sensor k relative to the arrival time of the same path p at the reference sensor 1; ii) the interpath delays θ_{mp_p} that measure the difference of the arrival time of path p at the reference sensor 1 relative to the arrival time of the reference path 1 at the same sensor.

Definition 5.1 (θ_{mp}): Denote by θ_{mp} the P-dimensional real vector of generic element

$$\theta_{mp_p} = \tau_{1p} - \tau_{11}, \qquad p = 1, \cdots, P. \tag{5.2}$$

Definition 5.2 (θ_s): Denote by θ_s the ($K \times P$) real matrix of generic element:

$$\theta_{s_{kp}} = \tau_{kp} - \tau_{1p}, \qquad k = 1, \cdots, K; \quad p = 1, \cdots, P. \tag{5.3}$$

This decomposition of the travel times into interpath and intersensor delays although resembling the set of interdelays used in [4] is distinct from it. In [4], the authors introduced an extended, linearly dependent set of parameters consisting of the same set of intersensor delays as used here, (5.3), plus, in addition, the set of all interpath delays at all sensors, not simply at the reference sensor as done here. In contrast with [4], the set of delays we use is linearly independent.

One of the goals of this study is the evaluation of the potential increase in performance due to the information in the multipath replicas. Using (5.1), we are able to keep the set of delays observable by the array (θ_s) and isolate in the vector of interpath delays (θ_{mp}) the parameters that define the temporal alignment of the several wavefronts.

Systems that use spatial/temporal models and those that are based only on spatial structure are naturally compared in terms of the sets θ_{mp} and θ_s: while the first kind of systems consider both θ_s and θ_{mp} as unknown parameters dependent on the source location, the latter consider only θ_s dependent on source location with θ_{mp} being treated as an unwanted parameter that has to be estimated along with source position and whose presence implies a degradation in performance.

For simplicity, we make the following assumption.

Assumption 5.1: The attenuation factors do not depend on sensor index

$$a_{kp} = a_p, \qquad k = 1, \cdots, K; \qquad p = 1, \cdots, P. \tag{5.4}$$

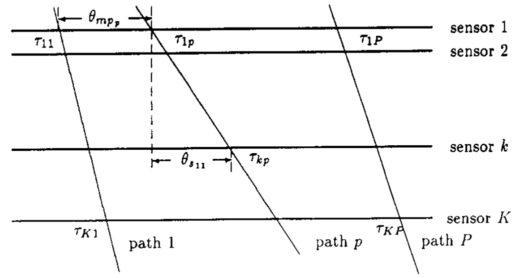

Fig. 2. Decomposition into multipath and intersensor components.

Definition 5.3 (A): Let A be the ($K \times P$) steering matrix, whose columns are the steering vectors for the P incoming paths. A describes the wavefront shapes at the receiving array corresponding to the P distinct paths:

$$A = \begin{bmatrix} 1 & \cdots & 1 \\ e^{j\omega(\tau_{21} - \tau_{11})} & \cdots & e^{j\omega(\tau_{2P} - \tau_{1P})} \\ \vdots & & \vdots \\ e^{j\omega(\tau_{K1} - \tau_{11})} & \cdots & e^{j\omega(\tau_{KP} - \tau_{1P})} \end{bmatrix}. \tag{5.5}$$

Definition 5.4 (b): Let b be the P-dimensional vector that groups all the interpath delays and attenuations, i.e., b describes the temporal structure of the signal impinging at reference sensor 1:

$$b^T = [a_1 \quad a_2 e^{j\omega(\tau_{12} - \tau_{11})} \quad \cdots \quad a_P e^{j\omega(\tau_{1P} - \tau_{11})}]. \tag{5.6}$$

Using these definitions

$$h = Ab \tag{5.7}$$

where A (steering matrix) depends only on the intersensor delays θ_s, and describes the wavefront curvatures at the array, and b depends on the attenuations $\{a_p\}$ and multipath delays $\{\theta_{mp}\}$.

Definition 5.5 (θ): Let θ be the ($P + KP$)-dimensional vector of real parameters

$$\theta^T = [\theta_{mp}^T \ \text{vec}\,(\theta_s)^T]. \tag{5.8}$$

According to the temporal and spatial references chosen, $\theta_{mp_1} = 0$, and $\{\theta_{s_{1p}}\}_{p=1}^{P} = 0$, so that we have in fact only $KP - 1$ nontrivial parameters. Since all information is encoded in the observation's covariance matrix, which is insensitive to absolute time reference, τ_{11} is not an observable parameter. To obtain simpler expressions, we include the trivial parameters θ_{mp_1} and $\{\theta_{s_{1p}}\}_{p=1}^{P}$ in the analysis, which is possible since their derivatives with respect to the source location are identically zero.

Using the chain rule of derivatives, we write the FIM for α in terms of $\mathfrak{F}(\theta)$:

$$\mathfrak{F}(\alpha) = \frac{\partial \theta^T}{\partial \alpha} \mathfrak{F}(\theta) \frac{\partial \theta}{\partial \alpha} \tag{5.9}$$

where $\partial\theta/\partial\alpha$ is the $(P(K+1) \times 2)$ matrix of first-order derivatives[3]:

$$\frac{\partial\theta^F}{\partial\alpha} = \left[\frac{\partial\theta_{mp}^T}{\partial\alpha}\ \frac{\partial\theta_s^T}{\partial\alpha}\right] \tag{5.10}$$

and the generic element of $\mathfrak{F}(\theta)$ is given by (3.3), with θ_i substituted for α_i.

Partition of the vector θ into multipath and intersensor parameters induces a corresponding partition on the $\mathfrak{F}(\theta)$:

$$\mathfrak{F}(\theta) = \begin{bmatrix} \mathfrak{F}(\theta_{mp}) & \mathfrak{F}(\theta_{mp}, \theta_s) \\ \mathfrak{F}(\theta_s, \theta_{mp}) & \mathfrak{F}(\theta_s) \end{bmatrix}. \tag{5.11}$$

Using

$$\frac{\partial h}{\partial\theta_{mp}} = j\omega AB \tag{5.12}$$

where $B = \text{diag}\,(b)$ (see (5.6) for the definition of b), and

$$\frac{\partial h}{\partial\theta_s} = j\omega[b_1 A_1 | \cdots | b_P A_P] \tag{5.13}$$

where the $(P \times P)$ diagonal matrices $A_i = \text{diag}\,(a_i)$, in the general expression (4.4), yields

$$\begin{aligned}\mathfrak{F}(\theta_{mp}) = \frac{N}{4\pi}\int \omega^2\{&K_1\,\text{Re}\,\{B^*A^HP_h^\perp AB\} \\ &+ K_2\,\text{Im}\,\{B^*A^HAb\} \\ &\cdot \text{Im}\,\{(B^*A^HAb)^T\}\}\,d\omega\end{aligned} \tag{5.14}$$

$$\begin{aligned}[\mathfrak{F}(\theta_s)]_{pq} = \frac{N}{4\pi}\int \omega^2\{&K_1\,\text{Re}\,\{b_p^*A_p^HP_h^\perp A_qb_q\} \\ &+ K_2\,\text{Im}\,\{b_p^*A_p^HAb\} \\ &\cdot \text{Im}\,\{(b_q^*A_q^HAb)^T\}\}\,d\omega\end{aligned} \tag{5.15}$$

$$\begin{aligned}[\mathfrak{F}(\theta_s, \theta_{mp})]_p = \frac{N}{4\pi}\int \omega^2\{&K_1\,\text{Re}\,\{b_p^*A_p^HP_h^\perp AB\} \\ &+ K_2\,\text{Im}\,\{b_p^*A_p^HAb\} \\ &\cdot \text{Im}\,\{(B^*A^HAb)^T\}\}\,d\omega.\end{aligned} \tag{5.16}$$

Using (5.10) and (5.11) we write $\text{CRB}(\alpha)^{-1}$ as the sum of four terms:

$$\begin{aligned}\text{CRB}^{-1}(\alpha) = &\frac{\partial\theta_{mp}^T}{\partial\alpha}\mathfrak{F}(\theta_{mp})\frac{\partial\theta_{mp}}{\partial\alpha} + \frac{\partial\theta_s^T}{\partial\alpha}\mathfrak{F}(\theta_s)\frac{\partial\theta_s^T}{\partial\alpha} \\ &+ \frac{\partial\theta_{mp}^T}{\partial\alpha}\mathfrak{F}(\theta_{mp}, \theta_s)\frac{\partial\theta_s}{\partial\alpha} + \frac{\partial\theta_s^T}{\partial\alpha} \\ &\cdot \mathfrak{F}(\theta_s, \theta_{mp})\frac{\partial\theta_{mp}}{\partial\alpha}.\end{aligned} \tag{5.17}$$

We note that in general the estimates of the spatial and temporal delays are not independent, i.e., the off-diagonal subblocks in $\mathfrak{F}(\theta)$ do not vanish.

[3]For simplicity we use from now on θ_s for vec (θ_s).

Let CRB_{mp}^{-1} denote the term in the right-hand side of (5.17) that depends only on the interpath delays' derivatives:

$$\begin{aligned}\text{CRB}^{-1}(\alpha)_{mp} = \frac{N}{4\pi}\int \omega^2\{&K_1\,\text{Re}\,\{\psi_{mp}^HA^HP_h^\perp A\psi_{mp}\} \\ &+ K_2\,\text{Im}\,\{\psi_{mp}^HA^HAb\} \\ &\cdot \text{Im}\,\{(\psi_{mp}^HA^HAb)^H\}\}\,d\omega\end{aligned} \tag{5.18}$$

where we have defined

$$\psi_{mp} = \text{diag}\,(b)\frac{\partial\theta_{mp}}{\partial\alpha}. \tag{5.19}$$

This term would equal $\text{CRB}(\alpha)^{-1}$ if the intersensor delays were known. It gives a measure of how relevant the multipath structure of the observations is to the estimation performance. In the following, we analyze the behavior of this term under different limits of spatial resolution. A detailed study of the frequency regions where these approximations are possible has been made in [3] for a linear uniform array and $P = 2$, with particular source/array geometries.

We want to stress that the expressions obtained here are dependent on the decomposition of the travel times that we used throughout this study (5.1). Instead of keeping all the intersensor delays and defining as temporal parameters the interpath delays at the reference sensor, we could do the opposite, defining instead a vector of interpath delays at each sensor and a single vector of intersensor delays that would describe the spatial alignment of the several paths. Not only do we think this decomposition more natural (since sensors are distinct physical entities that can be independently assessed, while path contributions are not individually measurable), but it also has the advantage of being adequate to our study of the improvement in performance of systems that use all the delays in the observed wavefield when compared to systems that do only spatial processing.

A. Interpretation of the Multipath Contribution to the FIM

The previous expressions show that the multipath contribution, $\text{CRB}\,(\alpha)_{mp}^{-1}$, to the Cramér–Rao bound depends on the matrix A^HA. This term is the generalized beampattern of the array that describes the amount of interference between each pair of paths.

We analyze the multipath contribution $\text{CRB}\,(\alpha)_{mp}^{-1}$, for two extreme beampatterns: i) $A^HA \simeq KI$, which occurs when the steering vectors are approximately orthogonal, i.e., the array is able to separate the paths, and ii) $A^HA \simeq K\mathbf{1}\mathbf{1}^T$, when the incoming wavefronts are undistinguishable, i.e., they are all within the main lobe of the array, their separation being smaller than the Rayleigh resolution limit. Condition i) will usually be valid at low frequencies, while condition ii) may well be violated at high frequencies. A more restricted version of this problem is considered in [4]. We will also consider an intermediate case (iii) below.

i) In the first case, ($A^H A \simeq KI$), it is easily shown, using the definition of $P_b^{\perp}$, that

$$A^H P_b^{\perp} A \simeq K P_b^{\perp} \tag{5.20}$$

where $P_b^{\perp}$ denotes the (orthogonal) projection matrix on the orthogonal complement of the vector b. Furthermore, in this case the "saturating term" in (5.18) is zero, and

$$\text{CRB}(\alpha)_{mp}^{-1} = K \frac{N}{2\pi} \int K_1 \omega^2 \frac{\partial \theta_{mp}^T}{\partial \alpha} \Lambda_a \left(I_P - \frac{aa^T}{\|a\|^2} \right) \cdot \Lambda_a \frac{\partial \theta_{mp}}{\partial \alpha} \, d\omega \tag{5.21}$$

which is K times (4.28) and $\Lambda_a = \text{diag}\{a\}$ is as in fact 4.2. Thus, when the impinging wavefronts are well resolved, the multipath contribution is equivalent to having K identical virtual arrays operating independently, each one with P sensors, observing a single impinging wavefront, with steering vector

$$b(\omega) = e^{j\omega\Theta_{mp}} a \tag{5.22}$$

where $\Theta_{mp} = \text{diag}(\theta_{mp})$. Note that the virtual array no longer has identical sensors: the vector a, of attenuation coefficients which in general has distinct entries, describes the gain of each of its sensors. Since the paths are well resolved by the array, the contribution of each path at each sensor can be estimated and considered as independently observed. In this analogy, the combined effects of array geometry and wavefront shape are subsummed by the vector $b(\omega)$ that describes the relative attenuations and delays for the P incoming paths at the reference sensor. Here, we may not control $b(\omega)$ by designing the receiving array, as it happens with the "physical" array.

We see that multipath results, in some sense, in an increase of the spatial aperture of the array. The efficiency with which this "synthesized" array locates the source depends, of course, on the particular locations of the receiver and the source, and also on the medium's characteristics.

We see that the geometric argument underlying the technique known as "multipath ranging" can be carried over to predicting the system's performance only in the limit of perfect spatial resolution. Multipath ranging [17] interprets each secondary path arriving at a sensor as a virtual sensor whose position relative to the physical sensor is determined by the differential delay between that path and the principal (reference) path.

ii) We analyze now the case of unresolved paths (approximately colinear steering vectors):

$$A^H A = K\mathbf{1}\mathbf{1}^T. \tag{5.23}$$

Using this approximation in (5.18), we get

$$\text{CRB}(\alpha)_{mp}^{-1} = K \frac{N}{2\pi} \int K_2 \omega^2 \frac{\partial \theta_{mp}^T}{\partial \alpha} \text{Im}\left\{ b^* \sum_p b_p \right\} \cdot \text{Im}\left\{ \sum_p b_p b^H \right\} \frac{\partial \theta_{mp}}{\partial \alpha} \, d\omega \tag{5.24}$$

which is K times the single sensor expression (4.29). This situation corresponds to K independent single sensors observing the superposition of the P incoming paths. Thus, when the paths are not resolved by the array, the multipath contribution reduces to the expected incoherent sum of the contribution of each single sensor.

iii) The two extreme cases above suggest that the multipath component may be interpreted as the contribution of K virtual arrays, whose sizes are equal to the number of spatially resolved replicas. To validate this hypothesis, we consider the intermediate case:

$$A^H A = K \begin{bmatrix} \mathbf{1}_{n_1}\mathbf{1}_{n_1}^T & \underline{0} & \cdots & \underline{0} \\ \underline{0} & \mathbf{1}_{n_2}\mathbf{1}_{n_2}^T & & \vdots \\ & \vdots & \ddots & \underline{0} \\ \underline{0} & \cdots & \underline{0} & \mathbf{1}_{n_r}\mathbf{1}_{n_r}^T \end{bmatrix} \tag{5.25}$$

where $\Sigma_i n_i = P$.

In this case, we have r "clusters" of rays, that are resolved by the array, although the n_i rays within "cluster" i are not resolved.

Using this approximation in (5.18), we get

$$\text{CRB}(\alpha)_{mp}^{-1} = K \frac{N}{2\pi} \int \left\{ K_1 \text{Re}\left\{ \frac{\partial d^H}{\partial \alpha} P_d^{\perp} \frac{\partial d}{\partial \alpha} \right\} + K_2 \text{Re}\left\{ \frac{\partial d^H}{\partial \alpha} d \right\} \text{Re}\left\{ d^T \frac{\partial d}{\partial \alpha} \right\} \right\} d\omega \tag{5.26}$$

where we have defined the r-dimensional vector

$$d \triangleq \begin{bmatrix} \mathbf{1}_{n_1}^T & & \\ & \ddots & \\ & & \mathbf{1}_{n_r}^T \end{bmatrix} b = \begin{bmatrix} \mathbf{1}_{n_1}^T b^1 \\ \vdots \\ \mathbf{1}_{n_r}^T b^r \end{bmatrix} \tag{5.27}$$

and

$$b^i = [0 \; | I_{n_i} | \; 0] b. \tag{5.28}$$

Comparing this with the general expression (4.4), we see that the multipath contribution is equivalent to that of K fictitious arrays of r sensors each, the component of the resultant vector for sensor i being the coherent sum of the unresolved replicas in "cluster" i.

Fact 5.1: Consider the single source version of (2.1), where the source signal has a known spectral density function. Furthermore, assume that the P incoming paths are grouped in r "clusters," so that the beampattern matrix can be approximated by (5.25). The multipath contribution to the Fisher information matrix is equivalent to the contribution of K fictitious arrays, each one with size equal to the number r of "clusters." The steering vector for each equivalent array is given by (5.27). Its component for sensor k is the superposition of the unresolved paths in the corresponding "cluster" k.

1) General Case: The three points discussed above are particular cases of the following fact, which is obtained using the spectral representation theorem for the beampattern matrix.

Fact 5.2: Consider the single source version of (2.1), where the source signal has a known spectral density function. Furthermore, consider the spectral representation of the beampattern matrix

$$A^H A = U \Lambda U^H \tag{5.29}$$

where U is a $(P \times r)$ matrix verifying $U^H U = I_r$, and Λ is the $(r \times r)$ diagonal matrix of the positive eigenvalues of $A^H A$. In general, $r \leq P$. Then, the multipath contribution to the Fisher information matrix is equivalent to the contribution of a fictitious array of r sensors, with resultant vector:

$$d = \Lambda^{1/2} U^H b. \tag{5.30}$$

Note that i) is obtained from fact 5.2 making $r = P$, $\Lambda = KI_P$ and $U = I_P$; ii) corresponds to $r = 1$, $\Lambda = K^2$, $U = 1/K^{1/2}\, \mathbf{1}_P$, and to recover iii) we make: $\Lambda = K$ diag $(n_i)_{i=1}^r$ and take U as the block-diagonal matrix $U =$ diag $(n_i^{-1/2} \mathbf{1}_{n_i})_{i=1}^r$.

B. Gain Due to Temporal Structure

Considering the limiting case of perfect spatial resolution, we determine the gain of systems that use the information in the set of interpaths' delays over systems that use only the information in the intersensor delays, θ_s. Refer to (5.17). Let CRB $(\alpha)_s^{-1}$ be the term of CRB $(\alpha)^{-1}$ dependent only on the intersensor delays' derivatives:

$$\text{CRB } (\alpha)_s^{-1} = \frac{\partial \theta_s^T}{\partial \alpha} \text{CRB } (\theta_s)^{-1} \frac{\partial \theta_s}{\partial \alpha}.$$

It would equal CRB $(\alpha)^{-1}$ if the interpath delays (i.e., vector b) were known.

If only the set θ_s is used in a constructive way to locate the source, i.e., if the differential delays θ_{mp} are modeled as unknown deterministic variables independent of source location, the CRB on the location vector α is given by

$$\text{CRB } (\alpha)_{spa}^{-1} = \frac{\partial \theta_s^T}{\partial \alpha} \text{CRB } (\theta_s)^{-1} \frac{\partial \theta_s}{\partial \alpha} \tag{5.31}$$

where CRB $(\theta_s)^{-1}$ is calculated using the formula for the inversion of a partitioned matrix on the complete FIM, (5.11), yielding

$$\text{CRB } (\theta_s)^{-1} = \mathfrak{F}(\theta_s) - \mathfrak{F}(\theta_s, \theta_{mp}) \mathfrak{F}(\theta_{mp})^{-1} \mathfrak{F}(\theta_{mp}, \theta_s). \tag{5.32}$$

We compute in this section the difference between the FIM for systems that use a complete spatial/temporal model of the incoming wavefield, given by (5.17) and that we will denote here by CRB $(\alpha)_{\text{spa/tem}}^{-1}$, and the FIM for systems that use only spatial information, given by (5.31)

$$\mathcal{G} = \text{CRB } (\alpha)_{\text{spa/tem}}^{-1} - \text{CRB } (\alpha)_{\text{spa}}^{-1}. \tag{5.33}$$

This difference gives us the gain in information due to constructive use of the temporal structure of the field.

Denote by $\mathcal{L}$ the loss term in (5.32):

$$\mathcal{L} = \mathfrak{F}(\theta_s, \theta_{mp}) \mathfrak{F}(\theta_{mp})^{-1} \mathfrak{F}(\theta_{mp}, \theta_s).$$

When the several paths are completely resolved, $A^H A = KI$, (situation i) in the last section), we show in Appendix C that the loss term is given by

$$\mathcal{L} = \frac{1}{C^2} \Phi_s^T P_a^\perp \Phi_s \tag{5.34}$$

where $P_a^\perp$ is the projection matrix on the orthogonal complement of $\boldsymbol{a}$ and $\Phi = [\Phi_{s1}, \cdots, \Phi_{sP}]$, where

$$[\Phi_s]_p^T = \int \omega^2 K_1 \text{ Re } \{b_p^* A_p^H A B \Lambda_a^{-1}\}$$

$$C^2 = K \int \omega^2 K_1 \, d\omega. \tag{5.35}$$

The matrix Φ_s depends only on the interpath delays at each sensor, i.e., of the temporal resolvability of the paths at each receiving sensor.

After several algebraic manipulations we obtain for the gain of the systems that use the complete model of the observed field:

$$\mathcal{G} = V^T P_a^\perp V \tag{5.36}$$

where

$$V = \frac{1}{C} \left[\Phi_s \frac{\partial \theta_s}{\partial \alpha} + \Lambda_a \frac{\partial \theta_{mp}}{\partial \alpha} \right].$$

In this last equation the trivial parameter θ_{mp_1} has been reinserted in θ_{mp}. It shows that the CRB for the complete model is not larger than the CRB for systems based only on spatial structure.

We point out that, as it should be expected, (5.36) involves the matrix $P_a^\perp$, that has rank $P - 1$, since only $P - 1$ additional degrees of freedom have been added to the model, defining the temporal alignment of the received paths at the reference sensor.

VI. Unknown Source Spectral Parameters

In this section we drop assumption 3.1 of known source spectral density function used in the two previous sections. We consider two distinct situations: (Section VI-A) $S_s(\omega)$ is completely arbitrary, and (Section VI-C) $S_S(\omega)$ has a known parametric form, with unknown deterministic parameters. In the first case, the FIM for the location parameters reduces to the first term of the FIM for the known spectral density function case. When there is a single sensor available to locate the source, the FIM is zero and no estimation is possible. In the second case, besides the first term of the FIM for known source spectrum, there is an additional term that depends on the uncertainty about the spectrum, and source estimation can still be possible with just a single sensor.

A. Nonparametric Spectra

When the source spectral density function, $S_s(\omega)$, is not known it must be estimated along with the location parameters of the source. The FIM for the vector of source locations still follows (3.3), but the CRB must now be computed from the FIM for the extended parameter vector, as follows.

Definition 6.1 (β): Let β be the extended parameter vector:

$$\beta^T = [\alpha^T \eta^T] \tag{6.1}$$

where α is the vector of source location parameters defined in Section III, definition 3.1, and η groups all the parameters that define the source spectral density.

Assumption 6.1 (Completely Unknown Spectrum): The source signal spectral density function is completely unknown to the receiver.

Consider a discretization of the frequency domain. Define element i of η as the value of the source spectrum at frequency ω_i:

$$\eta_i = S_s(\omega_i), \qquad i = 1, \cdots, L \tag{6.2}$$

where L is the number of discrete frequencies.

The CRB for α is the upper $(P(K+1) \times P(K+1))$ block of CRB $(\beta) = \mathfrak{F}(\beta)^{-1}$:

$$\mathfrak{F}(\beta) = \begin{bmatrix} \mathfrak{F}(\alpha) & \mathfrak{F}(\alpha, \eta) \\ \mathfrak{F}(\eta, \alpha) & \mathfrak{F}(\eta) \end{bmatrix} \tag{6.3}$$

where $\mathfrak{F}(\alpha)$ is the FIM for α when the source spectrum is known, given by (4.4).

Using the formulas for the inverse of a partitioned matrix, we obtain, assuming FIM(η) is nonsingular,

$$\text{CRB}(\alpha) = [\mathfrak{F}(\alpha) - \mathfrak{F}(\alpha, \eta)\mathfrak{F}(\eta)^{-1}\mathfrak{F}(\eta, \alpha)]^{-1}. \tag{6.4}$$

Using the Woodbury formula on (6.4), yields

$$\begin{aligned}\text{CRB}(\alpha) &= \mathfrak{F}(\alpha)^{-1} - \mathfrak{F}(\alpha)^{-1}\mathfrak{F}(\alpha, \eta) \\ &\cdot [\mathfrak{F}(\eta) + \mathfrak{F}(\eta, \alpha)\mathfrak{F}(\alpha)^{-1}\mathfrak{F}(\alpha, \eta)]^{-1} \\ &\cdot \mathfrak{F}(\eta, \alpha)\mathfrak{F}(\alpha)^{-1}.\end{aligned} \tag{6.5}$$

The last term in this expression represents the degradation due to the fact that η is not known and must also be estimated along with α.

Definition 6.2 ($\mathfrak{L}(\alpha, \eta)$): Denote by $\mathfrak{L}(\alpha, \eta)$ the loss term in (6.4):

$$\mathfrak{L}(\alpha, \eta) = \mathfrak{F}(\alpha, \eta)\mathfrak{F}(\eta)^{-1}\mathfrak{F}(\eta, \alpha). \tag{6.6}$$

With this definition

$$\text{CRB}(\alpha) = [\mathfrak{F}(\alpha) - \mathfrak{L}(\alpha, \eta)]^{-1}. \tag{6.7}$$

To determine the new entries in $\mathfrak{F}(\beta)$, we begin by noting that

$$\frac{\partial S_r(\omega)}{\partial S_s(\omega_i)} = h(\omega_i)h^H(\omega_i)\,\delta(\omega - \omega_i), \qquad i = 1, \cdots, L. \tag{6.8}$$

This fact implies that the summation over frequency in the FIM expression will degenerate into a single frequency contribution. We can now compute

$$[\mathfrak{F}(\eta)]_{ij} = \frac{N}{4\pi}\frac{1}{E^2(\omega_i)}\,\|h(\omega_i)\|^2\,\delta_{ij}. \tag{6.9}$$

Similarly, for the cross terms we obtain

$$[\mathfrak{F}(\alpha, \eta)]_{ij} = \frac{N}{2\pi}\frac{S_s(\omega_j)}{E^2(\omega_j)}\,\|h(\omega_j)\|^2\,\text{Re}\,\{h(\dot{\omega}_j)^H_{\alpha_i}h(\omega_j)\} \tag{6.10}$$

resulting, for the (i, j) element of $\mathfrak{L}(\alpha, \eta)$

$$\begin{aligned}[\mathfrak{L}(\alpha, \eta)]_{ij} &= \sum_l \mathfrak{F}(\alpha_i, \eta_l)\mathfrak{F}(\eta_l)^{-1}\mathfrak{F}(\eta_l, \alpha_j) \\ &= \frac{N}{2\pi}\sum_l \frac{2S_s(\omega_l)^2}{E(\omega_l)^2}\,\text{Re}\,\{\dot{h}(\omega_l)^H_{\alpha_i}h(\omega_l)\} \\ &\cdot \text{Re}\,\{h(\omega_l)^H\dot{h}(\omega_l)_{\alpha_j}\}\end{aligned} \tag{6.11}$$

which in the limit tends exactly to the second term of (4.4). We conclude thus that the "saturating term" of (4.4) is precisely the information that is lost when the source spectral density is not known.

Fact 6.1 (Single Source of Unknown Spectrum/Multipath CRB): Consider the single source version of model (2.1), where the source signal spectrum is not known. Then the Cramér–Rao bound for the error on the location parameters α is

$$\text{CRB}(\alpha)^{-1} = \frac{N}{2\pi}\int K_1\,\text{Re}\,\{\dot{h}^H_\alpha P_h^\perp \dot{h}_\alpha\}\,d\omega. \tag{6.12}$$

Noting that for $K = 1$, $P_h^\perp$ in (6.12) is zero, and consequently the CRB will grow to infinity, we get the following.

Fact 6.2 (Single Source with Unknown Spectrum/Single Sensor CRB): The Fisher information matrix of the source location parameters for a single sensor observation ($K = 1$) of the multipath propagation of a single stochastic Gaussian source signal with unknown spectrum is zero.

B. Dependence on the Beampattern

To study the dependence of the CRB for unknown spectrum on the beampattern matrix, we consider the decomposition of CRB $(\alpha)^{-1}$ on its four components, as in (5.17). Since the difference between known and unknown spectrum is the presence or absence of the "saturating term" of (4.4), each component is still given by the same expression, with $K_2 = 0$.

Thus, the interpretations in Section V carry over to this case. Namely, we can state the following:

Fact 6.3 (Single Source with Unknown Spectrum/Clusters of Paths CRB): Consider the single source version of (2.1), where the spectrum of the source signal is not known. Furthermore, assume that the P incoming paths form r "clusters," so that the beampattern matrix can be approximated by (5.25). Then, the multipath contribution *to the Fisher information matrix is equivalent to the contribution of K fictitious arrays, each one with size equal to the number of "clusters" r. The steering vector for each equivalent array, d, is given by (5.27). Its component for sensor k is the superposition of the unresolved paths in the corresponding "cluster" k.*

The CRB is given, for this case, by

$$\text{CRB}\ (\alpha)_{mp}^{-1} = K\frac{N}{2\pi}\int K_1\ \text{Re}\left\{\frac{\partial d^H}{\partial\alpha}P_d^{\perp}\frac{\partial d}{\partial\alpha}\right\}d\omega \quad (6.13)$$

where d was defined in (5.27). Making $r = 1$ in the last fact, we get the following.

Fact 6.4 (Single Source with Unknown Spectrum/Single Cluster CRB): Consider the situation of fact 6.3, when the number of clusters r is equal to one, i.e., all the paths impinging on the array are spatially close. Then the multipath contribution to the Fisher information matrix for the source location is approximately zero, showing that these parameters are not useful in determining source location. The performance is entirely dependent on the spatial processing.

Note that even in the absence of spectral information, the modeling of the interpath delays can be useful, as it is implied by the analysis in Section V, of the situation of perfect spatial resolution, that remains valid when the source spectrum is not known.

C. Parametric Spectra

We consider in this section the case where a parametric expression for $S_s(\omega)$ is available.

Assumption 6.2 (Parametric Form of Spectrum): The source spectrum is known to the receiver, except for the L-dimensional vector of unknown deterministic parameters η:

$$S_s(\omega) = S_s(\omega : \eta). \quad (6.14)$$

□

As for the case of completely unknown source spectrum, η must be estimated along with the location parameters α. The CRB is still given by expression (6.4) or (6.5). Since

$$\frac{\partial S_r(\omega)}{\partial\eta_i} = h(\omega)h^H(\omega)\frac{\partial S_s(\omega)}{\partial\eta_i} \quad (6.15)$$

in this case integration over frequency will not reduce to a single frequency contribution as before.

Simple algebraic manipulations lead to

$$\frac{\partial S_r(\omega)}{\partial\eta_i}S_r^{-1}(\omega) = \frac{1}{E}\frac{\partial S_s(\omega)}{\partial\eta_i}hh^H \quad (6.16)$$

which together with

$$\frac{\partial S_r(\omega)}{\partial\alpha_i}S_r^{-1}(\omega) = \frac{S_s}{S_n}\left[\frac{\partial h}{\partial\alpha_i}h^H\frac{S_n}{E} + h\frac{\partial h}{\partial\alpha_i}\left(I - \frac{S_s}{E}hh^H\right)\right] \quad (6.17)$$

imply the following expressions for the different subblocks of the FIM:

$$\mathfrak{F}(\eta) = \frac{N}{4\pi}\int\frac{\|h\|^4}{E^2}\frac{\partial S(\omega)^T}{\partial\eta}\frac{\partial S(\omega)}{\partial\eta}d\omega \quad (6.18)$$

$$\mathfrak{F}(\alpha, \eta) = \frac{N}{4\pi}\int 2\frac{S_s\|h\|^2}{E^2}\text{Re}\left\{\frac{\partial h^H}{\partial\alpha}h\right\}\frac{\partial S(\omega)}{\partial\eta}d\omega. \quad (6.19)$$

Using these equations, we get for the loss term (6.6)

$$\mathfrak{L}(\alpha, \eta) = \frac{N}{\pi}\int\frac{S_s\|h\|^2}{E^2}\text{Re}\left\{\frac{\partial h^H}{\partial\alpha}h\right\}\frac{\partial S(\omega)}{\partial\eta}d\omega \cdot\left[\int\frac{\|h\|^4}{E^2}\frac{\partial S(\omega)^T}{\partial\eta}\frac{\partial S(\omega)}{\partial\eta}d\omega\right]^{-1} \cdot\int\frac{S_s\|h\|^2}{E^2}\frac{\partial S(\omega)^T}{\partial\eta}\text{Re}\left\{h^H\frac{\partial h}{\partial\alpha}\right\}d\omega. \quad (6.20)$$

1) Single Unknown Parameter: Consider the case of a single parameter, i.e., $L = 1$, when $\mathfrak{F}(\eta)$ becomes a scalar. For this case the loss term is

$$\frac{\frac{N}{2\pi}\iint\frac{S_s(\omega_1)\|h(\omega_1)\|^2}{E^2(\omega_1)}\text{Re}\ \{\dot h_\alpha^H(\omega_1)h(\omega_1)\}\dot S_{s_\eta}(\omega_1)\dot S_{s_\eta}(\omega_2)\cdot\frac{S_s(\omega_2)\|h(\omega_2)\|^2}{E^2(\omega_2)}\text{Re}\ \{h(\omega_2)^H\dot h_\alpha(\omega_2)\}\,d\omega_1\,d\omega_2}{\int\frac{\|h(\omega)\|^4}{E^2(\omega)}\dot S_{s_\eta}(\omega)^2\,d\omega}. \quad (6.21)$$

Define

$$\phi(\omega) = \frac{1}{C}\frac{\|h(\omega)\|^2}{E(\omega)}\dot S_{s_\eta}(\omega) \quad (6.22)$$

where

$$C^2 = \int\frac{\|h(\omega)\|^4}{E^2(\omega)}\dot S_{s_\eta}^2(\omega)\,d\omega. \quad (6.23)$$

Define also

$$X(\omega_1, \omega_2) = \phi(\omega_1)\phi(\omega_2). \quad (6.24)$$

Let $\mathcal{S}_\phi$ be the one-dimensional space spanned by the function $\phi(\omega)$. Then, the integral projection operator in $\mathcal{S}_\phi$ is $P_{\mathcal{S}_\phi}$:

$$P_{\mathcal{S}_\phi}[v(\omega)] = \int X(\omega, \omega_1)v(\omega_1)\,d\omega_1. \quad (6.25)$$

Finally, we define

$$v(\omega) = \frac{S_s(\omega)}{E(\omega)}\text{Re}\ \{\dot h_\alpha^H(\omega)h(\omega)\} \in \boldsymbol{R}^2. \quad (6.26)$$

With these definitions, we can write the loss term as

$$\mathfrak{L} = \frac{N}{\pi}\int\int v(\omega_1)X(\omega_1, \omega_2)v(\omega_2)\,d\omega_1\,d\omega_2. \quad (6.27)$$

Since $P_{\mathcal{S}_\phi}$ is a projection operator

$$P_{\mathcal{S}_\phi}[P_{\mathcal{S}_\phi}[v(\omega)]] = P_{\mathcal{S}_\phi}[v(\omega)]. \quad (6.28)$$

$\mathcal{L}$ can be written

$$\mathcal{L} = \frac{N}{\pi} \langle P_{S_\phi}[v(\omega)], P_{S_\phi}[v(\omega)] \rangle \quad (6.29)$$

where the inner product is the usual $\mathcal{L}^2$ inner product.

We can now write the CRB for the case of parametric spectrum which we denote by CRB_{par}, as

$$\text{CRB}(\alpha)_{\text{par}}^{-1} = \mathfrak{F}(\alpha) - \frac{N}{\pi} \langle P_{S_\phi}[v(\omega)], P_{S_\phi}[v(\omega)]^T \rangle. \quad (6.30)$$

Remember that $\mathfrak{F}(\alpha)$ is the FIM under assumption 3.1, of known spectrum, and can be written as (see fact 4.1, equation (4.4)):

$$\mathfrak{F}(\alpha) = \text{CRB}(\alpha)_{\text{unk}}^{-1} + \frac{N}{\pi} \langle v(\omega), v(\omega)^T \rangle \quad (6.31)$$

where $\text{CRB}(\alpha)_{\text{unk}}^{-1}$ is the FIM under assumption 6.1 (see fact 6.1, equation (6.12)). Using these two equations, we can write

$$\text{CRB}(\alpha)_{\text{par}}^{-1} = \text{CRB}(\alpha)_{\text{unk}}^{-1} + \mathcal{G}(\alpha, \eta) \quad (6.32)$$

where $\mathcal{G}(\alpha, \eta)$ is the information gain with respect to the complete unknown spectrum case:

$$\mathcal{G}(\alpha, \eta) = \frac{N}{\pi} \langle P_{S_\phi}^{\perp}[v(\omega)], P_{S_\phi}^{\perp}[v(\omega)]^T \rangle. \quad (6.33)$$

From this form of $\mathcal{G}$ we can conclude the following.

Fact 6.5: Let $\mathcal{G}$ be defined by (6.33). Then

$$0 \leq \mathcal{G} \leq \frac{N}{\pi} \langle v(\omega), v(\omega)^T \rangle. \quad (6.34)$$

The proof of this fact is trivial, considering a generic quadratic form of $\mathcal{G}$ and using the properties of norm and of projection operators.

The two extreme cases in fact 6.5 are particularly interesting, since they correspond to the two situations previously analyzed:

i) No spectral information $\mathcal{G} = 0$, i.e, $\text{FIM}(\alpha)_{\text{par}} = \text{FIM}(\alpha)_{\text{unk}}$. To have the above relation, it is necessary that the vector of functions $v(\omega)$ be colinear with its projection in S_ϕ. Using their definitions

$$\frac{S_s(\omega)}{E(\omega)} \text{Re}\,\{\dot{h}_{\alpha_i}(\omega)^* h(\omega)\} = \frac{C_1}{C} \frac{\|h\|^2}{E(\omega)} \dot{S}_\eta(\omega) \quad (6.35)$$

where C_1 is an arbitrary constant. This equation is equivalent to

$$\frac{\partial}{\partial \alpha_i} [\|h\|^2 S_s(\omega)] = C_* \frac{\partial}{\partial \eta} [\|h\|^2 S_s(\omega)] \quad (6.36)$$

where C_* is an arbitrary constant.

We get the intuitively pleasant result that no spectrum-based information retrieval is possible when the variation of the observed signal energy is the same with respect to the location parameters or the spectral parameters.

ii) Complete spectral information $\mathcal{G} = N/\pi \langle v(\omega), v(\omega)^T \rangle$, i.e., $\text{FIM}(\alpha)_{\text{par}} = \text{FIM}(\alpha)$. Again, using the same kind of arguments, we can conclude that to have no information loss, the vector $v(\omega)$ must be orthogonal to the function $\phi(\omega)$. This condition is equivalent to

$$\int \frac{\partial}{\partial \alpha_i} [\|h\|^2 S_s(\omega)] \frac{1}{E^2(\omega)} \frac{\partial}{\partial \eta} [\|h\|^2 S_s(\omega)]\, d\omega = 0 \qquad i = 1, 2. \quad (6.37)$$

This equation means that the variation of the signal energy with respect to the spectral parameter must be orthogonal to its variation with respect to the location of the source so that uncertainty about η does not affect the estimation of α.

2) Arbitrary number L of Spectral Parameters: The case of an arbitrary number L of unknown spectral parameters can be treated in the same way as we did for $L = 1$. Define the L-dimensional vector:

$$\phi_L(\omega) = \frac{\|h(\omega)\|^2}{E(\omega)} \dot{S}_{s_\eta}(\omega)^T. \quad (6.38)$$

Note that $\mathfrak{F}(\eta)$ is the $(L \times L)$ Gram matrix of the functions $\{\phi_{L_i}\}_{i=1}^{L}$

$$\mathfrak{F}(\eta) = \Gamma_L = \langle \phi_L(\omega), \phi_L(\omega)^T \rangle \quad (6.39)$$

which we assumed nonsingular. Define the equivalent to the kernel $X(\omega_1, \omega_2)$ for this case:

$$X_L(\omega_1, \omega_2) = \phi_L(\omega_1)^T \Gamma_L^{-1} \phi_L(\omega_2). \quad (6.40)$$

Let S_{ϕ_L} be the spaced spanned by the functions $\{\phi_{L_i}\}_{i=1}^{L}$. This subspace has dimension L. Let $v(\omega)$ have the same definition (6.26). Then, the loss term $\mathcal{L}$ is given by

$$\mathcal{L} = \frac{N}{\pi} \iint v(\omega_1) X(\omega_1, \omega_2) v(\omega_2)^T\, d\omega_1\, d\omega_2 \quad (6.41)$$

which is exactly the same expression as (6.27). The difference between the two cases lies in the dimensionality of S_ϕ, which is 1 for a single unknown parameter and $n \leq L$ in the general case. The gain term $\mathcal{G}$ is

$$\mathcal{G} = \frac{N}{\pi} \langle P_{S_{\phi_L}}^{\perp}[v(\omega)], P_{S_{\phi_L}}^{\perp}[v(\omega)]^T \rangle \quad (6.42)$$

where $P_{S_{\phi_L}}$ is the projection operator in the orthogonal complement of S_{ϕ_L}. Fact 6.5 still holds for this general case.

The analysis of the two extreme cases i) and ii) must now be done taking into consideration the new dimensionality of S_{ϕ_L}.

i) Now, instead of requiring colinearity with a single function of ω, we must require that all the components of $v(\omega)$ belong to S_{ϕ_L}, i.e., to have $\mathcal{G} = 0$, there must exist a $(2 \times L)$ matrix T such that

$$v(\omega) = T\phi_L(\omega). \quad (6.43)$$

Note that if for a given subset of the unknown parameters this condition is satisfied, then it will be trivially satisfied for the complete vector η, showing that having additional unknown parameters cannot remove ambiguities.

Consider the discrete frequency version of the FIM. To avoid confusion, let the number of frequencies be denoted by W. Within this framework, the kernel $X(\omega_1, \omega_2)$ instead of being defined in an infinite dimensional space, would be defined in a W-dimensional space. All the above equations hold, with a reinterpretation of the inner product. If $W = L$, the subspace $\mathcal{S}_{\phi L}$ has full dimension W, and its orthogonal complement is trivially equal to the zero vector. In this case, condition (6.43) is satisfied and $\mathcal{G} = 0$ always. The case of unknown spectrum of Section VI-A can be considered as the limiting case where both W and L tend to ∞.

ii) The condition for no information loss is that all the elements of $v(\omega)$ belong to $\mathcal{S}_{\phi L}^{\perp}$:

$$\int v_i(\omega)\phi_{L_j}(\omega)\, d\omega = 0, \qquad \forall i, j. \tag{6.44}$$

Consider a fixed spectral parameter vector η_0 of dimension L. If we add another unknown parameter η_{L+1}, the subspace $\mathcal{S}_{\phi L+1}^{\perp}$ is a proper subspace of $\mathcal{S}_{\phi}^{\perp} L$, and consequently $\mathcal{G}_L \geq \mathcal{G}_{L+1}$, as it should be expected.

Fact 6.6 (Single Source Parametric Spectrum/Multipath CRB): Consider the single source version of model (2.1), and assumption 6.2. Then, the Cramér–Rao bound for the location parameter α is given by

$$\text{CRB}(\alpha)^{-1} = \text{CRB}(\alpha)_{\text{unk}}^{-1} + \mathcal{G}(\alpha, \eta) \tag{6.45}$$

where CRB $(\alpha)_{unk}$ is the CRB under assumption 6.1 and $\mathcal{G}(\alpha, \eta)$ satisfies fact 6.5, and is given by

$$\mathcal{G} = \frac{N}{\pi} \langle P_{\mathcal{S}_{\phi_L}}^{\perp}[v(\omega)], P_{\mathcal{S}_{\phi_L}}^{\perp}[v(\omega)]^T \rangle \tag{6.46}$$

where $P_{\mathcal{S}_{\phi_L}}^{\perp}[\cdot]$ is the projection operator into the orthogonal complement of $\mathcal{S}_{\phi L}$

$$\mathcal{S}_{\phi L} = \text{span}\{\phi_{L_i};\ i = 1, \cdots, L\}. \tag{6.47}$$

The functions ϕ_{L_i} and $v(\omega)$ were defined in (6.38) and (6.26), respectively.

VII. Conclusions

We derived the Cramér–Rao bound for the general problem of multipath and multiple stochastic sources. We considered the cases of known source spectral densities and unknown power spectral densities. Our study evaluates the increase in performance provided by the multipath structure. It shows that the contribution of the multipath delays to the Fisher information matrix can be understood as the result of the spatial processing of a virtual array, whose geometry is dependent on the multipath structure and whose size depends on the numbers of spatially resolved replicas. The influence of uncertainty about the source spectrum was studied. If the source spectrum is completely unknown, estimation is not possible with a single sensor. However, if a parametric form of the spectrum is known, there is still some gain due to the exploitation of the signal structure at each sensor, and estimation may be possible even with a single sensor. We analyzed the Cramér–Rao bound expression for several important simple cases, and in doing so we recovered expressions previously reported in the literature.

Appendix A
Derivation CRB (α)

Using

$$\frac{\partial S_r}{\partial \alpha_i} = \frac{\partial h_{\alpha_i}}{\partial \alpha_i} s_i^H H^H + H s_i \frac{\partial h_{\alpha_i}^H}{\partial \alpha_i} \tag{A.1}$$

and (3.7) yields

$$\frac{\partial S_r}{\partial \alpha_i} S_r^{-1} = \frac{1}{S_n}[S_n \dot{h}_{\alpha_i} s_i^H S_s^{-1}[S_n S_s^{-1} + H^H H]^{-1} H^H + H s_i \dot{h}_i^H [I - H(S_n S_s^{-1} + H^H H)^{-1} H^H]. \tag{A.2}$$

Since

$$s_i^H S_s^{-1} = e_i^T \tag{A.3}$$

where e_i denotes the ith canonical vector, we get

$$\frac{\partial S_r}{\partial \alpha_i} S_r^{-1} = \dot{h}_{\alpha_i} e_i^T (S_n S_s^{-1} + H^H H)^{-1} H^H + \frac{1}{S_n} H s_i \dot{h}_{\alpha_i}^H [I - H(S_n S_s^{-1} + H^H H)^{-1} H^H]. \tag{A.4}$$

Finally, we obtain

$$\begin{aligned}\frac{\partial S_r}{\partial \alpha_i} S_r^{-1} \frac{\partial S_r}{\partial \alpha_j} S_r^{-1} &= \dot{h}_{\alpha_i} e_i^T (S_n S_s^{-1} + H^H H)^{-1} H^H \dot{h}_{\alpha_j} e_j^T \\ &\quad \cdot (S_n S_s^{-1} + H^H H)^{-1} H^H \\ &\quad + \dot{h}_{\alpha_i} e_i^T (S_n S_s^{-1} + H^H H)^{-1} H^H H s_j \dot{h}_j^H S_r^{-1} \\ &\quad + H s_i \dot{h}_{\alpha_i}^H S_r^{-1} \dot{h}_{\alpha_j} e_j^T (S_n S_s^{-1} + H^H H)^{-1} H^H \\ &\quad + H s_i \dot{h}_{\alpha_i}^H S_r^{-1} H s_j \dot{h}_{\alpha_j}^H S_r^{-1}.\end{aligned} \tag{A.5}$$

Using the properties of the trace operator

$$\begin{aligned}\text{tr}\left\{\frac{\partial S_r}{\partial \alpha_i} S_r^{-1} \frac{\partial S_r}{\partial \alpha_j} S_r^{-1}\right\} &= e_i^T (S_n S_s^{-1} + H^H H)^{-1} \\ &\quad \cdot H^H \dot{h}_{\alpha_j} e_j^T (S_n S_s^{-1} + H^H H)^{-1} H^H \dot{h}_{\alpha_i} \\ &\quad + e_i^T (S_n S_s^{-1} + H^H H)^{-1} H^H H s_j \dot{h}_j^H S_r^{-1} \dot{h}_{\alpha_i} \\ &\quad + \dot{h}_{\alpha_i}^H S_r^{-1} \dot{h}_{\alpha_j} e_j^T (S_n S_s^{-1} + H^H H)^{-1} H^H H s_i \\ &\quad + \dot{h}_{\alpha_i}^H S_r^{-1} H s_j \dot{h}_{\alpha_j}^H S_r^{-1} H s_i.\end{aligned} \tag{A.6}$$

This equation can be written in matrix form using the definition of the Hadamard product (see (3.12) in the main text)

$$\operatorname{tr}\left\{\frac{\partial S_r}{\partial \alpha_i} S_r^{-1} \frac{\partial S_r}{\partial \alpha_j} S_r^{-1}\right\}$$
$$= [(S_n S_s^{-1} + H^H H)^{-1} H^H D]_{ij}$$
$$\odot\ [(S_n S_s^{-1} + H^H H)^{-1} H^H D]_{ji}$$
$$+ [(S_n S_s^{-1} + H^H H)^{-1} H^H H S_s]_{ij} \odot [D^H S_r^{-1} D]_{ji}$$
$$+ [(H^H S_r^{-1} D]_{ij} \odot [(S_n S_s^{-1} + H^H H)^{-1} H^H H S_s]_{ji}$$
$$+ [D^H S_r^{-1} H S]_{ij} \odot [D^H S_r^{-1} H S_s]_{ji} \tag{A.7}$$

where $\odot$ denotes the Hadamard (elementwise) product of matrices (see (3.12) in the main text), and D is defined by (3.14). Using in the previous equation the following easily verified relation:

$$S_r^{-1} H S_s = H(S_n S_s^{-1} + H^H H)^{-1} \tag{A.8}$$

we obtain, after some algebraic manipulations, the following expression for the generic element in the integrand of (3.3):

$$\operatorname{tr}\left\{\frac{\partial S_r}{\partial \alpha_i} S_r^{-1} \frac{\partial S_r}{\partial \alpha_j} S_r^{-1}\right\}$$
$$= [2 \operatorname{Re}\{(S_s H^H H(S_n S_s^{-1} + H^H H)^{-1} \otimes (11^T))$$
$$\odot\ (D^H S_r^{-1} D)(D^H H(S_n S_s^{-1} + H^H H)^{-1})$$
$$\odot\ (D^H H(S_n S_s^{-1} + H^H H)^{-1})^T\}]_{ij} \tag{A.9}$$

where $\otimes$ denotes Kronecker product, see (3.13). The Kronecker product with the outer product of two 2-dimensional "one-forms" enforces the same multiplier for each (2×2) subblock of D corresponding to a pair of sources, in agreement with the fact that the term multiplying $\dot{h}_{\alpha_i}\dot{h}_{\alpha_j}$ does not depend on whether we take derivatives with respect to range ($\alpha = R$) or depth ($\alpha = Y$).

Using (A.9) in the FIM equation (3.3) yields (3.11).

Appendix B
Derivation of CRB for Single Sensor

Using (4.1) and (4.3) in (3.11) we get

$$\mathrm{CRB}^{-1}(\alpha) = \frac{N}{2\pi}\int \operatorname{Re}\left\{\frac{S_s^2\|h\|^2}{S_n E}\dot{h}^H\left(I - \frac{S_s}{E}hh^H\right)\dot{h}\right.$$
$$\left. + \dot{h}^H h\left(\frac{S_s}{E}\right)^2 h^T\dot{h}\right\}. \tag{B.1}$$

Noting that, using the definition of E (4.2)

$$\frac{S_s}{E} = \frac{1}{\|h\|^2} - \frac{S}{E\|h\|^2} \tag{B.2}$$

and regrouping terms, we obtain

$$\mathrm{CRB}^{-1}(\alpha) = \frac{N}{2\pi}\int K_1 \operatorname{Re}\{\dot{h}^H P_h^{\perp}\dot{h}\}$$
$$+ K_2 \operatorname{Re}\{\dot{h}^H h\}\operatorname{Re}\{h^H \dot{h}\}\, d\omega \tag{B.3}$$

where K_1 and K_2 are defined in fact 4.1, equations (4.5) and (4.6), respectively, which is (4.4) in the paper.

Appendix C
Derivation of the Loss Term, (5.34)

The expression of the loss term involves the inverse of the FIM for the multipath delays, given by (5.21). Until now we were able to keep the P-dimensional vector of interpaths delays, including its first trivially zero component. To be able to invert $\mathfrak{F}(\theta_{mp})$, however, we must exclude θ_{mp_1} from the analysis, and consider (just in this Appendix) the following redefinition of θ_{mp}:

$$\theta_{mp} = [\theta_{mp_2} \cdots \theta_{mp_P}].$$

The matrix of derivatives is now

$$\frac{\partial h}{\partial \theta_{mp}} = j\omega A\begin{bmatrix} 0^T \\ \overline{B}\end{bmatrix}$$

where 0 is a $P - 1$-dimensional vector with all its components equal to zero, and $\overline{B}$ is the lower right-hand side $P - 1$-dimensional subblock of B. Define, correspondingly, $\overline{a}$ as the vector that groups the attenuations of paths 2 through P. Simple algebraic manipulations give

$$\mathfrak{F}(\theta_{mp}) = K\frac{N}{4\pi}\int \omega^2 K_1 \Lambda_{\overline{a}} S_a \Lambda_{\overline{a}}\, d\omega$$

where S_a is the nonsingular $(P-1)\times(P-1)$ matrix

$$S_a = I_{P-1} - \frac{\overline{a}\,\overline{a}^T}{\|a\|^2}$$

and $\Lambda_{\overline{a}} = \operatorname{diag}\{\overline{a}\}$. Note that S_a is not the projection matrix on the orthogonal complement of $\overline{a}$. Its inverse is

$$S_a^{-1} = I_{P-1} + \frac{\overline{a}\,\overline{a}^T}{a_1^2}.$$

For the cross term, we obtain

$$\mathfrak{F}(\theta_{s_p}, \theta_{mp}) = \frac{N}{4\pi}\int \omega^2 K_1 \operatorname{Re}\{b_p^* A_p^H \Psi\}\, d\omega\, R_a \Lambda_{\overline{a}}$$
$$= \Phi_{s_p}^T R_a \Lambda_{\overline{a}}$$

where Φ_{s_p} is defined by (5.35) in the main text, and Ψ is the diagonal matrix of generic element $[\Psi]_{ij} = \delta_{ij} e^{-j\omega_n \theta_{mp_i}}$, and the $P \times (P-1)$ matrix R_a is defined by

$$R_a \triangleq \begin{bmatrix} 0^T \\ I \end{bmatrix} - \frac{a\overline{a}^T}{\|a\|^2}.$$

To establish (5.34) we need to prove that

$$R_a S_a^{-1} R_a^T = P_a^{\perp}$$

which involves some simple algebraic manipulations.

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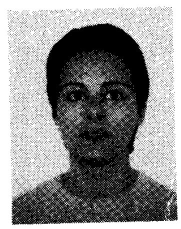

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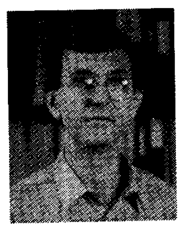

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