Cramer-Rao Bounds for Passive Range and Depth in a Vertically Inhomogeneous Medium

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Abstract
We establish Cramer-Rao Bounds for passive location in a multipath environment with an array of multiple sensors. The source signature is wideband stationary and the background noise is white. We derive general expressions for the bounds and then compare the performance gain contributed by the inter-path delays over location techniques based on wavefront curvature only (i.e., spatial processing across the array of sensors).

1. Introduction
There is an increasing interest in incorporating detailed medium descriptions in passive location mechanisms. This improves the performance and eliminates ambiguities arising in simplified propagation models, such as isovelocity models.

Here, we study the performance (Cramer-Rao Bound (CRB)) of passive systems, under the following simplifying assumptions:
1. The ocean is a vertically inhomogeneous medium;
2. The source signal is a stationary wideband Gauss signal of known spectral density function;
3. The observations correspond to a multisensor array of known geometry and location;
4. Source and receiver have no relative horizontal motion.

Assumption 1 implies the existence of multiple paths for the energy propagating from a point source to a point receiver, leading to the following general expression for the signal received at a sensor:

$$x(t) = \sum_{p=1}^{P} a_p s(t - \tau_p)$$  \hspace{1cm} (1)

where $s(t)$ is the signal emitted by the source.

The information about the source location is encoded in the propagation parameters $P, \{a_p\}$, and $\{\tau_p\}$. Here, we focus on the $\tau_p$'s. In the CRB study, we consider only situations where the number of propagation paths $P$ is constant in a neighborhood of the source, and we assume that the variation of the attenuation along different sensors with the source position is negligible.

The paper is organized as follows: first, the basic equations of the propagation model are described, in section 2. In section 3, the expressions for the CRB of the propagation parameters are presented, assuming perfect knowledge of the environmental parameters. Finally, in section 4, we plot the CRB for configurations of practical interest.

2. Propagation Model
We assume that the ocean is a horizontally homogeneous medium, with flat boundaries, and that the sound velocity profile (SVP) is a bilinear function of depth, as in Fig. 1. With this SVP, the rays bent towards the depth corresponding to the minimum of the SVP, (duct axis). The rays are classified according to their interference with the medium's boundaries, into purely refracted rays (SOAR), surface (SR) or bottom (BR) reflected rays, and surface/bottom reflected rays (SRBR).

We give in this section the basic equations of the propagation model, assuming that both source and receiver are above the duct axis. For the case of source and receiver at the same depth, simpler equations are given in [4].

The equations are parametrized according to the number $K$ of complete bounces around the duct axis.
We present the expressions for the delays and their derivatives with respect to the location parameters (range and depth).

The delay from a point source located at \( \{ R, y \} \) to a point receiver located at depth \( y_{\text{rec}} \) along a path with launching angle \( \theta_r \) is,

\[
\tau(\theta_r) = \frac{c_0}{v_0} \mathcal{F}(c_r) - \frac{c_0}{v_t} \mathcal{F}(c_r) + A_1 \mathcal{F}(c_{\text{surf}}) + \left( \frac{c_0}{v_t} \right) \mathcal{F}(c_{\text{surf}})
\]

where\(^1\), \( \mathcal{F}(c) = \ln \left( 1 + \sqrt{1 - c^2} \right)/c \), and,

\[
\gamma = \text{sign}(\theta_r), \quad \delta = \text{sign}(\theta_r)
\]

and,

\[
A_{i}(K) \begin{cases} 2(K + 1) \left( \frac{1}{\mu_i} - \frac{1}{\mu_p} \right), & \text{SOFAR, SR, SBR} \\
2K, & \text{DIRECT RAYS} \\
2(K + 1), & \text{SR, SBR} \\
2(K + 2), & \gamma = 1, \delta = -1 \\
0, & \text{DIRET SR} \\
A_2, & 0
\end{cases}
\]

\[
A_3(K) \begin{cases} 0, & \text{DIRET SR} \\
A_3(K + 1), & \text{SOFAR, SR} \\
2, & \text{DIRECT RAYS}
\end{cases}
\]

In the previous equations, \( \theta_r, \theta_{\text{surf}}, \theta_{\text{dut}}, \theta_{\text{dut}} \) and \( \theta_r \) are the ray angles at the source (launching angle), the duct, the bottom, and the receiver, respectively. We denote the cosine of these angles by \( c_\theta, c_{\text{surf}}, c_{\text{dut}} \). These angles are related through Snell’s law. The launching angle is derived from the following equation:

\[
R = \frac{c_0}{v_0} \tan \theta_r - \frac{c_0}{v_t} \tan \theta_r + A_1 \frac{v_{\text{min}}}{v_t} \tan \theta_{\text{surf}} - A_2 \frac{v_{\text{surf}}}{v_t} \tan \theta_{\text{dut}} + A_3 \frac{v_{\text{surf}}}{v_t} \tan \theta_{\text{surf}} - A_4 \frac{v_{\text{dut}}}{v_t} \tan \theta_{\text{dut}}
\]

The derivatives of the delays with respect to the location parameters are:

\[
\frac{\partial \tau}{\partial R} = \left( \frac{\partial \tau}{\partial R} \right)_{y_{\text{rec}}}, \quad \left( \frac{\partial \tau}{\partial R} \right)_{y_{\text{rec}}}
\]

\[
\frac{\partial \tau}{\partial y_{\text{rec}}} = \left( \frac{\partial \tau}{\partial y_{\text{rec}}} \right)_{y_{\text{rec}}}, \quad \left( \frac{\partial \tau}{\partial y_{\text{rec}}} \right)_{y_{\text{rec}}}
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\]

and,

\[
\frac{\partial \tau_{\text{rec}}}{\partial \theta_r} = \delta \frac{\partial \mathcal{F}(c_r)}{\partial c_r} - A_1 \frac{\partial \mathcal{F}(c_{\text{surf}})}{\partial c_r} + A_2 \frac{\partial \mathcal{F}(c_{\text{surf}})}{\partial c_r} - A_3 \frac{\partial \mathcal{F}(c_{\text{surf}})}{\partial c_r} + A_4 \frac{\partial \mathcal{F}(c_{\text{surf}})}{\partial c_r}
\]

\[
\frac{\partial \tau_{\text{rec}}}{\partial \theta_r} = \tan(\theta_r) - \frac{c_0}{c_r} \frac{\partial \mathcal{F}(c_r)}{\partial c_r} + A_1 \frac{\partial \mathcal{F}(c_{\text{surf}})}{\partial c_r} + A_2 \frac{\partial \mathcal{F}(c_{\text{surf}})}{\partial c_r} - A_3 \frac{\partial \mathcal{F}(c_{\text{surf}})}{\partial c_r} + A_4 \frac{\partial \mathcal{F}(c_{\text{surf}})}{\partial c_r}
\]

In the above equations, \( \mathcal{F}(c) = -[c \sqrt{1 - c^2}]^{-1} \)

3. Cramer Rao Bound

In this section, we present expressions for the CRB of the vector of location parameters: \( \alpha \equiv [R, y] \). The CRB is related to \( J(\alpha) \), the Fisher Information Matrix (FIM) of the vector \( \alpha \), by

\[
J(\alpha)_{ij} = \frac{\partial}{\partial \alpha_i} \frac{\partial}{\partial \alpha_j} \ln \mathcal{L}(\alpha)
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where \( \mathcal{L}(\alpha) \) is the spectral density matrix (SDM) of the vector of observations,

\[
y_k(t) = x_k(t) + w(t), \quad 1, \ldots, K
\]

which for the coherent case is

\[
\mathcal{S}_h(\omega) = \mathcal{S}_h(\omega)h(\omega)h(\omega)\dagger + \Sigma(\omega)
\]

For simplicity, we consider here only the case of spatially incoherent noise, i.e., \( \Sigma(\omega) = \mathbf{I} \).

In (14), the complex vector \( h(\omega) \) has generic element \( h_k(\omega) = \mathbf{a}_k^\dagger e^{j \theta_k} \), where \( \mathbf{a}_k \) is the attenuation along path \( i \), and \( \tau_{\text{rec}} \) is the travel time from source to sensor \( k \) along path \( i \). To be able to analyze the impact of both the temporal (multipath) and spatial (geometric) structures of the incoming wavefield, we decompose each individual travel time:

\[
\tau_{\text{rec}} = \tau_{\text{temp}} + \theta_{\text{geom}}\theta_{\text{rec}}
\]

With this decomposition, \( h(\omega) = \mathbf{A}(\omega)h(\omega) \) where

\[
\mathbf{A}(\omega) = [a_1(\omega) \cdots a_p(\omega)]
\]

\[
b(\omega)^\dagger = [a_1 e^{j \theta_{\text{geom}}\theta_{\text{rec}}} \cdots a_p e^{j \theta_{\text{geom}}\theta_{\text{rec}}}]
\]

\[
a_{i}(\omega) = [a_1(\omega) \cdots a_P(\omega)]
\]

\[
b(\omega)^\dagger = [a_1 e^{j \theta_{\text{geom}}\theta_{\text{rec}}} \cdots a_P e^{j \theta_{\text{geom}}\theta_{\text{rec}}}]
\]
between the P coherent replicas, and depends only on the inter-path delays. Note that $\mathcal{S}_\alpha(\omega)$ is independent of $\tau_{11}$, and that the delays $\theta_{mp}$ are relative to the reference path 1.

The location parameters $\alpha$ do not appear explicitly in (14). Actually, $\alpha$ affects $\mathcal{S}_\alpha(\omega)$ through the set of inter-delays $\theta_{mp}$ and $\theta_\alpha$. We group all the inter-delays into a $KP - 1$ vector:

$$\theta^T = [\theta_{mp}^T, \text{vec}(\Theta_\alpha)^T]$$

where $\Theta_\alpha = [\xi_1^\alpha, \cdots, \xi_P^\alpha]$ and $\xi_p$ groups all the inter-sensor delays corresponding to the p-th path.

The FIM's for $\alpha$ and $\theta$ are related by

$$J(\alpha) = \frac{\partial \theta^T}{\partial \alpha} J(\theta) \frac{\partial \theta}{\partial \alpha}$$

where $\partial \theta/\partial \alpha$ is the matrix of first order derivatives, and $J(\theta)$ is the FIM for the inter-delays $\theta$.

Using the chain rule once more, to express the derivatives with respect to the inter-sensor delays in terms of the elements of the complex vector $\mathbf{b}(\omega)$, and after algebraic manipulations, the following expression for $J(\theta)$ is obtained:

$$J(\theta) = \frac{K_1}{4\tau} \int \left\{ \mathbf{K}_1 \text{Re} \left\{ \frac{\partial \mathbf{b}(\omega)}{\partial \theta} \right\} \mathbf{P}_1 \frac{\partial \mathbf{b}}{\partial \theta} \right\} + K_2 \mathbf{R} \left\{ \frac{\partial \mathbf{b}(\omega)}{\partial \theta} \right\} \mathbf{R}^H \left\{ \frac{\partial \mathbf{b}(\omega)}{\partial \theta} \right\} \right\} d\omega$$

(19)

and $\mathbf{P}_1$ is the projection matrix in the orthogonal complement of the vector $\mathbf{b}(\omega)$:

$$\mathbf{P}_1 = \mathbf{I}_K - \frac{1}{||\mathbf{b}(\omega)||^2} \mathbf{b}(\omega) \mathbf{b}(\omega)^H$$

(20)

and $K_1$ and $K_2$ are

$$K_1 = 2 \frac{\mathcal{S}_\alpha(\omega)^2}{\pi^2 |\omega|^2} ||\mathbf{h}(\omega)||^2$$

$$K_2 = 4 \frac{\mathcal{S}_\alpha(\omega)^2}{\mathbf{E}^T}$$

(21)

where $\mathbf{E} = \mathbf{E}_\alpha + \mathbf{S}_\gamma(\omega) ||\mathbf{h}(\omega)||^2$. The derivation of (19) is lengthy and is given in [3].

We note that the above expression yields as a particular case the expression for the CRB for a single sensor given in [5], and presents close relations with the expression for the narrowband uncorrelated case given in [6].

When there is only a single path, $\mathbf{b}(\omega) = e^{j\omega D(\omega)} \mathbf{1}$ where $\mathbf{1}$ is a one-form, and $D(\theta) = \text{diag}(\theta_{11})$. It is easy to show that $\partial \mathbf{b}(\omega)/\partial \theta = j\omega e^{j\omega D(\omega)}$, which implies, $\text{Re}(\partial \mathbf{b}(\omega)^H/\partial \theta \mathbf{b}(\omega)) = 0$, leaving just the first term in (19). Simple algebraic manipulations give

$$\frac{\partial \mathbf{b}(\omega)^H}{\partial \theta} \mathbf{P}_1 \frac{\partial \mathbf{b}(\omega)}{\partial \theta} = \omega^2 (\mathbf{I}_K - \frac{1}{M} \mathbf{1} \mathbf{1}^H)$$

(22)

which is the expression in [5].

Comparison of (19) with the expression given in [6], shows that in the case of multiple incoherent sources, the second term is not present, while the remaining term has essentially the same meaning in both cases. We can, therefore, attribute the second term to the multipath (coherent) structure of the observations. This additional contribution to the CRB is affected by the factors $K_1$, which, contrary to $K_1$ does not increase indefinitely with the signal to noise ratio, i.e., the additional information in the temporal structure of the data given by the second term in (19) attains a limit.

According to (17) the FIM for the inter-delays is partitioned as:

$$J(\theta) = \begin{bmatrix} J(\theta_{mp}) & J(\theta_{mp} : \theta_\alpha) \\ J(\theta_{mp} : \theta_\alpha) & J(\theta_\alpha) \end{bmatrix}$$

(23)

Using the above equations,

$$\frac{\partial \mathbf{b}(\omega)}{\partial \theta_{mp}} = j\omega \mathbf{A}(\omega) \text{diag}(\mathbf{b}(\omega))$$

$$\frac{\partial \mathbf{b}(\omega)}{\partial \theta_{\alpha}} = j\omega [b_1 \text{diag}(\mathbf{a}_1), \cdots, b_P \text{diag}(\mathbf{a}_P)]$$

(24)

which, together with the general equation (19) yield for the multipath sub-block of $J(\theta)$,

$$J(\theta_{mp}) = \frac{K_1}{4\tau} \omega^2 \left\{ \mathbf{K}_1 \text{Re} \left\{ \text{diag}(\mathbf{b}^H) \mathbf{A}^H \mathbf{P}_1 \text{diag}(\mathbf{a}_1) \mathbf{b}) \right\} + K_2 \mathbf{R} \left\{ \text{diag}(\mathbf{b}^H) \mathbf{A}^H \mathbf{A} \mathbf{b} \right\} \right\} d\omega$$

(25)

and for the inter-sensor sub-blocks, corresponding to the paths $p$ and $q$,

$$J(\theta_{\alpha p}) = \omega^2 \left\{ \mathbf{K}_1 \text{Re} \left\{ b_p \text{diag}(\mathbf{a}_p) \mathbf{P}_1 \text{diag}(\mathbf{a}_q) \mathbf{b}) \right\} + K_2 \mathbf{R} \left\{ b_p \text{diag}(\mathbf{a}_p) \mathbf{A} \mathbf{b} \right\} \right\} d\omega$$

(26)

For the terms involving both types of parameters,

$$J(\theta_{mp} : \theta_\alpha) = \frac{K_1}{4\tau} \omega^2 \left\{ \mathbf{K}_1 \text{Re} \left\{ \text{diag}(\mathbf{b}^H) \mathbf{A}^H \mathbf{P}_1 \text{diag}(\mathbf{a}_1) \mathbf{b}) \right\} + K_2 \mathbf{R} \left\{ \text{diag}(\mathbf{b}^H) \mathbf{A}^H \mathbf{A} \mathbf{b} \right\} \right\} d\omega$$

(27)

We note that for $K = 1$, i.e., for single sensor observations, the first term in (25) is zero, and we are left with the single sensor CRB formula for the delay estimates given in [2].

The contributions to CRB($\alpha$) are:

$$\text{CRB}^{-1}(\alpha)_{mp} = \frac{2\tau}{\omega^2} \int \omega \left\{ \mathbf{K}_1 \text{Re} \left\{ \psi_{mp}^H \mathbf{A}^H \mathbf{P}_1 \psi_{mp} \right\} + K_2 \mathbf{R} \left\{ \psi_{mp}^H \mathbf{A}^H \mathbf{A} \mathbf{b} \right\} \right\} d\omega$$

(28)

for the multipath block, where we defined $\psi_{mp} \equiv \text{diag}(\mathbf{b})\partial \theta_{mp}/\partial \alpha$ and for the inter-sensor block,

$$\text{CRB}^{-1}(\alpha)_{\alpha} = \frac{2\tau}{\omega^2} \int \omega \left\{ \mathbf{K}_1 \text{Re} \left\{ \psi_{\alpha p}^H \mathbf{P}_1 \psi_{\alpha q} \right\} + K_2 \mathbf{R} \left\{ \psi_{\alpha p}^H \mathbf{A} \mathbf{b} \right\} \right\} d\omega$$

(29)
where, \( y_r \triangleq \sum_{m} b_r \delta \theta_r / \partial a \). Similar expressions are obtained for the cross-terms.

We see from the previous expressions that the multipath contribution depends heavily on the Gram matrix for the steering vectors: \( \Lambda = \Lambda^H \). We analyze (25) in two extreme cases: \( \Lambda \approx K I \), which corresponds to orthogonal direction vectors, and the case \( \Lambda \approx K 11^T \), when the incoming wavefronts are undistinguishable for the array.

In the first case, \( \Lambda \approx K I \), and noting that the matrix product in the first term of \( J(\theta_{mn}) \),

\[
\Lambda^H P_1 \Lambda = KP_{n,1}
\]

it is easily checked that

\[
CRB^{-1}_{mp} = K \frac{N}{4\pi} \int \omega^2 \frac{\partial \theta_r}{\partial a} \left( I_r - \frac{\hat{a}^T}{|a|^2} \right) \frac{\partial \theta_r}{\partial a} d\omega
\]

that is equivalent to having \( K \) independent arrays of \( P \) sensors receiving a single wavefront with steering vector \( b(\omega) \).

In the other extreme case, \( \Lambda \approx K 11^T \), we find:

\[
CRB^{-1}_{mp} = K \frac{N}{2\pi} \int \omega^2 \frac{\partial \theta_r}{\partial a} \text{Im} \left\{ \sum_r b_r^T b_r \right\} \left( \sum_r b_r b_r^H \right) \frac{\partial \theta_r}{\partial a} d\omega
\]

which corresponds to having \( K \) independent sensors observing the \( P \) incoming wavefronts. In these two extreme cases, we were led to the existence of a "virtual" array, whose size depends on the number of spatially resolved sources, observing a fictitious wavefront with direction vector described by the vector \( b(\omega) \). The geometry of this "virtual" array is determined both by the multipath delays \( \theta_{mp} \) and by their derivatives.

Finally, this result shows the importance of methods that exploit the double (temporal/spatial) structure of the observations, when the arrivals are correlated. Furthermore, the analogy now established with the classic array processing brings into this study the vast body of results of that area. This will be explored elsewhere.

5. Plots

Figure 2 shows plots of the normalized \( CRB_{mp} (R) = 10 \log \left( \frac{NCRB}{R} \right) / R^2 \), for several values of the signal-to-noise ratio \( 10 \log \left( S_n / \sigma^2 \right) \) and signal bandwidth. The solid line corresponds to the total \( CRB \), given by (19). The dashed line is the \( CRB \) considering only the spatial structure, i.e., given by (29). We see that in this case, the contribution of the inter-path delays to the \( CRB \) is fundamental to attain a satisfactory performance. Similar plots are obtained for \( CRB (y_r) \). Lack of space precludes full discussion of the behavior of the \( CRB \) here, and will be reported elsewhere.

In figure 2, the following channel parameters were used: \( \phi_0 = -0.035 \text{ s}^{-1} \), \( \psi_1 = 0.035 \text{ s}^{-1} \), \( v_{m,1} = 1800 \text{ m/s} \), duct depth = 914m, bottom depth = 5000m. The source is at depth \( y_s = 10 \text{ m} \). The receiving array is a vertical linear uniform array of 4 sensors, with inter-sensor spacing of 10m and its first sensor is located at \( R = 3 \text{ km} \), \( y_1 = 100 \text{ m} \).

References