DUAL ALGORITHM FOR ARMA SPECTRUM ESTIMATION

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ABSTRACT

The present work describes an ARMA estimation algorithm that differs from the known available techniques. It substitutes the autocorrelation estimation sequence by the sequence of estimated reflection coefficients. These are reliably provided by the Burg technique [1]. Then it fits to the process both a sequence of higher order linear predictors (e.g., Levinson algorithm), and a sequence of higher order linear innovations filters (e.g., by recursive inversion). Finally, it obtains the MA coefficients from the linear relations satisfied by the corresponding coefficients of the successive higher order linear predictors, and likewise obtains the AR coefficients from the linear relations satisfied by the corresponding coefficients of the successive higher order innovation filters.

We stress that the procedure does not use the sample autocorrelation lags; it uses instead the sequence of sample reflection coefficients, from which it estimates independently of each other and in a dual way, the MA and the AR components of the process.

1. INTRODUCTION

The use of autoregressive moving-average (ARMA) models in spectral estimation has received increased atention in the last few years (see e.g. [2]-[4]). Most of the reported techniques involve several steps, the first of which constructs a sample autocovariance function. Another characteristic of those algorithms is the dependence of the MA component estimation upon the AR component estimation.

This work addresses the ARMA estimation problem, presenting an estimation procedure that does not share the two above mentioned common features. On the one hand, it departs from the usual approach of using the sample autocovariance lag sequence, estimating instead from the data the sequence of reflection coefficients. The simulation results presented here use the Burg technique [1] for the estimation of the reflection coefficients. On the other hand, the AR and MA coefficients are obtained in a dual way. The algorithm constructs from the reflection coefficients two sequences of successively higher order linear filters - one is a sequence of linear predictors, the other is a sequence of innovations filters. The MA part is then obtained by exploring the linear relations that are satisfied by corresponding coefficients of the sequence of linear predictors, while the AR part is obtained similarly but using in turn the coefficients of the increasing order innovations filters. The estimation algorithm dualizes the roles of the AR and MA components, performing the same kind of operations for both components.

The estimation algorithm is based on a finite sample of lenght T drawn from the Gaussian, stationary stochastic process $\{y_n\}$, with the ARMA signal model

$$y_n + \sum_{i=1}^{p} a_i \, y_{n-i} = \sum_{i=0}^{q} b_i \, e_{n-i}, \tag{1}$$

where $\{e_n\}$ is a Gaussian, white, zero mean, unit variance noise sequence. The linear, time-invariant representation (1) is assumed to be stable, minimum phase, with no common roots on the numerator and denominator polynomials of its transfer function.

The estimation algorithm discussed here assumes apriori knowledge of the number of poles p and zeros q, and that $p \geq q$. We are presently testing an extension of the present procedure that jointly estimates the orders p and q as well as the parameters.

In section 2 we briefly review the underlying theory concerning the estimation algorithm, which is detailed in [6]. Here, the emphasis is on the algorithm aspects of the estimation procedure. Some simulated examples are presented in section 3. Further examples are in [5]. In [7] some preliminary results of this algorithm are compared with those of an estimation procedure based on d-step ahead predictors. A statistical analysis computing analytically the asymptotics of the bias and of the error covariance has been carried out and will be presented elsewhere.

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2. DUAL ESTIMATION ALGORITHM

The dual estimation algorithm herein presented is derived from the exact knowledge of the increasing order prediction and innovation filter coefficients associated with the process $\{y_n\}$. We will see that both the AR and the MA components are obtained from the solution of a system of linear equations built up from those coefficients. In the presence of a sample of lenght T of the observed process, those exact values are replaced by a suitable estimate. This problem is discussed later.

To present the algorithm, we will assume that an infinite sample is given, i.e. that we know exactly the above referred filter coefficients. Let

$$a_i^N\ ,\quad i=1,\dots,N,\qquad a_0^N = 1, \tag{2}$$

be the Nth order, one-step ahead prediction error filter coefficients associated with $\{y_n\}$, and denote by

$$c_N = a_N^N \tag{3}$$

the reflection coefficient of order N and by d_N^2 the variance of the Nth order prediction error. Collecting the increasing order coefficients (2) in a matrix format, yields the lower triangular matrix of order N+1, W_N^{-1} as in (4),

$$W_N^{-1} = \begin{bmatrix} 1 & & & & \\ c_1 & 1 & & \underline{0} & \\ c_2 & a_1^2 & 1 & & \\ \vdots & \vdots & & \ddots & \\ c_N & a_{N-1}^N & \dots & a_1^N & 1 \end{bmatrix}. \tag{4}$$

The elements on line i $(0 \leq i \leq N)$ of this matrix are the coefficients of the prediction error filter of order i. The inverse of the nonsingular matrix (4) will be denoted by W_N and its elements writen as in (5),

$$W_N = \begin{bmatrix} 1 & & & & \\ w_1^1 & 1 & & \underline{0} & \\ w_2^2 & w_1^2 & 1 & & \\ \vdots & \vdots & & \ddots & \\ w_N^N & w_{N-1}^N & \dots & w_1^N & 1 \end{bmatrix}. \tag{5}$$

The elements on line i of W_N, w_j^i $(0 \leq j \leq i \leq N)$, are the ith order innovation filter coefficients. They may be obtained from (4) through a matrix inversion. The lower triangular, unity diagonal structure of (4) allows a recursive implementation of this matrix operation.

By the Wold decomposition, the elements on line N of matrix W_N^{-1} constitute, as N goes to infinity, a long AR(N) representation of the ARMA(p,q) process under study. With a dual argument, a long MA representation of the process is obtained, for an high value of N, from the Nth order innovation filter, i.e. the elements on line N of the matrix W_N.

Several spectral estimation techniques use long AR or long MA models as an intermediate or a final step in the estimation procedure. These approaches correspond to exploring the matrices W_N and W_N^{-1} by lines. On the contrary, the estimation algorithm presented on this section looks upon the columns of those matrices. In fact, both the AR and the MA components of the ARMA process are computed from the linear dependencies exhibited by the elements on each column of the matrices W_N and W_N^{-1}.

For $N \geq p$, the elements on each of the first N-q columns of the matrix W_N are linearly dependent. The coefficients of those linear dependencies are the same for all the columns, being the parameters of the AR component. This result, proved in [6], is writen as the following system of linear equations,

$$\begin{cases} w_N^N + a_1\, w_{N-1}^{N-1} + a_2\, w_{N-2}^{N-2} + \dots + a_p\, w_{N-p}^{N-p} = 0 \\ w_{N-1}^N + a_1\, w_{N-2}^{N-1} + a_2\, w_{N-3}^{N-2} + \dots + a_p\, w_{N-p-1}^{N-p} = 0 \\ \quad \dots \\ w_p^N + a_1\, w_{p-1}^{N-1} + a_2\, w_{p-2}^{N-2} + \dots + a_p \qquad = 0 \\ \quad \dots \\ w_{q+1}^N + a_1\, w_q^{N-1} + \dots + a_{q+1} = 0\ . \end{cases} \tag{6}$$

Note that, in (6), the elements of the matrix W_N on each linear equation belong to the same column of that matrix.

A dual result, relating the MA component of the process and the elements on each of the first columns of the matrix W_N^{-1} is presented. First, introduce the normalized version of (4), i.e. the matrix $\tilde{W}_N^{-1}$ with elements

$$\tilde{a}_j^i = a_j^i \,/\, d_i \qquad 0 \leq j \leq i \leq N \tag{7}$$

where d_i is the standard deviation of the prediction error of order i. For N>p, the elements on each of the first N-p columns of the normalized matrix $\tilde{W}_N^{-1}$, defined by (4) and (7), are linearly dependent. The coefficients of the linear dependencies are the same for all those columns, leading to the following system of linear equations [5]-[6],

$$\begin{cases} \alpha_o(N)\, \tilde{c}_N + \alpha_1(N)\, \tilde{c}_{N-1} + \dots + \alpha_q(N)\, \tilde{c}_{N-q} = 0 & (8) \\ \alpha_o(N)\, \tilde{a}^N_{N-1} + \alpha_1(N)\, \tilde{a}^{N-1}_{N-2} + \dots + \alpha_q(N)\, \tilde{a}^{N-q}_{N-q-1} = 0 \\ \dots \\ \alpha_o(N)\, \tilde{a}^N_{p+1} + \alpha_1(N)\, \tilde{a}^{N-1}_{p} + \dots + \alpha_q(N)\, \tilde{a}^{N-q}_{p-q+1} = 0 \end{cases}$$

where,

$$\alpha_o(N) = d_N \quad (9)$$

and, [6],

$$\lim_{N \to \infty} \alpha_i(N) = b_i \quad i=0,1,\dots,q \quad (10)$$

Note that the elements of $\tilde{W}_N^{-1}$ on each of the linear equations in (8) belong to the same column of that matrix. This fact, together with (10), states that the coefficients of the linear dependency defined by the elements on each of the first N-p columns of $\tilde{W}_N^{-1}$ are, asymptotically, those of the MA component of the process. The result expressed by the first equation in (8), dealing with the increasing order reflection coefficients, was presented first in [8].

The systems of linear equations (6) and (8) play a central role in the dual ARMA estimation algorithm. They were established assuming the exact knowledge of the prediction and innovation filter coefficients. Being given a sample of the process with lenght T, those values are replaced by suitable estimates that will be denoted by $\hat{a}^i_j$, $\hat{W}^i_j$, $\hat{d}^2_j$. We use the Burg technique [1] and the Levinson algorithm to estimate the reflection coefficients, the prediction error filter coefficients and the power of the prediction error directly from data. For each value of N we can thus construct the estimated matrix $\hat{W}_N^{-1}$ and $\hat{d}_N^2$.

An inversion of $\hat{W}_N^{-1}$ leads to $\hat{W}_N$. This operation is implemented recursively in N due to the triangular structure of the first matrix and the fact that $\hat{W}_{N-1}^{-1}$ is an upper left submatrix of $\hat{W}_N^{-1}$.

Replacing in (6), (7), (8) and (9) the esmated values of $\hat{W}^i_j$, $\hat{a}^i_j$ and $\hat{d}^2_j$, the solution of the resulting linear system of equations yields the estimate of the AR component and the estimate of the values of $\alpha_i(N)$ $(0 \leq i \leq q)$ that converge, with N, to the MA coefficients (see (10)).

The dual ARMA estimation procedure is summarized in the following algorithm:

For $N=0,1,\dots,N_{max}$

- Compute $\hat{c}_N$ and $\hat{d}_N^2$ from $\{y_o, y_1, \dots, y_{T-1}\}$ using the Burg technique.
- Compute $\hat{a}^N_i$ $(1 \leq i \leq N-1)$ through the Levinson algorithm.
- Construct $\hat{W}_N^{-1}$ by adding a new line to $\hat{W}_{N-1}^{-1}$.
- Obtain $\hat{W}_N$ by matrix inversion from $\hat{W}_N^{-1}$.
- Solve (6) to obtain $\hat{a}_i(N)$, $i=1,\dots,p$ } Only for $N \geq p+q$
- Solve (8) to obtain $\hat{\alpha}_i(N)$, $i=1,\dots,q$ } Only for $N \geq p+q$

End

In the above algorithm, $\hat{a}_i(N)$ and $\hat{\alpha}_i(N)$ denote the estimates of the corresponding coefficients using N-q and N-p linear equations established from the innovation and prediction filters of orders N, N-1, ..., N-p and N, N-1, ..., N-q, respectively.

At this point, some comments ought to be done, strenghting the main features of the algorithm. First, the AR and MA components are estimated independently of each other, being each one obtained from the solution of a system of linear equations. The asymptotic nature of the MA estimation is the price to be paid for the corresponding linearity. In [6] it is proved that the rate of convergence of the $\alpha_i(N)$ parameters to the b_i coefficients is governed by the second power of the zeros of the original process. Second, for N>p+q, both (6) and (8) represent an oversized system of equations, with the statistical relevance of compensating the estimation errors on the prediction and innovation filter coefficients. Third, both the AR and MA (asymptotically) are obtained by a Modified Yule-Walker square root type algorithm. An implementation of this algorithm updating recursively (on the number of reflection coefficients constructed from the data) the MA and AR estimates may be obtained. See [5] and [6] for details.

3. SIMULATION RESULTS

The performance evaluation of the dual estimation algorithm presented in the previous section is described here by simulated examples. The spectrum associated with an ARMA(4,2) process with

poles: $0.85 \exp(\pm j70^o)$, $0.85 \exp(\pm j110^o)$

zeros: $0.95 \exp(\pm j90^o)$

$b_o = 1$

is compared with the estimated spectrum computed from a sample of lenght T.

In fig. 1 we display the spectrum and its estimates obtained with T=1000 data points, for two different values of N, the number of estimated reflection coefficients computed by the Burg technique. The asymptotic nature of the MA component estimation (see equations (8) and (10)) is displayed in fig. 1 where an increase on N improves the estimation of the spectral notch.

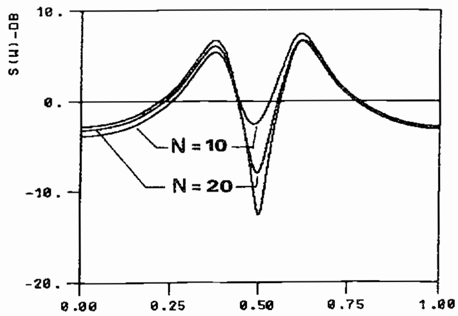

Fig.1 - True and Estimated Spectrum, T=1000 data points, N=10, 20.

The quality of the spectrum estimation as a function of the sample lenght T, is shown in fig. 2 obtained with N=20 and T=500 and 5000 data points. A decrease in T leads to a degradation of the estimation of both the zeros and the poles of the original process. This is due to the degradation of the prediction and innovation filter coefficients estimation.

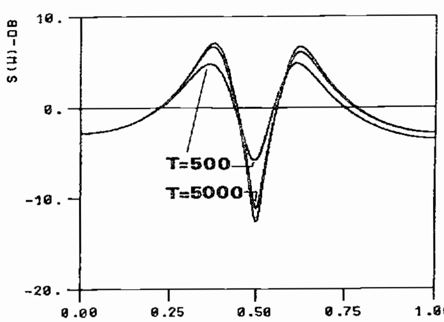

Fig.2 - True and Estimated Spectrum, N=20, T=500 and 5000 data points.

Figure 3-a) and b) show for 10 Monte-Carlo runs the dispersion of the spectral estimates for two values of the sample lenght T. When T is increased from 1000 (fig.3-a)) to 5000 (fig.3-b)) data points, the spread is significantly reduced.

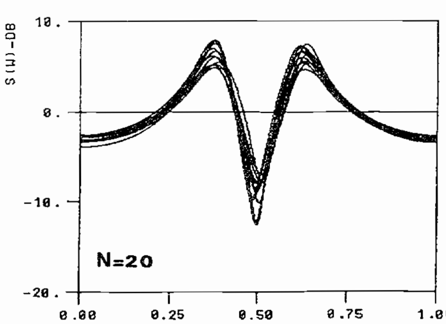

Fig.3-a) - Estimated Spectrum for T=1000 data points and 10 independent Monte-Carlo runs.

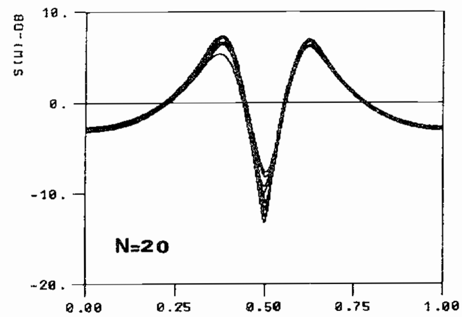

Fig.3-b) - Estimated Spectrum for T=5000 data points and 10 independent Monte-Carlo runs.

We have shown that asymptotically the variance of the errors on the ARMA parameters vanishes with 1/T. Details will be provided in a forthcoming paper.

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