Low Complexity LDPC Codes for Partial Response Channels

Hongwei Song
Vijayakumar Bhagavatula

Data Storage Systems Center
Carnegie Mellon University
Pittsburgh, PA 15213

November 20, 2002
Motivation

- As hard disk drive densities are pushed towards Tb/in$^2$, channels must cope with
  - Low SNRs (7 to 12 dB)
  - High ISI (PW50/T >3)
- Data rates needed are Gbps and faster
- Corrected BER requirements are 10$^{-12}$ to 10$^{-15}$
- Current PRML approaches will not be good enough
- Codes must be of high rate (8/9 and higher)
- Investigating the use of low-complexity, high-rate LDPC codes to meet above goals
Systematic construction of column weight $j=2$ LDPC codes

Girth and distance property of constructed $j=2$ codes.

Density evolution analysis of LDPC codes concatenated with partial response channels

Simulation results
Quasi-cyclic $j=2$ LDPC Codes

- Typically, LDPC codes with column weights $> 2$ are considered; here we look at $j=2$ codes.
- Low complexity encoding and decoding implementation (bandwidth and memory).
- Lower computational complexity due to smaller column weight $j=2$.
- Short cycle free, i.e., girth $g=8$ or $12$.
- Minimum distance $d_{\text{min}}$ and its multiplicity $A(d_{\text{min}})$ can be easily computed.

Parity check matrix
Quasi-cyclic $j=2$ LDPC Code Construction

- Disjoint difference sets (DDS) LDPC codes
- Permutation matrix based LDPC codes
- Graph based LDPC codes
- Random interleaver used with structured codes

$$H = \begin{bmatrix} H_1 & H_2 & \cdots & H_t \end{bmatrix}$$

$$H = \begin{bmatrix} I & I & I & \cdots & I \\ I & \sigma^1 & \sigma^2 & \cdots & \sigma^{p-1} \end{bmatrix}$$
Lemma 1. A $j=2$ LDPC code having girth $g=12$ and block length $n = k(k^2 - k + 1)$ exists, if $k-1$ is a prime number, where $k$ is the row weight.

Lemma 2. A $j=2$ LDPC code constructed in Lemma 1 has $A(d_{\text{min}}) = k(k - 1)^3(k^2 - k + 1) / 6$ number of codewords with minimum distance $d_{\text{min}} = 6$, where $k$ is row weight.

Must avoid Case 1 and Case 2 to achieve girth 12
The parity bits can be computed from the information bits as follows:

\[
\begin{align*}
    p_i &= x_{i_1} \oplus x_{i_2} \oplus \cdots \oplus x_{i_{k-1}} \\
    q_i' &= p_{i_1'} \oplus p_{i_2'} \oplus \cdots \oplus p_{i_{k-1}'} \\
    t &= q_1 \oplus q_2 \oplus \cdots \oplus q_{k-1}
\end{align*}
\]

\[
\# op = (1 + 2(k - 1) + 2(k - 1)^2)(k - 2) \approx 2n \quad \text{linear encoding complexity!}
\]
Word Error Rates for $j=2$ codes over AWGN

- ML union bound

$$P_E \leq \sum_{d=d_{\text{min}}}^{N} A(d)Q\left(\sqrt{2d \cdot R \cdot E_b / N_0}\right) \approx A(d_{\text{min}})Q\left(\sqrt{2d_{\text{min}} \cdot R \cdot E_b / N_0}\right)$$

- Iterative soft decoding is fairly close to ML decoding for these codes

Rate $R = 1 - (2/k)$
Density Evolution Analysis

- Track the evolution of the probability density function (pdf)
- Assume infinite block length and Gaussian pdf, define \( SNR = \frac{m^2}{\sigma^2} \)

\[
SNR^C_{out} = f_C(SNR^C_{in}, E_b / N_0)
\]
\[
SNR^L_{out} = f_L(SNR^L_{in})
\]

Empirically measured histograms of an EPR4 channel detector at SNR = 5.0 dB
Extrinsic Information Transfer Functions

\[ \text{SNR}^C_{\text{out}} = f_C(\text{SNR}^C_{\text{in}}, \frac{E_b}{N_0} = 4.75\,\text{dB}) \]

\[ \text{SNR}^L_{\text{out}} = f_L(\text{SNR}^L_{\text{in}}) \]

- Transfer functions of PR channel (computed via Monte-Carlo) without precoder flatten out and saturate to the same output.
- LDPC code with column weight \( j=2 \) provides higher output SNR than codes with \( j>2 \) for low input SNR; situation reverses at high input SNR.
Larger girth estimates and simulation results are closer than for small girth.
BER for PR Channels

\[ \frac{E_b}{N_0} = 4.0 : 0.5 : 7.0 dB \]

3 Channel decoding iterations

\[ SNR = SNR_{out}^{L} \left( fixed \right) + SNR_{in}^{L} \left( fixed \right) \]

\[ BER = Q \left( \sqrt{SNR} \right) \]

- PR4 predicted BER and simulated BER match well
- At high SNRs, all three PR targets yield similar BER performance
Single Convolutional Code as Outer Code

- BER curves based on single rate 8/9 RSC \((31,23)_8\) concatenated with PR
- At low SNR, PR1 and PR4 are better
- Different PR channels have similar BER performance at high SNR
- \(j=2\) LDPC code worse than the convolutional code for low SNR, better at high SNR; both suffer error floor
- $j=2$ LDPC code with proper precoding outperforms LDPC codes with $j \geq 3$, while precoding adds little complexity to encoding and decoding.
- LDPC codes with $j \geq 3$ have narrow “tunnel” compared to $j=2$ code, which results in significantly different block error statistics.
- Inclusion of precoding narrows the “tunnel”.
- $j=2$ LDPC code exhibits block error statistics more compatible with an outer Reed-Solomon (RS) code, due to wider “tunnel”.
Conclusions

- Systematic construction of column weight $j=2$ LDPC codes
- Investigated girth and distance properties
- Sum-product algorithm is fairly close to ML decoding for these codes.
- $j=2$ LDPC code exhibits block error statistics more compatible with an outer RS code, attractive for magnetic recording channels.