Appendix A
Fault Detection Filter Background

A LINEAR TIME-INVARIANT SYSTEM with q failure modes and no disturbances or sensor noise can be modeled (Beard 1971), (White and Speyer 1987), (Massoumnia 1986) by

$$\dot{x} = Ax + Bu + \sum_{i=1}^{q} F_i m_i \tag{a.1a}$$

$$y = Cx. \tag{a.1b}$$

All system variables belong to real vector spaces $x \in \mathcal{X}$, $u \in \mathcal{U}$, $y \in \mathcal{Y}$ and $m_i \in \mathcal{M}_i$ with $n = \dim \mathcal{X}$, $p = \dim \mathcal{U}$, $m = \dim \mathcal{Y}$ and $q_i = \dim \mathcal{M}_i$. The input $u \in \mathcal{U}$ is known as is the output $y \in \mathcal{Y}$. The failure modes $m_i \in \mathcal{M}_i$ are vectors that are unknown and arbitrary functions of time and are zero when there is no failure. The failure signatures $F_i : \mathcal{M}_i \mapsto \mathcal{F}_i \subseteq \mathcal{X}$ are maps that are known, fixed and unique. A failure mode m_i models the time-varying amplitude of a failure while a failure signature F_i models the directional characteristics of a failure. Assume the F_i are monic so that $m_i \neq 0$ implies $F_i m_i \neq 0$. Actuator and plant faults are modeled with F_i as the appropriate direction from A or B.

For example, a stuck actuator is modeled with F_i as the column of A associated with the actuator dynamics and with $m_i(t) = -u_i(t) + u_{ic}$ where u_{ic} is some constant.

Sensor faults are most naturally modeled as an additive term in the measurement equation as follows where E_i is a column vector of zeros except for a one in the i^{th} position and where μ_i is an arbitrary time-varying real scalar.

$$y = Cx + E_i\mu_i \tag{a.2}$$

It can be shown that the E_i sensor fault form of (a.2) may be converted to an equivalent F_i form (a.1) with no need for appended dynamics (Beard 1971), (White and Speyer 1987), (Douglas 1993). This is demonstrated shortly.

A.1 The Detection Filter Problem

Consider a full-order observer of the form

$$\dot{\hat{x}} = (A + LC)\hat{x} + Bu - Ly \tag{a.3a}$$

$$z = C\hat{x} - y. \tag{a.3b}$$

The state estimation error $e = \hat{x} - x$ dynamics are

$$\dot{e} = (A + LC)e - \sum_{i=1}^{q} F_i m_i \tag{a.4}$$

If (C, A) is observable and L is chosen so that $A + LC$ is stable, then in steady-state and in the absence of disturbances and modeling errors, the residual r is nonzero only if a failure mode m_i is nonzero and is almost always nonzero whenever m_i is nonzero. It follows that any stable observer can detect the occurrence of a fault. Simply monitor the residual z and when it is nonzero a fault has occurred. A more difficult task is to determine which fault has occurred and that is what a fault detection filter is designed to do.

A fault detection filter is an observer with the property that when an unknown input or fault is nonzero, $m_i(t) \neq 0$, the error $e(t)$ remains in a (C, A)-invariant subspace $\mathcal{W}_i$ which contains the reachable subspace of $(A + LC, F_i)$. Thus, the residual remains in the output subspace $C\mathcal{W}_i$. Furthermore, the output subspaces $C\mathcal{W}_1, \ldots, C\mathcal{W}_q$ are independent so that $z \in \sum_{i=1}^{q} C\mathcal{W}_i$ has a unique representation $z = z_1 + \cdots + z_q$ with $z_i \in C\mathcal{W}_i$. The fault is identified by projecting z onto each of the output subspaces $C\mathcal{W}_i$. The following statement of the detection filter problem, sometimes called the Beard-Jones detection filter problem, is essentially the same as that found in (Beard 1971) and (White and Speyer 1987) but is stated in the geometric language of (Massoumnia 1986).

Definition A.1 (Detection Filter Problem). Given the system (a.1), with state-space $\mathcal{X}$ and measurement-space $\mathcal{Y}$, the detection filter problem is to find a set of subspaces $\mathcal{W}_i \subseteq \mathcal{X}$, $i = 1, \ldots, q$ such that for some map $L : \mathcal{Y} \mapsto \mathcal{X}$ the following conditions are met:

$$
\begin{aligned}
(A + LC)\mathcal{W}_i &\subseteq \mathcal{W}_i && \text{Subspace invariance.} \\
\mathcal{F}_i &\subseteq \mathcal{W}_i && \text{Failure inclusion.} \\
C\mathcal{W}_i \cap (\sum_{j \neq i} C\mathcal{W}_j) &= 0 && \text{Output separability.}
\end{aligned}
$$

It can be shown (Massoumnia 1986), (White and Speyer 1987) that the last condition, output separability, implies that the subspaces $\mathcal{W}_1, \ldots, \mathcal{W}_q$ are independent when (C, A) is observable

A.2 Sensor Fault Models

It is now shown how the E_i sensor fault form of (a.2) is converted to an equivalent F_i form with no need for appended dynamics. While this is also shown in (Beard 1971), (White and Speyer 1987) and (Douglas 1993), the following original demonstration is more easily extended to time varying systems. Let F_i be any map that satisfies

$$CF_i = E_i$$

and define a new state estimation error $\bar{e}$ as

$$\bar{e} = e - F_i \mu_i$$

This is a Goh transformation on the error space (Jacobson 1971). The residual is then.

$$r = C\bar{e}$$

Using (a.4), the dynamics of $\bar{e}$ are

$$\dot{\bar{e}} = (A + LC)\bar{e} + AF_i\mu_i - F_i\dot{\mu}_i \tag{a.5}$$

and a sensor fault E_i in (a.2) is equivalent to a two-dimensional fault F_i

$$\dot{x} = Ax + Bu + F_i m_i \qquad \text{with } F_i = \left[F_i^1, F_i^2\right]$$

where the directions F_i^1 and F_i^2 are given by

$$E_i = CF_i^1 \tag{a.6a}$$

$$F_i^2 = AF_i^1 \tag{a.6b}$$

An interpretation of the effect of a sensor fault on observer error dynamics follows from (a.5) where F_i^1 is the sensor fault rate $\dot{\mu}_i$ direction and F_i^2 is the sensor fault magnitude μ_i direction. This interpretation suggests a possible simplification when information about the spectral content of the sensor fault is available. If it is known that a sensor fault has persistent and significant high frequency components, such as in the case of a noisy sensor, the fault direction could be approximated by the F_i^1 direction alone. Or, if it is known that a sensor fault has only low frequency components, such as in the case of a bias, the fault direction could be approximated by the F_i^2 direction alone. For example, if a sensor were to develop a bias, a transient would be likely to appear in all fault directions but, in steady-state, only the residual associated with the faulty sensor should be nonzero.

In the case where the dynamics (a.1) are time varying, the error dynamics (a.5) become

$$\dot{\bar{e}} = (A + LC)\bar{e} + (AF_i - \dot{F}_i)\mu_i - F_i\dot{\mu}_i$$

so that once again, a sensor fault E_i in (a.2) is equivalent to a two-dimensional fault F_i

$$\dot{x} = Ax + Bu + F_i m_i \qquad \text{with } F_i = \left[F_i^1, F_i^2\right]$$

but where the directions F_i^1 and F_i^2 are given by

$$
\begin{aligned}
E_i &= CF_i^1 \\
F_i^2 &= AF_i^1 - \dot{F}_i^1
\end{aligned}
$$

A.3 Solving The Detection Filter Problem

It should be pointed out that for any subspace $\mathcal{F}_i \subseteq \mathcal{X}$ there is a minimal (C, A)-invariant subspace $\mathcal{F}_i \subseteq \mathcal{W}_i^* \subseteq \mathcal{X}$. A recursive algorithm, the (C, A)-invariant subspace algorithm, for computing a minimal invariant subspace is suggested by (Wonham 1985) and restated in the following theorem.

Theorem A.1 (CAISA). Let $\mathcal{W}(\mathcal{F})$ be a family of (C, A)-invariant subspaces where $\mathcal{F} \subseteq \mathcal{W} \in \mathcal{W}(\mathcal{F})$. Then, there exists a minimal (C, A)-invariant subspace $\mathcal{W}^* \in \mathcal{W}(\mathcal{F})$ where for any $\mathcal{W} \in \mathcal{W}(\mathcal{F})$, $\mathcal{W}^* \subseteq \mathcal{W}$. Furthermore, $\mathcal{W}^* = \lim \mathcal{W}^k$ where $\mathcal{W}^k$ is given by the recursive algorithm

$$
\begin{aligned}
\mathcal{W}^0 &= \emptyset \\
\mathcal{W}^{k+1} &= \mathcal{F} + A\left(\mathcal{W}^k \cap \operatorname{Ker} C\right)
\end{aligned}
$$

Proof. The proof given in (Wonham 1985) follows from the result of (Willems 1982) that the set $\mathcal{W}(\mathcal{F})$ is closed under subspace intersection. ◀

Note that the algorithm given in Theorem A.1 implies that for $\dim \mathcal{F}_i = 1$, the minimal (C, A)-invariant subspace $\mathcal{W}_i^*$ is spanned by $\{F_i, AF_i, \ldots, A^{\mu_i} F_i\}$ where μ_i is the smallest integer such that $CA^{\mu_i} F_i \neq 0$. For one-dimensional faults, the algorithm of Theorem A.1 is a very simple way to find $\mathcal{W}_i^*$.

Theorem A.1 also suggests a check for output separability. Let $\{f_{i_1}, \ldots, f_{i_{qi}}\}$ be any set of basis vectors for $\mathcal{F}_i$. An output separability check is that

$$
\operatorname{rank}\left[CA^{\beta_{11}} f_{1_1}, \ldots, CA^{\beta_{i_j}} f_{i_j}, \ldots, CA^{\beta_{qq_q}} f_{q_{q_q}}\right] = p \tag{a.7}
$$

where $p = \sum q_i$ is the total number of basis vectors for the q failure spaces $\mathcal{F}_i$ and β_{i_j} is the smallest integer such that $CA^{\beta_{1_1}} f_{1_1} \neq 0$. Note that if (a.7) is not satisfied, then usually, the designer needs to discard some failures from the design set.

In the case where the dynamics (a.1) are time varying, an output separability check is that

$$\operatorname{rank}\left[Cb_{1_1}^{\beta_{1_1}}(t), \ldots, Cb_{i_j}^{\beta_{i_j}}(t), \ldots, Cb_{q_{qq}}^{\beta_{q_{qq}}}(t)\right] = p, \qquad \forall t \in [t_0, t_1] \tag{a.8}$$

where β_i is the smallest integer such that the following iteration:

$$b_{i_j}^1(t) = f_{i_j}(t) \tag{a.9a}$$

$$b_{i_j}^{k+1}(t) = Ab_{i_j}^k(t) + \dot{b}_{i_j}^k(t) \tag{a.9b}$$

results in a vector $b_{i_j}^k(t)$ such that $Cb_{i_j}^k(t) \neq 0$ for all $t \in [t_0, t_1]$. Note that (a.9) are the product of a Goh transformation on the output error space.

It is assumed that the system matrices $A(t)$, $C(t)$ and $F_i(t)$ are such that the number of iterations of (a.9) needed for the full rank condition is constant over the entire interval $[t_0, t_1]$, that is, the time variations of the system do not change the dimensionality of the detection problem. This restricts the applicability of this analysis to a subclass of time varying systems, but it avoids pathological cases. Assumptions such as this seem to be unavoidable when dealing with time varying systems. See, for example, (Clements and Anderson 1978).

When $Cf_{i_j} = 0$, both output separability tests fail immediately. However, this is not indicative of whether or not the system is output separable. As we will see in the next section, $Cf_{i_j} = 0$ is a sign that a f_{i_j} possess a higher-order detection space, meaning that it takes more than one vector to span this space. From Theorem A.1, one of these must lie outside the kernal of C and is, thus, the vector which must be used in the output separability test.

To ensure stability, the invariant subspaces $\mathcal{W}_i$ are usually chosen as a set of mutually detectable, minimal unobservability subspaces or detection spaces (Beard 1971) as they are also called in the context of fault detection. An unobservability subspace $\mathcal{T} \subseteq \mathcal{X}$ or UOS is a subspace with the property that $\mathcal{T}$ is the unobservable subspace of the pair $(HC, A+LC)$ for some L and H. This means not only that $\mathcal{T}$ is (C, A)-invariant but also that the spectrum of $(A + LC)$ induced on the factor space $\mathcal{X}/\mathcal{T}$ may be placed arbitrarily within a conjugate symmetry constraint and with respect to L such that $(A + LC)\mathcal{T} \subseteq \mathcal{T}$. Furthermore, when (C, A) is observable, the entire spectrum of $(A + LC)$ is arbitrary. If $\mathcal{T}(\mathcal{F})$ is the set of (C, A)-unobservability subspaces that contain $\mathcal{F}$, then it can be shown that $\mathcal{T}(\mathcal{F})$ has a smallest element denoted $\mathcal{T}^*$ (Willems 1982). The detection space is usually found as a minimal UOS, $\mathcal{T}^*$, because there is no known parameterization of all UOS and algorithms exist to compute the minimal UOS (White and Speyer 1987), (Massoumnia 1986).

One method for computing $\mathcal{T}^*$ is suggested by (Wonham 1985) as a numerically stable method for finding supremal controllability subspaces. These are the dual of minimal unobservability subspaces or detection spaces. There are two steps. First, for a fault F_i, find the minimal (C, A)-invariant subspace $\mathcal{W}_i^*$ using the recursive (C, A)-invariant subspace algorithm as explained above. Next, calculate the invariant zero directions of the triple (C, A, F_i), if any. Denote the invariant zero directions as $\mathcal{V}_i$. Then

$$\mathcal{T}_i^* = \mathcal{W}_i^* \oplus \mathcal{V}_i$$

Detection space calculations are described in detail in (Wonham 1985) with amplification and examples given in (Douglas 1993).

Finally, a mutually detectable set of unobservability subspaces $\{\mathcal{T}_1^*, \ldots, \mathcal{T}_q^*\}$ is one which satisfies Definition A.1 such that the sum $\sum_{i=1}^{q} \mathcal{T}_i^*$ is also an UOS. While for any one UOS $\mathcal{T}_i$, the spectrum of $(A + LC)$ induced on $\mathcal{X}/\mathcal{T}_i$ may be placed arbitrarily with respect to L, it is not necessarily true that the factor space spectrum is arbitrary when several UOS are considered simultaneously. When a set of UOS $\mathcal{T}_1^*, \ldots, \mathcal{T}_q^*$ is mutually detectable, the spectrum of $(A + LC)$ induced on $\mathcal{X}/\sum_{i=1}^{q} \mathcal{T}_i^*$ is arbitrary and, when (C, A) is observable, the entire spectrum of $(A + LC)$ is arbitrary.

A.4 The Restricted Diagonal Detection Filter Problem

In (Massoumnia 1986), the Beard-Jones detection filter problem is shown to be a special case of the *restricted diagonal detection filter problem* (RDDFP). First, define the complementary failure map $\hat{F}_i$ as

$$\hat{F}_i = [F_1, \ldots, F_{i-1}, F_{i+1}, \ldots, F_q] \tag{a.10}$$

The RDDFP, which is the dual of the *restricted decoupling problem* (Wonham 1985), is to find a set of q unobservability subspaces $\hat{\mathcal{T}}_1, \ldots, \hat{\mathcal{T}}_q$ such that

$$\mathcal{F}_i \cap \hat{\mathcal{T}}_i = 0$$
$$\hat{\mathcal{F}}_i \subseteq \hat{\mathcal{T}}_i$$

In the Beard-Jones detection filter, the idea is to confine each fault to an invariant subspace and then monitor that subspace through the residual for fault activity. In the RDDFP, the idea is to confine all the faults but one to an unobservable subspace, then monitor the observable factor space for activity caused by the remaining fault. By the definition of an unobservability subspace, there exists a projector H_i and a gain L such that $\hat{\mathcal{T}}_i$ is the unobservable subspace of the pair $(H_i C, A + LC)$. The signal

$$z_i = H_i(y - C\hat{x}) \tag{a.11}$$

is decoupled from all faults except F_i. Furthermore, $\mathcal{F}_i \cap \hat{\mathcal{T}}_i = 0$ implies that F_i is input observable so that $F_i m_i \neq 0$ implies that $z_i \neq 0$. Also, by construction, $\hat{H}_i$ satisfies

$$\operatorname{Ker} \hat{H}_i C = \hat{\mathcal{T}}_i + \operatorname{Ker} C$$

An explicit construction of $\hat{H}_i$ is to form CM_i as in (a.7)

$$CM_i \;=\; \left[CA^{\beta_{1_1}} f_{1_1}, \ldots, CA^{\beta_{i_j}} f_{i_j}, \ldots, CA^{\beta_{q_q q}} f_{q_{qq}}\right]$$

Then

$$\hat{H}_i \;=\; I - (CM_i)[(CM_i)^T (CM_i)]^{-1} (CM_i)^T$$

In the case where the dynamics (a.1) are time varying, $\hat{H}_i(t)$ may be constructed by forming $CM_i(t)$ as in (a.8)

$$CM_i = \left[Cb_{1_1}^{\beta_{1_1}}(t), \ldots, Cb_{i_j}^{\beta_{i_j}}(t), \ldots, Cb_{q_{qq}}^{\beta_{q_{qq}}}(t)\right]$$

where β_i is the smallest integer such that the following iteration:

$$\begin{aligned} b^1_{i_j}(t) &= \hat{f}_{i_j}(t) \\ b^{k+1}_{i_j}(t) &= Ab^k_{i_j}(t) + \dot{b}^k_{i_j}(t) \end{aligned}$$

results in a vector $b^k_{i_j}(t)$ such that $Cb^k_{i_j}(t) \neq 0$ for all $t \in [t_0, t_1]$. This time, $\hat{f}_{i_j}$ is taken to be vector from a basis for $\hat{\mathcal{F}}_i$.

It is easy to show that a Beard-Jones detection filter is always a restricted diagonal detection filter. For example, suppose a Beard-Jones detection filter is formed as a set of mutually detectable unobservability subspaces $\mathcal{T}^*_1, \ldots, \mathcal{T}^*_q$. Let

$$\hat{\mathcal{T}}^*_i = \sum_{j \neq i} \mathcal{T}^*_j \tag{a.12}$$

Then, by the definition of mutual detectability, $\hat{\mathcal{T}}^*_i$ is itself a minimal unobservability subspace for the fault group $\hat{F}_i$.

Appendix B

Parameter Robustness By Left Eigenvector Assignment

ONCE the detection spaces are found, the next step is to find a fault detection filter gain. The gain is not unique and several methods exist for finding one. Eigenstructure assignment algorithms, which are the most accessible, are described in (Douglas and Speyer 1995b) and (White and Speyer 1987). An $\mathcal{H}_\infty$ disturbance bounded fault detection filter described in (Douglas and Speyer 1995a) is reviewed in Appendix C. The procedure applied in this report is a left eigenvector assignment algorithm introduced in (Douglas and Speyer 1996) and (Douglas 1993). This procedure is used because it extends directly to one that hedges against sensitivity to parameter uncertainty. Noise robustness algorithms such as the $\mathcal{H}_\infty$-bounded fault detection filter of (Douglas and Speyer 1995a) and Appendix C are not used here because disturbances and sensor noise are not yet included in the vehicle model. Furthermore, later, when they are included, the reduced-order fault detection filters provide a natural way to accommodate noise without the need for redesigning the filter.

The left eigenvector assignment algorithm works by assigning an eigenstructure in the dual space to a set of intersecting detection space annihilators. This means that left eigenvectors, which annihilate the detection spaces, are placed instead of right eigenvectors, which span the detection spaces, as is done in (White and Speyer 1987). Since the detection space annihilators intersect, care must be taken to ensure that the assigned eigenvectors are consistent.

Before proceeding, it is necessary to establish a dual relation between unobservability and controllability subspaces. First, introduce the following notation. $\mathcal{X}'$ denotes the dual space of $\mathcal{X}$ and if $C : \mathcal{X} \mapsto \mathcal{Y}$, then C' denotes the dual map $C'\mathcal{Y}' \mapsto \mathcal{X}'$. Writing C^T, the transpose of matrix C, for the dual map C' implies that bases have been chosen for $\mathcal{X}$ and $\mathcal{Y}$. Now, in (Wonham 1985) it is shown that if $\mathcal{T} \subseteq \mathcal{X}$ is a (C, A)-unobservability subspace then the annihilator of $\mathcal{T}$ denoted here by $\mathcal{T}^\perp \subseteq \mathcal{X}'$ is an (A', C')-controllability subspace in the dual system. Second, if $\mathcal{T}$ is a (C, A)-unobservability subspace, the observable part of the system is characterized by the factor space $\mathcal{X}/\mathcal{T}$ and the induced system maps. Furthermore, for any subspace $\mathcal{T} \subseteq \mathcal{X}$, the annihilator of $\mathcal{T}$ and the factor space $\mathcal{X}/\mathcal{T}$ are isomorphic, $\mathcal{T}^\perp \simeq (\mathcal{X}/\mathcal{T})'$.

The dual relation between unobservability and controllability subspaces is useful because any result found for controllability subspaces can be applied easily to the unobservability subspaces of a detection filter. Consider the results of (Moore and Laub 1978) which are paraphrased as follows. The first statement describes a set of vectors in the kernal of C that can be assigned as closed-loop eigenvectors.

Theorem B.1. Let $A : \mathcal{X} \mapsto \mathcal{X}$, $B : \mathcal{U} \mapsto \mathcal{X}$ and $C : \mathcal{X} \mapsto \mathcal{Y}$. Then a set of linearly independent vectors $\{v_1, \ldots, v_k \mid v_i \in \operatorname{Ker} C \subseteq \mathcal{X}\}$ satisfies $(A + BK)v_i = \lambda_i v_i$ for some $K : \mathcal{X} \mapsto \mathcal{U}$ and distinct self-conjugate complex numbers $\lambda_1, \ldots, \lambda_k$ if and only if v_i and v_j are conjugate pairs when λ_i and λ_j are and there exists a set of vectors $\{w_1, \ldots, w_k | w_i \in \mathcal{U}\}$ such that

$$\begin{bmatrix} A - \lambda_i I & B \\ C & 0 \end{bmatrix} \begin{bmatrix} v_i \\ w_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

It follows immediately that for a monic B, a set of vectors $\{v_1, \ldots, v_k\}$ satisfies theorem B.1 if and only if $Kv_i = w_i$.

The second result also from (Moore and Laub 1978) characterizes the set of eigenvectors that span a supremal (A, B)-controllability subspace $\mathcal{R}^*$.

Theorem B.2. Let $\lambda_1, \ldots, \lambda_k$ be a set of distinct, self-conjugate complex numbers that satisfy

1) $k \geq \dim(\mathcal{R}^*)$ where $\mathcal{R}^*$ is the supremal (A, B)-controllability subspace in $\operatorname{Ker} C$

2) at least one λ_i is real

3) no λ_i or $\operatorname{Re}(\lambda_i)$ is a transmission zero of (C, A, B)

Let V_i and W_i solve

$$\begin{bmatrix} A - \lambda_i I & B \\ C & 0 \end{bmatrix} \begin{bmatrix} V_i \\ W_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then $\mathcal{R}^* = \operatorname{Im} V_1 + \cdots + \operatorname{Im} V_k$.

Given the dual relationship between controllability and unobservability subspaces, the application of Theorems B.1 and B.2 to detection filter design is immediate. First, consider just one detection space $\mathcal{T}_i^*$. Characterize the left eigenvectors that annihilate $\mathcal{T}_i^*$ and find a detection filter gain L_i that produces $\mathcal{T}_i^*$. Next establish a consistency requirement on a detection filter gain L that is to produce q detection spaces $\mathcal{T}_1^*, \ldots, \mathcal{T}_q^*$.

If $\mathcal{T}_i^* \subseteq \mathcal{X}$ with dimension ν_i is a detection space for fault F_i, the annihilator $(\mathcal{T}_i^*)^\perp$ is the supremal controllability subspace of the dual system with $(\mathcal{T}_i^*)^\perp \subseteq \operatorname{Ker} F_i'$ and has dimension $n - \nu_i$. Let $\hat{\Lambda}_i = \{\lambda_{i_1}, \ldots, \lambda_{i_{n-\nu_i}}\}$ be a set of distinct self-conjugate complex numbers that does not include any of the invariant zeros of the triple (F_i', A', C'). By Theorem B.2 the annihilator of $\mathcal{T}_i^*$ satisfies

$$(\mathcal{T}_i^*)^\perp = \operatorname{Im} V_{i_1} + \cdots + \operatorname{Im} V_{i_{n-\nu_i}}$$

where the V_{i_j} are found, along with W_{i_j}, by solving

$$\begin{bmatrix} A^T - \lambda_{i_j} I & C^T \\ F_i^T & 0 \end{bmatrix} \begin{bmatrix} V_{i_j} \\ W_{i_j} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{b.1}$$

where $j = 1, \ldots, n - \nu_i$ and where $\lambda_{i_j} \in \hat{\Lambda}_i$. A set of linearly independent closed-loop left eigenvectors $v_{i_1}, \ldots, v_{i_{n-\nu_i}}$ that spans $(\mathcal{T}_i^*)^\perp$ satisfies Theorem B.1 and is found by solving

$$\begin{bmatrix} A^T - \lambda_{i_j} I & C^T \\ F_i^T & 0 \end{bmatrix} \begin{bmatrix} v_{i_j} \\ w_{i_j} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{b.2}$$

Since $v_{i_j} \in \operatorname{Im} V_{i_j}$ (b.1), the left eigenvectors may not be unique but they are constrained to be arranged in conjugate pairs when the given closed-loop eigenvalues λ_{i_j} are in conjugate pairs.

Now find a detection filter gain L_i. By the remark following Theorem B.1, L_i^T satisfies

$$L_i^T v_{i_j} = w_{i_j} \tag{b.3}$$

and $(A^T + C^T L_i^T) v_{i_j} = \lambda_{i_j} v_{i_j}$ for each $j = 1, \ldots, n - \nu_i$. Form two matrices $\hat{V}_i$ and $\hat{W}_i$

$$\hat{V}_i = \left[v_{i_1}, \ldots, v_{i_{n-\nu_i}} \right] \tag{b.4a}$$

$$\hat{W}_i = \left[w_{i_1}, \ldots, w_{i_{n-\nu_i}} \right] \tag{b.4b}$$

and solve $L_i^T \hat{V}_i = \hat{W}_i$. A real solution for L_i^T always exists because the v_{i_j} are linearly independent and the assigned closed-loop poles λ_{i_j} and eigenvectors v_{i_j} when complex are arranged in conjugate pairs. Finally, L_i, the detection filter gain found as the transpose

$$\hat{V}_i^T L_i = \hat{W}_i^T \tag{b.5}$$

satisfies $(A + L_i C)\mathcal{T}_i^* \subseteq \mathcal{T}_i^*$ and places the spectrum of $(A + L_i C)$ induced on $\mathcal{X}/\mathcal{T}_i^*$ as $\sigma(A + L_i C | \mathcal{X}/\mathcal{T}_i^*) = \hat{\Lambda}_i$.

Because the detection filter has q detection spaces $\mathcal{T}_1^*, \ldots, \mathcal{T}_q^* \subseteq \mathcal{X}$, the detection filter gain L has to satisfy (b.5) for $i = 1, \ldots, q$ or

$$L^T \left[\hat{V}_1, \ldots, \hat{V}_q \right] = \left[\hat{W}_1, \ldots, \hat{W}_q \right] \tag{b.6}$$

Since the $\hat{V}_i$ and $\hat{W}_i$ represent $\sum_{i=1}^q (n - \nu_i)$ pairs of vectors (v_{i_j}, w_{i_j}), care must be taken to construct the $\hat{V}_i$ and $\hat{W}_i$ conformably. If (b.6) is to have a solution for L, there can be no more than n distinct pairs (v_{i_j}, w_{i_j}) and of these, the v_{i_j} must be linearly independent and arranged in conjugate pairs if a solution is to be unique and real.

Finding a set of left eigenvectors consistent with (b.6) is not difficult but requires careful bookkeeping. Since $(\mathcal{T}_i^*)^\perp$ and $(\mathcal{X}/\mathcal{T}_i^*)'$ are isomorphic, the closed-loop spectrum induced on the factor space $\mathcal{X}/\mathcal{T}_i^*$ is

$$\sigma(A + L_iC|\mathcal{X}/\mathcal{T}_i^*) = \sigma(A' + C'L_i'|(\mathcal{T}_i^*)^\perp) = \hat{\Lambda}_i$$

If Λ_i is the spectrum of $(A + L_iC)$ restricted to the invariant subspace $\mathcal{T}_i^*$

$$\Lambda_i = \sigma(A + LC|\mathcal{T}_i^*)$$

then the spectrum of $(A + L_iC)$ is just

$$\Lambda = \sigma(A + L_iC) = \Lambda_i \cup \hat{\Lambda}_i \tag{b.7}$$

Now, the subspaces $\mathcal{T}_1^*, \ldots, \mathcal{T}_q^*$ are independent when the faults are output separable and (C, A) is observable (Massoumnia 1986), (White and Speyer 1987), so

$$\Lambda = \Lambda_1 \cup \cdots \cup \Lambda_q \cup \Lambda_0$$

where Λ_0 is a set of $\nu_0 = n - \nu_1 - \cdots - \nu_q$ eigenvalues associated with the complementary space $\hat{\mathcal{X}}_0 = \mathcal{X}/\sum_{i=1}^q \mathcal{T}_i^*$, $\nu_0 = \dim(\hat{\mathcal{X}}_0)$,

$$\Lambda_0 \;=\; \sigma(A + LC|\mathcal{X}/\sum_{i=1}^{q} \mathcal{T}_i^*)$$

It follows from (b.7) that

$$\hat{\Lambda}_i \;=\; \bigcup_{\substack{k=0 \\ k\neq i}}^{q} \Lambda_k \tag{b.8}$$

Since the sets of assigned closed-loop poles $\hat{\Lambda}_i$ intersect, the sets of vectors v_{i_j} and w_{i_j} that solve (b.2) should also form intersecting sets compliant with (b.8). By (b.8), if $\lambda_{i_j} \in \Lambda_i$ for $i \neq 0$, then $\lambda_{i_j} \in \hat{\Lambda}_{k\neq i}$ and the v_{i_j} and w_{i_j} that satisfy (b.2) now must satisfy

$$\begin{aligned} 0 &= (A^T - \lambda_{i_j} I)v_{i_j} + C^T w_{i_j} \\ 0 &= F_1^T v_{i_j} \end{aligned}$$

$$
\begin{aligned}
&\vdots \\
0 &= F_{i-1}^T v_{i_j} \\
0 &= F_{i+1}^T v_{i_j} \\
&\vdots \\
0 &= F_q^T v_{i_j}
\end{aligned}
$$

For $i = 0$ and $\lambda_{i_j} \in \Lambda_0$, then $\lambda_{i_j} \in \hat{\Lambda}_k$ for $k = 1, \dots, q$ and the v_{i_j} and w_{i_j} that satisfy (b.2) now must satisfy

$$
\begin{aligned}
0 &= (A^T - \lambda_{i_j} I) v_{i_j} + C^T w_{i_j} \\
0 &= F_1^T v_{i_j} \\
&\vdots \\
0 &= F_q^T v_{i_j}
\end{aligned}
$$

The fault detection filter gain computation algorithm suggested by (b.2)-(b.6) and modified to force consistency among eigenvectors which span the intersecting detection space annihilators, is as follows.

Algorithm B.1.

1) Find the dimensions of the detection spaces $\nu_i = \dim \mathcal{T}_i^*$ for $i = 1, \dots, q$ and the dimension of the complementary space $\nu_0 = n - \sum_{i=1}^q \nu_i$.

2) Define the complementary fault sets

$$\hat{F}_i = \begin{cases} [F_1, \dots, F_q] & \text{for } i = 0 \\ [F_1, \dots, F_{i-1}, F_{i+1}, \dots, F_q] & \text{for } 1 \le i \le q \end{cases} \tag{b.9}$$

Define $(q+1)$ sets of distinct self-conjugate complex numbers $\Lambda_0, \Lambda_1, \dots, \Lambda_q$ where $\dim \Lambda_i = \nu_i$ and where no elements of Λ_i are zeros of the triple $(C, A, \hat{F}_i)$. By the remarks at the end of Appendix A, each of these sets may be specified arbitrarily except for conjugate symmetry when (C, A) is observable and when the detection spaces $\mathcal{T}_i^*$ are mutually detectable.

3) For $i = 0, \ldots, q$ and $j = 1, \ldots, \nu_i$ and for $\lambda_{i_j} \in \Lambda_i$ solve

$$\begin{bmatrix} A^T - \lambda_{i_j} I & C^T \\ \hat{F}_i^T & 0 \end{bmatrix} \begin{bmatrix} v_{i_j} \\ w_{i_j} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{b.10}$$

for pairs (v_{i_j}, w_{i_j}) where the v_{i_j} are linearly independent for all i, j. Let

$$\tilde{V}_i = \left[v_{i_1}, \ldots, v_{i_{\nu_i}}\right] \tag{b.11a}$$

$$\tilde{W}_i = \left[w_{i_1}, \ldots, w_{i_{\nu_i}}\right] \tag{b.11b}$$

4) Solve for the detection filter gain L as

$$\left[\tilde{V}_0, \tilde{V}_1, \ldots, \tilde{V}_q\right]^T L = \left[\tilde{W}_0, \tilde{W}_1, \ldots, \tilde{W}_q\right]^T \tag{b.12}$$

APPENDIX C
An $\mathcal{H}_\infty$ Bounded Fault Detection Filter

ANALYTICAL REDUNDANCY METHODS for fault detection and identification use a modeled dynamic relationship between system inputs and measured system outputs to form a residual process. Nominally, faults are detected as the residual process is nonzero only when a fault has occurred and is zero at other times. An example of a residual process for an observable system when no disturbances or sensor noise are present is the innovations process of any stable linear observer. A detection filter is a linear observer with the gain constructed so that when a fault occurs, the residual responds in a known and fixed direction. Thus, when a nonzero residual is detected, a fault can be announced and identified at the same time. Since process disturbances and sensor noise also produce a nonzero residual, the ambiguity must be resolved with an appropriate threshold.

An objective of a detection filter design in the presence of disturbances is to reduce the component of the residual due to the disturbance without at the same time degrading the component of the residual due to the fault. This suggests as a cost function, a ratio of transfer matrix norms (Frank and Wünnenberg 1989), (Lee 1994). In the numerator is the transfer matrix from the disturbance to the detection filter residual and in the denominator is the transfer matrix from the fault to the detection filter residual. This formulation works well when only one fault is to be detected. Generalized eigenvector solutions are found using a parity equation approach in (Frank and Wünnenberg 1989) and an optimization approach in (Lee 1994). Unfortunately, for the detection filter structure where several faults are isolated simultaneously, no similar problem formulation is available.

The approach taken here follows two steps. First, bound the $\mathcal{H}_\infty$ norm of the transfer matrix from the disturbance to the detection filter fault isolation residuals. Next, for each multi-dimensional fault isolation residual and working within the noise bound constraint, enhance the component due to the fault signal to be isolated. This is done by maximizing the ratio of the residual component due to a fault to the residual component due to the noise.

In the case of one-dimensional faults, the primary effect of the first step is to bound noise transmission through the complementary space, the state subspace independent of all detection spaces. The second step is not usually needed. This is because, generically, a fault detection space is given by the fault direction itself, which means the detection space is spanned by a single fixed eigenvector. The associated eigenvalue is the only degree of freedom left so there is no way to increase the residual component due to a fault without at the same time increasing the residual component due to the noise. In practical applications, plant and actuator failures usually are modeled as one-dimensional faults. Sensor faults generically require a two dimensional detection space so a design freedom exists where a residual component due to a fault could be enhanced.

This Appendix is organized as follows. Section C.1 shows that the detection filter gain is not unique and, given a set of invariant subspaces that solve the detection filter problem, parameterizes the set of detection filter gains. Section C.2 defines a disturbance robust detection filter problem and Section C.3 provides a stabilizing and $\mathcal{H}_\infty$ bounding detection filter gain by solving a modified algebraic Riccati equation. Section C.4 enhances the residual component due to the associated isolated fault signal by solving a generalized eigenvalue problem. Section C.5 provides an application to a simplified aircraft elevon and accelerometer fault detection filter where wind and sensor noise is present. The example illustrates how a numerical integration approach can be applied to solve the modified Riccati equation. Section C.6 contains a few concluding remarks.

C.1 Detection Filter Gain Parameterization

Given a set of subspaces $\mathcal{W}_1, \ldots, \mathcal{W}_q$ that solve the detection filter problem, the next problem is to characterize the set of maps $L : \mathcal{Y} \mapsto \mathcal{X}$ such that $L \in \cap_{i=1}^{q} \underline{L}(\mathcal{W}_i)$ where

$$\underline{L}(\mathcal{W}_i) \triangleq \{L \mid (A + LC)\mathcal{W}_i \subseteq \mathcal{W}_i\}$$

A first step is to find a set $\underline{L}(\mathcal{W})$ for any one (C, A)-invariant subspace $\mathcal{W}$. Proposition C.3 parameterizes $L \in \underline{L}(\mathcal{W})$ in two parameters $\alpha : C\mathcal{W} \mapsto \mathcal{W}$ and $\beta : \mathcal{Y} \mapsto \mathcal{X}$. Then, given a set of (C, A)-invariant subspaces $\mathcal{W}_1, \ldots, \mathcal{W}_q$ that solve the detection filter problem, Proposition C.4 parameterizes $L \in \cap_{i=1}^{q} \underline{L}(\mathcal{W}_i)$ in $q + 1$ parameters $\alpha_1, \ldots, \alpha_q$ and β. First, a Lemma from (White and Speyer 1987, Lemma 1), except for the geometric language, is restated to provide a solution to a generalized inverse problem. Lemma C.2 provides a few well-known properties of projections.

Lemma C.1. Let $B : \mathcal{U} \mapsto \mathcal{X}$, $C : \mathcal{X} \mapsto \mathcal{Y}$ and $D : \mathcal{U} \mapsto \mathcal{Y}$ where B is monic. Then a general solution of $CB = D$ for C is given by

$$C = D\tilde{P}_B + K(I - P_B) \tag{c.1}$$

where $P_B : \mathcal{X} \mapsto \mathcal{X}$ is any projection such that $\Im P_B = \Im B$, $\tilde{P}_B : \mathcal{X} \mapsto \mathcal{U}$ is the natural projection where $B\tilde{P}_B = P_B$ and $K : \mathcal{X} \mapsto \mathcal{Y}$ is arbitrary.

Lemma C.2. Let $C : \mathcal{X} \mapsto \mathcal{Y}$ and let $P : \mathcal{X} \mapsto \mathcal{X}$ be any projection. Then $\operatorname{Ker} P \subseteq \operatorname{Ker} C$ if and only if $C = CP$. Now let $\operatorname{Ker} P = \operatorname{Ker} C$ and let V decompose P as $VV^T = P$ and $V^T V = I$. Then CV is monic with $\Im CV = \Im C$.

An easy way to find a projector P that satisfies Lemma C.2 is to find the singular value decomposition of C. For $C = U\Sigma V^T$ where Σ is a diagonal matrix of nonzero singular values, the V of the lemma are the right singular vectors of C. Thus $P = VV^T$ and $CV = U\Sigma V^T V = U\Sigma$ is monic with $\Im C = \Im U\Sigma$.

Proposition C.3. Let $\mathcal{W} \subset \mathcal{X}$ be a (C, A)-invariant subspace with insertion map $W : \mathcal{W} \mapsto \mathcal{X}$. Let $P : \mathcal{W} \mapsto \mathcal{W}$ be any projection where $\operatorname{Ker} P = \operatorname{Ker} CW$ and let $\hat{F}$ decompose P as $\hat{F}\hat{F}^T = P$ and $\hat{F}^T\hat{F} = I$. Let $H : \mathcal{Y} \mapsto \mathcal{Y}$ be another projection where $\Im H = C\mathcal{W}$ and let $\tilde{H}$ be the associated natural projection that satisfies $CW\hat{F}\tilde{H} = H$ and $\tilde{H}CW\hat{F} = I$. Then $L : \mathcal{Y} \mapsto \mathcal{X}$ satisfies $(A + LC)W = WA_W$ for some $A_W : \mathcal{W} \mapsto \mathcal{W}$ if and only if

$$L = (-AW\hat{F} + W\alpha)\tilde{H} + \beta(I - H) \tag{c.2}$$

for some $\alpha : C\mathcal{W} \mapsto \mathcal{W}$ and $\beta : \mathcal{Y} \mapsto \mathcal{X}$.

Proof. $(\Rightarrow)$ Assume L satisfies $(A + LC)W = WA_W$ for some map A_W. Then

$$LCW = -AW + WA_W$$

and

$$LCW\hat{F} = -AW\hat{F} + WA_W\hat{F} \tag{c.3}$$

Now $\hat{F}$ is defined so that $\hat{F}\hat{F}^T$ is a projection with $\operatorname{Ker} CW = \operatorname{Ker} \hat{F}\hat{F}^T$ and $\hat{F}^T\hat{F} = I$. Therefore, by Lemma C.2, $CW\hat{F}$ is monic and by (c.3) and Lemma C.1

$$L = (-AW\hat{F} + WA_W\hat{F})\tilde{H} + \beta(I - H)$$

So $(A + LC)W = WA_W$

$$\Rightarrow \quad L = (-AW\hat{F} + W\alpha)\tilde{H} + \beta(I - H)$$

where $\alpha = A_W\hat{F}$ and β is anything.

$(\Leftarrow)$ Suppose $L = (-AW\hat{F} + W\alpha)\tilde{H} + \beta(I - H)$. Now $HCW\hat{F} = CW\hat{F}$ and $\tilde{H}CW\hat{F} = I$ so $LCW\hat{F} = (-AW\hat{F} + W\alpha)$ and

$$(A + LC)W\hat{F} = W\alpha \tag{c.4}$$

$\hat{F}$ is defined so that $\hat{F}\hat{F}^T$ is a projector with $\operatorname{Ker} CW = \operatorname{Ker} \hat{F}\hat{F}^T$ and $\hat{F}^T\hat{F} = I$. Therefore, by Lemma C.2, $CW = CW\hat{F}\hat{F}^T$ and it follows that $CW(I - \hat{F}\hat{F}^T) = 0$ and

$$\Im\left[W(I - \hat{F}\hat{F}^T)\right] \subseteq \mathcal{W} \cap \operatorname{Ker} C \tag{c.5}$$

Since for any (C, A)-invariant subspace $\mathcal{W}$ it is true that $A(\mathcal{W} \cap \operatorname{Ker} C) \subseteq \mathcal{W}$, it follows from (c.5) that for some $\tilde{A}_W$

$$AW(I - \hat{F}\hat{F}^T) = W\tilde{A}_W \tag{c.6}$$

and

$$(A + LC)W(I - \hat{F}\hat{F}^T) = W\tilde{A}_W$$

By (c.4), $(A + LC)W\hat{F}\hat{F}^T = W\alpha\hat{F}^T$. So

$$(A + LC)W = W\left(\alpha\hat{F}^T + \tilde{A}_W\right)$$

and $L = (-AW\hat{F} + W\alpha)\tilde{H} + \beta(I - H)$

$$\Rightarrow \quad (A + LC)W = WA_W$$

where $A_W = \alpha\hat{F}^T + \tilde{A}_W$ and where $\tilde{A}_W$ satisfies (c.6). Note that $\tilde{A}_W = \tilde{A}_W(I - \hat{F}\hat{F}^T)$ so

$$A_W = \alpha\hat{F}^T + \tilde{A}_W(I - \hat{F}\hat{F}^T)$$

By Lemma C.1 A_W is a particular solution to $\alpha = A_W\hat{F}$.

The remark following Lemma C.2 shows that $\hat{F}$ is the set of right singular vectors of CW.

Proposition C.4. Let $\mathcal{W}_1, \ldots, \mathcal{W}_q \subset \mathcal{X}$ be a set of (C, A)-invariant subspaces that solve the detection filter problem and let the $W_i : \mathcal{W}_i \mapsto \mathcal{X}$ be the insertion maps. Let P_i, $\hat{F}_i$, H_i and $\tilde{H}_i$ associated with $\mathcal{W}_i$ be as in Proposition C.3 but partially specify the kernal of H_i and $\tilde{H}_i$ as $\sum_{j \neq i} C\mathcal{W}_j \subseteq \operatorname{Ker} H_i = \operatorname{Ker} \tilde{H}_i$. Also, define the projection $H_0 = (I - \sum_{i=1}^q H_i)$ and the associated natural projection $\tilde{H}_0$. Finally, define a set of maps

$$\underline{L}(\mathcal{W}_i) = \{L : \mathcal{Y} \mapsto \mathcal{X} \mid (A + LC)\mathcal{W}_i \subseteq \mathcal{W}_i\}$$

Then $L \in \cap_{i=1}^{q} \underline{L}(\mathcal{W}_i)$ if and only if

$$L = \sum_{i=1}^{q}(-AW_i\hat{F}_i + W_i\alpha_i)\tilde{H}_i + \beta\tilde{H}_0 \tag{c.7}$$

for some $\alpha_0 : \Im H_0 \mapsto \mathcal{X}$ and $\alpha_i : C\mathcal{W}_i \mapsto \mathcal{W}_i$ where $i = 1, \ldots, q$.

Proof. ($\Rightarrow$) Assume $L \in \underline{L}(\mathcal{W}_i)$. Then L satisfies $(A + LC)W_i = W_iA_{W_i}$ for some $A_{W_i} : \mathcal{W} \mapsto \mathcal{W}$ for $i = 1, \ldots, q$. So

$$LCW_i = -AW_i + W_iA_{W_i}$$

and

$$LCW_i\hat{F}_i = -AW_i\hat{F}_i + W_iA_{W_i}\hat{F}_i$$

and

$$\begin{aligned} & L\left[CW_1\hat{F}_1, \ldots, CW_q\hat{F}_q\right] = \\ & \quad \left[\left(-AW_1\hat{F}_1 + W_1A_{W_1}\hat{F}_1\right), \ldots, \left(-AW_q\hat{F}_q + W_qA_{W_q}\hat{F}_q\right)\right] \end{aligned} \tag{c.8}$$

The $\hat{F}_i$ are defined so that $\hat{F}_i\hat{F}_i^T$ is a projector with $\operatorname{Ker} CW_i = \operatorname{Ker} \hat{F}_i\hat{F}_i^T$ and $\hat{F}_i^T\hat{F}_i = I$. Therefore, Lemma C.2 shows that $\Im CW_i = \Im CW_i\hat{F}_i$ and $CW_i\hat{F}_i$ is monic. Since the $\mathcal{W}_1, \ldots, \mathcal{W}_q$ solve the detection filter problem, they are output separable, which means the output subspaces $C\mathcal{W}_1, \ldots, C\mathcal{W}_q$ are independent. Therefore, $[CW_1\hat{F}_1, \ldots, CW_q\hat{F}_q]$ is monic.

In Proposition C.3 $\operatorname{Ker} H$ is not specified and is not important. Here however, $H_iCW_j = 0$ so if H is the projection $H = \sum_{i=1}^{q} H_i$ then

$$H\left[CW_1\hat{F}_1, \ldots, CW_q\hat{F}_q\right] = \left[CW_1\hat{F}_1, \ldots, CW_q\hat{F}_q\right]$$

A natural projection $\tilde{H}$ associated with H is

$$\tilde{H} = \begin{bmatrix} \tilde{H}_1 \\ \vdots \\ \tilde{H}_q \end{bmatrix}$$

because

$$\begin{aligned}\left[CW_1\hat{F}_1,\ldots,CW_q\hat{F}_q\right]\tilde{H} &= \sum_{i=1}^{q} CW_i\hat{F}_i\tilde{H}_i \\ &= \sum_{i=1}^{q} H_i \\ &= H\end{aligned}$$

and

$$\tilde{H}\left[CW_1\hat{F}_1,\ldots,CW_q\hat{F}_q\right] = \operatorname{diag}\left(\tilde{H}_iCW_i\hat{F}_i\right) = I$$

Since, $[CW_1\hat{F}_1,\ldots,CW_q\hat{F}_q]$ is monic and H and $\tilde{H}$ meet the requirements of Lemma C.1, the general solution of (c.8) for L is

$$\begin{aligned}L &= \left[\left(-AW_1\hat{F}_1 + W_1A_{W_1}\hat{F}_1\right),\ldots,\left(-AW_q\hat{F}_q + W_qA_{W_q}\hat{F}_q\right)\right]\tilde{H} + \hat{\beta}(I-H) \\ &= \sum_{i=1}^{q}(-AW_i\hat{F}_i + W_iA_{W_i}\hat{F}_i)\tilde{H}_i + \hat{\beta}(I-H) \\ &= \sum_{i=1}^{q}(-AW_i\hat{F}_i + W_i\alpha_i)\tilde{H}_i + \hat{\beta}(I-H)\end{aligned}$$

where $\alpha_i = A_{W_i}\hat{F}_i$ and $\hat{\beta}$ is anything. Finally, it follows directly from the definitions of H and $\tilde{H}_0$ that for any $\hat{\beta}$, there exists β such that $\hat{\beta}(I-H) = \beta\tilde{H}_0$. So,

$$L = \sum_{i=1}^{q}(-AW_i\hat{F}_i + W_i\alpha_i)\tilde{H}_i + \beta\tilde{H}_0$$

($\Leftarrow$) Assume

$$\begin{aligned}L &= \sum_{i=1}^{q}(-AW_i\hat{F}_i + W_i\alpha_i)\tilde{H}_i + \beta\tilde{H}_0 \\ &= \sum_{i=1}^{q}(-AW_i\hat{F}_i + W_i\alpha_i)\tilde{H}_i + \hat{\beta}(I-H)\end{aligned}$$

where the equality follows from the definitions of H and $\tilde{H}_0$. Since $H_iH_j = 0$,

$$(I-H) = (I - \sum_{i=1}^{q} H_i) = (I - \sum_{j\neq i} H_j)(I - H_i)$$

Then

$$
\begin{aligned}
L &= \sum_{i=1}^{q}(-AW_i\hat{F}_i + W_i\alpha_i)\tilde{H}_i + \beta(I - \sum_{i=1}^{q} H_i) \\
&= \sum_{i=1}^{q}(-AW_i\hat{F}_i + W_i\alpha_i)\tilde{H}_i + \beta(I - \sum_{j\neq i} H_j)(I - H_i) \\
&= (-AW_i\hat{F}_i + W_i\alpha_i)\tilde{H}_i + \left[\sum_{j\neq i}(-AW_j\hat{F}_j + W_j\alpha_j)\tilde{H}_j + \beta(I - \sum_{j\neq i} H_j)\right](I - H_i)
\end{aligned}
$$

Therefore, L has the form

$$
L = (-AW_i\hat{F}_i + W_i\alpha_i)\tilde{H}_i + \beta_i(I - H_i)
$$

where

$$
\beta_i = \sum_{j\neq i}(-AW_j\hat{F}_j + W_j\alpha_j)\tilde{H}_j + \beta(I - \sum_{j\neq i} H_j)
$$

By Proposition C.3, $L \in \underline{L}(\mathcal{W}_i)$ for each $\mathcal{W}_i$ which means $L \in \cap_{i=1}^{q}\underline{L}(\mathcal{W}_i)$.

C.2 A Disturbance Robust Detection Filter Problem

Section C.1 showed that a detection filter gain associated with a set of detection filter solution spaces $\mathcal{W}_1, \ldots, \mathcal{W}_q$ is easy to find, but generally is not unique. In this section, the $\mathcal{W}_1, \ldots, \mathcal{W}_q$ are found as for the deterministic case, but the nonuniqueness of the detection filter gain is treated as a degree-of-freedom in the detection filter design. This leads to the definition of a noise robust detection filter problem where the objective is to find a detection filter gain that minimizes or bounds a norm of the transfer matrix from the disturbance to the residual.

The linear time-invariant system of (a.1) with q failure modes is extended to include disturbances as

$$
\dot{x} = Ax + B\omega + B_u u + \sum_{i=1}^{q} F_i m_i \tag{c.9a}
$$

$$
y = Cx + D\omega. \tag{c.9b}
$$

The input ω includes dynamic disturbances and sensor noise and is square integrable over $[0, \infty)$.

The error dynamics and residual of a full-order filter have the same form as the observer (a.3, a.4)

$$\dot{e} = (A + LC)e - (B + LD)\omega - \sum_{i=1}^{q} F_i m_i \tag{c.10a}$$

$$r = C\hat{x} - y = Ce - D\omega. \tag{c.10b}$$

Since only forcing terms differentiate the residual process of the observer (a.3, a.4) from (c.10), the detection filter structure does not change with the introduction of disturbances and sensor noise. However, with the residual driven by an unknown signal, a nonzero residual does not necessarily mean a fault has occurred.

An objective of a detection filter design in the presence of disturbances is to reduce the component of the residual due to the disturbance without at the same time degrading the component of the residual due to the fault. This suggests as a cost function, a ratio of transfer matrix norms (Frank and Wünnenberg 1989). The transfer matrix from the disturbance to the residual is in the numerator and the transfer matrix from the fault to the residual is in the denominator. Unfortunately, this formulation requires some assumption about the functional form of the fault because a transfer matrix norm does not convey much information about the size of a transfer matrix output when nothing can be said about the input. Since it is a standard and reasonable assumption that process and sensor noise is white or nearly so, only the transfer matrix from the disturbance to the detection filter residual is retained in the definition of a noise robust detection filter problem.

Before continuing, it is necessary to carefully define what is meant by the component of the residual due to the fault. Define z_i as a projection of the observer residual (c.10) onto the output subspace $C\mathcal{W}_i$. Let $H_i : \mathcal{Y} \mapsto \mathcal{Y}$ be any projection onto $C\mathcal{W}_i$ and along the $C\mathcal{W}_{j\neq i}$ so that $C\mathcal{W}_i = \Im H_i$ and $\sum_{j\neq i} C\mathcal{W}_j \subseteq \operatorname{Ker} H_i$. Let $\tilde{H}_i$ be the associated natural projection and define $\tilde{z}_i$, a fault residual, as

$$\tilde{z}_i = \tilde{H}_i r \tag{c.11}$$

Using $\tilde{H}_i$ rather than H_i in (c.11) doesn't change any information given by the fault residual but is convenient later when certain matrix inverses are needed.

Now consider that for a system with q faults as in (c.9), there are q transfer matrices from the system disturbance to each of the fault residuals $\tilde{z}_i$ (c.11). There are several ways to procede. One approach is to define a multi-objective problem where a detection filter gain L is found that in some way simultaneously bounds or makes small all the transfer matrix norms $\|T_{\tilde{z}_i\omega}\|$, for example, a Pareto optimal solution. Another is to abandon the structure of the full-order detection filter for a system of q *residual generators* (Massoumnia et al. 1989). The q reduced-order filter gains are found independently of one another with the penalty that the order of the combined system usually is somewhat larger than the full-order detection filter. The approach taken here is to combine the fault residuals into a single detection filter output as follows.

Define a combined fault residual $z \in (C\mathcal{W}_1 \times \cdots \times C\mathcal{W}_q)$ by forming a map H from the $\tilde{H}_i$ in the expected way:

$$z = Hr, \qquad H^T = \left[\tilde{H}_1^T, \ldots, \tilde{H}_q^T\right] \tag{c.12}$$

The combined fault residual z provides the same information as the fault residuals, but it combines the $\tilde{z}_1, \ldots, \tilde{z}_q$ so that a single cost function can be defined for the detection filter. A noise robust detection filter problem is to find a set of subspaces $\mathcal{W}_i$ that solve the detection filter problem of Definition A.1. Then, given the $\mathcal{W}_i$ and the associated filter gain sets

$$\underline{L}(\mathcal{W}_i) = \{L_i \mid (A + L_iC)\mathcal{W}_i \subseteq \mathcal{W}_i\}$$

find a filter gain $L \in \cap\underline{L}(\mathcal{W}_i)$ that bounds or minimizes some norm $\|T_{z\omega}\|$ where $T_{z\omega}$ is the transfer matrix from the disturbance ω to the combined fault residual z of (c.12).

Note that L is found in a two-step process. First, a set of subspaces $\mathcal{W}_i$ is found that satisfies Definition A.1. Then a map L is found from the set $\cap\underline{L}(\mathcal{W}_i)$. The alternative is to find L from the union of sets $\cap\underline{L}(\mathcal{W}_i)$, where the union is taken over all sets of subspaces $\mathcal{W}_i$ that satisfy Definition A.1. While the latter statement certainly is more general, it is impractical because there is no known parameterization of all (C, A)-invariant subspaces $\mathcal{W}_i$.

C.3 An $\mathcal{H}_\infty$ Bounded Detection Filter

The main result of this section is a proposition that provides an $\mathcal{H}_\infty$ norm bounding detection filter gain. Before this result is stated, a more general $\mathcal{H}_\infty$ norm bounding theorem is needed. Consider an observer with error dynamics and output

$$\dot{e} = (A+LC)e + (B+LD)\omega \tag{c.13a}$$

$$z = C_z e + D_z \omega \tag{c.13b}$$

The following theorem and corollary provide a filter gain L that stabilizes the filter and bounds the $\mathcal{H}_\infty$ norm of the transfer matrix from ω to z. This standard result is mainly from Lemma 1 of (Willems 1971) so no proof is provided here.

Theorem C.5. Consider a system G with the form (c.13), where $(A - BD^T(DD^T)^{-1}C)$ has no purely imaginary eigenvalues and where $(DD^T)^{-1}$ exists. Suppose there exists a scalar real constant $\gamma > 0$ and a symmetric positive definite real matrix $Y > 0$ that satisfies the following algebraic Riccati equation

$$\begin{aligned} 0 \;=\; & (A+LC)Y + Y(A+LC)^T + (B+LD)(B+LD)^T \\ & + \gamma^{-2}(YC_z^T + BD_z^T)(YC_z^T + BD_z^T)^T \end{aligned} \tag{c.14}$$

Then $(A + LC)$ is stable and $\|G\|_\infty \leq [\gamma^2 + \sigma^2_{\max}(D_z)]^{1/2}$ where $\sigma_{\max}(D_z)$ is the largest singular value of D_z.

When the terms of (c.14) are manipulated to isolate L, a corollary which provides an L that stabilizes G and bounds $\|G\|_\infty$ follows immediately.

Corollary C.6. Suppose a symmetric positive definite real matrix $Y > 0$ satisfies the following algebraic Riccati equation

$$\begin{aligned} 0 \;=\; & \left[A - BD^T(DD^T)^{-1}C + \gamma^{-2}BD_z^TC_z\right]Y \\ & + Y\left[A - BD^T(DD^T)^{-1}C + \gamma^{-2}BD_z^TC_z\right]^T \\ & + B\left[I - D^T(DD^T)^{-1}D + \gamma^{-2}D_z^TD_z\right]B^T \\ & - Y\left[C^T(DD^T)^{-1}C - \gamma^{-2}C_z^TC_z\right]Y \end{aligned} \tag{c.15}$$

Then for

$$L = -(YC^T + BD^T)(DD^T)^{-1} \tag{c.16}$$

$(A + LC)$ is stable and $\|G\|_\infty \leq [\gamma^2 + \sigma_{\max}^2(D_z)]^{1/2}$ where $\sigma_{\max}(D_z)$ is the largest singular value of D_z.

Standard results strengthen Corollary C.6 by replacing (c.15) with conditions on an associated Hamiltonian matrix and adding a system detectability requirement (Kučera 1972, Doyle 1984). That is not done here because in the next proposition, the Riccati equation (c.15) is modified to provide a detection filter gain and has no associated Hamiltonian matrix.

In the detection filter problem, L is constrained to generate a set of q invariant subspaces $\mathcal{W}_1, \ldots, \mathcal{W}_q$. There is no reason to expect that L, at the same time, should satisfy (c.16). In the next proposition, (c.16) is modified so that L satisfies both constraints. When the modified relation is substituted for L in (c.14) and L is eliminated, the result is an algebraic Riccati equation with an extra term. The modified Riccati equation has no associated Hamiltonian and conditions for the uniqueness or even the existence of a solution are unknown. However, (Veillette et al. 1992) reports success in finding iterative numerical solutions to a similar relation arising from a decentralized control problem. An example in the next section illustrates the application a numerical integration approach.

Before stating the main proposition, it is convenient to rearrange the detection filter error dynamics by combining the error dynamics (c.10) with the detection filter gain (c.7).

Then the problem of choosing the parameters α_0 and $\alpha_1, \ldots, \alpha_q$ has the same form as the problem of choosing a set of $q+1$ constant feedback gains for the system

$$\dot{e} = \hat{A}e - \hat{B}\omega - \sum_{i=1}^{q} F_i m_i + W_1 u_1 + \cdots + W_q u_q + u_0 \tag{c.17a}$$

$$y_1 = \tilde{H}_1 Ce - \tilde{H}_1 D\omega, \qquad u_1 = \alpha_1 y_1 \tag{c.17b}$$

$$\vdots \tag{c.17c}$$

$$y_q = \tilde{H}_q Ce - \tilde{H}_q D\omega, \qquad u_q = \alpha_q y_q \tag{c.17d}$$

$$y_0 = \tilde{H}_0 Ce - \tilde{H}_0 D\omega, \qquad u_0 = \alpha_0 y_0 \tag{c.17e}$$

where

$$\hat{A} = A + \hat{L}C \tag{c.17f}$$

$$\hat{B} = B + \hat{L}D \tag{c.17g}$$

$$\hat{L} = -\sum_{i=1}^{q} AW_i \hat{F}_i \tilde{H}_i \tag{c.17h}$$

Proposition C.7. Consider the system G with output given by (c.12)

$$G = \left[\begin{array}{c|ccccc} \hat{A} & -\hat{B} & W_1 & \cdots & W_q & I \\ \hline HC & -HD & 0 & \cdots & 0 & 0 \\ \tilde{H}_1 C & -\tilde{H}_1 D & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \tilde{H}_q C & -\tilde{H}_q D & 0 & \cdots & 0 & 0 \\ \tilde{H}_0 C & -\tilde{H}_0 D & 0 & \cdots & 0 & 0 \end{array}\right]$$

Define

$$C_2 = \begin{bmatrix} \tilde{H}_1 C \\ \vdots \\ \tilde{H}_q C \\ \tilde{H}_0 C \end{bmatrix} \quad D_{21} = \begin{bmatrix} \tilde{H}_1 D \\ \vdots \\ \tilde{H}_q D \\ \tilde{H}_0 D \end{bmatrix} \quad V = D_{21} D_{21}^T$$

and the partitioning matrices $\Pi_1, \ldots, \Pi_q$ and Π_0

$$\Pi_1 = \begin{bmatrix} I \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad \cdots \quad \Pi_q = \begin{bmatrix} 0 \\ \vdots \\ I \\ 0 \end{bmatrix} \quad \Pi_0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I \end{bmatrix}$$

such that

$$
\begin{aligned}
\Pi_i^T[\ C_2,\ \ D_{21}\ ] &= [\ \tilde{H}_i C,\ \ \tilde{H}_i D\ ],\\
\Pi_0^T[\ C_2,\ \ D_{21}\ ] &= [\ \tilde{H}_0 C,\ \ \tilde{H}_0 D\ ]
\end{aligned}
$$

Now define a set of projections $P_{W_1}, \dots, P_{W_q}$ where $\Im P_{W_i} = \Im W_i$ and define a set of associated natural projections $\tilde{P}_{W_i}$, which satisfy $W_i \tilde{P}_{W_i} = P_{W_i}$. Assume $(\hat{A} - \hat{B} D_{21}^T V^{-1} C_2)$ has no eigenvalues on the imaginary axis. Let $\gamma > 0$ be a constant real scalar and suppose there exists $Y > 0$ such that

$$
\begin{aligned}
0 \;=\; & \left[\hat{A} - \hat{B} D_{21}^T V^{-1} C_2 + \gamma^{-2} \hat{B} D^T H^T C_2\right] Y + Y \left[\hat{A} - \hat{B} D_{21}^T V^{-1} C_2 + \gamma^{-2} \hat{B} D^T H^T C_2\right]^T \\
& + \hat{B} \left[I - D_{21}^T V^{-1} D_{21} + \gamma^{-2} D^T H^T H D\right] \hat{B}^T \\
& - Y \left[C_2^T V^{-1} C_2 - \gamma^{-2} C^T H^T H C\right] Y \\
& + \left(\sum_{i=1}^{q} (I - P_{W_i})(Y C_2^T + \hat{B} D_{21}^T) V^{-1} \Pi_i \tilde{H}_i D\right) \\
& \quad \times \left(\sum_{i=1}^{q} (I - P_{W_i})(Y C_2^T + \hat{B} D_{21}^T) V^{-1} \Pi_i \tilde{H}_i D\right)^T
\end{aligned}
\qquad \text{(c.18)}
$$

Then

$$
\begin{aligned}
\alpha_1 &= -\tilde{P}_{W_1} (Y C_2^T + \hat{B} D_{21}^T) V^{-1} \Pi_1 \\
&\ \ \vdots \\
\alpha_q &= -\tilde{P}_{W_q} (Y C_2^T + \hat{B} D_{21}^T) V^{-1} \Pi_q \\
\alpha_0 &= -(Y C_2^T + \hat{B} D_{21}^T) V^{-1} \Pi_0
\end{aligned}
$$

stabilizes G and bounds the transfer matrix $T_{z\omega}$ as $\|T_{z\omega}\|_\infty \leq [\gamma^2 + \sigma_{\max}^2(HD)]^{1/2}$ where $\sigma_{\max}(HD)$ is the largest singular value of HD.

Proof. The transfer matrix $T_{z\omega}$ is

$$
T_{z\omega} = \left[\begin{array}{c|c} A_T & -B_T \\ \hline HC & -HD \end{array}\right]
$$

where

$$
\begin{aligned}
A_T &= \hat{A} + \sum_{i=1}^{q} W_i \alpha_i \tilde{H}_i C + \alpha_0 \tilde{H}_0 C \\
B_T &= \hat{B} + \sum_{i=1}^{q} W_i \alpha_i \tilde{H}_i D + \alpha_0 \tilde{H}_0 D
\end{aligned}
$$

By Theorem C.5 and since $(\hat{A} - \hat{B}D_{21}^T V^{-1} C_2)$ has no eigenvalues on the imaginary axis, it is sufficient to show that $S = 0$ for some $Y > 0$ where

$$
\begin{aligned}
S &= A_T Y + Y A_T^T + B_T B_T^T + \\
&\quad \gamma^{-2}(YC^T + \hat{B}D^T)H^T H(YC^T + \hat{B}D^T)^T
\end{aligned}
$$

The rest of the proof involves algebraic manipulations that put S in the form of the modified algebraic Riccati equation (c.18).

C.4 Fault Enhancement

As discussed in the introduction, it is not enough to bound the residual component due to the process disturbances and sensor noise since this might, at the same time, make the fault residual component small. The approach taken here is to enhance each fault residual component while maintaining the disturbance and sensor noise bound.

Consider a cost function given as the fault signal to noise ratio

$$
J_i = \frac{\|T_{z_i m_i}\|_\infty}{\|T_{z_i \omega}\|_\infty} \tag{c.19}
$$

This is the same cost function as given in (Frank and Wünnenberg 1989) for a set of parity equations. Combining the filter gain of Proposition C.7 with results from (Doyle et al. 1989) provide a Youla parameterization of stable and $\mathcal{H}_\infty$ norm bounded transfer matrices. This could be applied to the fault detection filter by restricting the Youla parameter to those which maintain the invariant subspace structure. Maximizing (c.19) with respect to a restricted set of Youla parameters is a very difficult problem. A more tractable problem may be defined as follows.

First, consider the fault detection filter transfer matrix for the fault isolation residual z_i. By the filter unobservability subspace structure, only the fault m_i influences residual z_i, so a reduced-order realization is written. The subscript i is dropped for notational convenience.

$$\begin{aligned}\dot{\bar{e}}(t) &= \bar{A}\bar{e}(t) + \bar{F}m(t) + \bar{B}\omega(t)\\ z(t) &= \bar{C}\bar{e}(t) + D_m m(t) + D_\omega \omega(t)\end{aligned}$$

The error $\bar{e}$ lies in the factor space $\bar{e} \in \mathcal{X}/\sum_{j\neq i}\mathcal{T}_j$, the observable factor space with respect to z. All maps are taken as induced on this factor space. Now consider signals $m(t)$ and $\omega(t)$ as elements of $\mathcal{L}_2(-\infty,0]$ spaces of appropriate dimensions and define the controllability operators

$$\begin{aligned}\psi_m : \mathcal{L}_2(-\infty,0] \mapsto \mathcal{R}^n &\triangleq \int_{-\infty}^{0} e^{-\bar{A}\tau}\bar{F}m(\tau)d\tau\\ \psi_\omega : \mathcal{L}_2(-\infty,0] \mapsto \mathcal{R}^n &\triangleq \int_{-\infty}^{0} e^{-\bar{A}\tau}\bar{B}\omega(\tau)d\tau\end{aligned}$$

Then $z = z_m + z_\omega$ where z_m and z_ω are residual components due to $m(t)$ and $\omega(t)$ given by

$$\begin{aligned}z_m(t) &= \bar{C}e^{\bar{A}t}\bar{e}_{0_m} = \bar{C}e^{\bar{A}t}\psi_m m\\ z_\omega(t) &= \bar{C}e^{\bar{A}t}\bar{e}_{0_\omega} = \bar{C}e^{\bar{A}t}\psi_\omega \omega\end{aligned}$$

A detection filter fault enhancement problem may be stated as follows. Consider the residual components z_m and z_ω as elements of $\mathcal{L}_2[0,T)$ spaces where T is an observation *window*. Find a constant mapping q^T that maximizes the cost

$$J = \left[\max_{\omega}\left(\frac{\|q^T z_m\|^2_{\mathcal{L}_2[0,T)}}{\|\omega\|^2_{\mathcal{L}_2(-\infty,0]}}\right)\right]^{-1}\left[\max_{m}\left(\frac{\|m\|^2_{\mathcal{L}_2(-\infty,0]}}{\|q^T z_m\|^2_{\mathcal{L}_2[0,T)}}\right)\right] \tag{c.20}$$

Note that $q^T z_m$ and $q^T z_\omega$ are scalars. When maximized with respect to q^T, this cost penalizes large residual components due to a disturbance ω and small residual components due to a fault m.

The choice of the observation window T and the fault detection threshold is a design decision based on the functional form of the expected faults and disturbances. A detailed discussion is found in (Emami-Naeini et al. 1988). However, it is worthwhile to point out that a window of zero length, $T = 0$, is not practical. First, since faults and disturbances enter the residual directly through D_m and D_ω, it is not possible to distinguish a fault from a disturbance at any one point in time. Second, the operators that map signals $m(t)$ and $\omega(t) \in \mathcal{L}_2(-\infty, 0]$ to the respective residual components at time $t = 0$ are given by

$$\begin{aligned} \bar{\psi}_m : \mathcal{L}_2(-\infty,0] \mapsto \mathcal{R}^m &\triangleq \bar{C}\psi_m m(t) + D_m m(0) \\ \bar{\psi}_\omega : \mathcal{L}_2(-\infty,0] \mapsto \mathcal{R}^m &\triangleq \bar{C}\psi_\omega \omega(t) + D_\omega \omega(0) \end{aligned}$$

These operators are not bounded. For example, let

$$m_h(t) = \begin{cases} 1/\sqrt{h} & -h \le t \le 0 \\ 0 & t < -h \end{cases} \tag{c.21}$$

Then $m_h(t) \in \mathcal{L}_2(-\infty, 0]$ and $\|m_h\| = 1$ for all h but $\bar{\psi}_m m_h \to \infty$ as $h \to 0$. Hence, further restrictions on m and ω need to be made before a cost function such as the following could be used.

$$\frac{\|\bar{C}\psi_m m + D_m m(0)\|_{\mathcal{R}^m}}{\|\bar{C}\psi_\omega \omega + D_\omega \omega(0)\|_{\mathcal{R}^m}}$$

A well-known result (Doyle et al. 1989) is that for a given initial state $\bar{e}_{0_\omega}$, the smallest signal $\omega \in \mathcal{L}_2(-\infty, 0]$ that produces $\bar{e}_{0_\omega}$ has a norm given by

$$\inf_{\omega \in \mathcal{L}_2(-\infty,0]} \{\|\omega\|^2 | \bar{e}(0) = \bar{e}_{0_\omega}\} = \bar{e}_{0_\omega}^T X_\omega^{-1} \bar{e}_{0_\omega} \tag{c.22}$$

where X_ω is the controllability gramian given as the solution to the steady-state Lyapunov equation

$$0 = \bar{A}X_\omega + X_\omega \bar{A}^T + \bar{B}\bar{B}^T$$

If q were known, an initial state $\bar{e}_{0_\omega}$ could be found by maximizing the ratio

$$\begin{aligned} J_\omega &= \sup_{\omega \in \mathcal{L}_2(-\infty,0]} \frac{\|q^T z_\omega\|^2_{\mathcal{L}_2[0,T)}}{\|\omega\|^2_{\mathcal{L}_2(-\infty,0]}} \\ &= \max_{\bar{e}_{0_\omega} \neq 0} \frac{\bar{e}_{0_\omega}^T \left[\int_0^T e^{\bar{A}^T\tau}\bar{C}^T q q^T \bar{C} e^{\bar{A}\tau} d\tau\right] \bar{e}_{0_\omega}}{\bar{e}_{0_\omega}^T X_\omega^{-1} \bar{e}_{0_\omega}} \end{aligned}$$

This is solved as an eigenvalue problem

$$J_\omega = \lambda_{\max}\left[\int_0^T e^{\bar{A}^T\tau}\bar{C}^T q q^T \bar{C} e^{\bar{A}\tau} d\tau X_\omega\right]$$

where $\bar{e}_{0_\omega}$ is the eigenvector associated with the largest eigenvalue $\lambda_{\max}$. Note that in the case where $T = \infty$, J' is the Hankel norm of the transfer matrix.

Since q is not known, consider a worst case $\bar{e}_{0_\omega}$ as the eigenvector associated with

$$\left[\int_0^T e^{\bar{A}^T\tau}\bar{C}^T\bar{C}e^{\bar{A}\tau}d\tau X_\omega\right]\bar{e}_{0_\omega} = \lambda_{\omega_{\max}}\bar{e}_{0_\omega} \tag{c.23}$$

Similarly, a worst-case fault maximizes the ratio

$$\begin{aligned} J_m &= \sup_{m\in\mathcal{L}_2(-\infty,0]} \frac{\|m\|^2_{\mathcal{L}_2(-\infty,0]}}{\|q^T z_m\|^2_{\mathcal{L}_2[0,T)}} \\ &= \max_{\bar{e}_{0_m}\neq 0} \frac{\bar{e}_{0_m}^T X_m \bar{e}_{0_m}}{\bar{e}_{0_m}^T\left[\int_0^T e^{\bar{A}^T\tau}\bar{C}^T q q^T \bar{C} e^{\bar{A}\tau}d\tau\right]\bar{e}_{0_m}} \end{aligned}$$

where $\bar{e}_{0_m}$ is the eigenvector associated with

$$\left[\int_0^T e^{\bar{A}^T\tau}\bar{C}^T\bar{C}e^{\bar{A}\tau}d\tau X_m\right]\bar{e}_{0_m} = \lambda_{m_{\max}}\bar{e}_{0_m} \tag{c.24}$$

Now maximize (c.20) with respect to q using $\bar{e}_{0_\omega}$ and $\bar{e}_{0_m}$ from (c.23) and (c.24). This is solved as another eigenvalue problem.

$$J = \max_{q\neq 0}\frac{\|q^T z_m\|^2_{\mathcal{L}_2[0,T)}}{\|q^T z_\omega\|^2_{\mathcal{L}_2[0,T)}} = \lambda_{\max} \tag{c.25}$$

where

$$\begin{aligned} &\left(\bar{C}\int_0^T e^{\bar{A}\tau}\bar{e}_{0_m}\bar{e}_{0_m}^T e^{\bar{A}^T\tau}d\tau\bar{C}^T\right)^T q = \\ &\lambda_{\max}\left(\bar{C}\int_0^T e^{\bar{A}\tau}\bar{e}_{0_\omega}\bar{e}_{0_\omega}^T e^{\bar{A}^T\tau}d\tau\bar{C}^T\right)^T q \end{aligned} \tag{c.26}$$

Finally, the controllability gramians in (c.26) for the case $T = \infty$ may be found as solutions to a pair of steady-state Lyapunov equations. Let

$$\begin{aligned} X_{0_m} &= \int_0^T e^{\bar{A}\tau}\bar{e}_{0_m}\bar{e}_{0_m}^T e^{\bar{A}^T\tau}d\tau \\ X_{0_\omega} &= \int_0^T \bar{e}^{\bar{A}\tau}e_{0_\omega}\bar{e}_{0_\omega}^T e^{\bar{A}^T\tau}d\tau \end{aligned}$$

Then

$$\begin{aligned} 0 &= \bar{A}X_{0_m} + X_{0_m}\bar{A}^T + \bar{e}_{0_m}\bar{e}_{0_m}^T \\ 0 &= \bar{A}X_{0_\omega} + X_{0_\omega}\bar{A}^T + \bar{e}_{0_\omega}\bar{e}_{0_\omega}^T \end{aligned}$$

C.5 Application to an Aircraft Fault Detection System

This example considers a simplified aircraft fault detection filter. The dynamics of an F16XL are linearized about a trimmed level flight condition at 10,000 feet altitude and Mach 0.9. The five-state model includes longitudinal dynamics only, no lateral dynamics and no actuator dynamics. A first-order Dryden wind gust model is included.

$$\begin{aligned} \dot{x} &= Ax + B_\omega \omega + B_\delta \delta \\ y &= Cx + D\nu \end{aligned}$$

The states are

u	longitudinal body axis velocity (ft/sec)
w	normal body axis velocity (ft/sec)
q	pitch rate (deg/sec)
θ	pitch angle (deg)
w_g	wind gust (ft/sec)

the measurements are

q	pitch rate (deg/sec)
α	angle of attack (deg)
A_z	normal acceleration (ft/sec^2)
A_x	longitudinal acceleration (ft/sec^2)

the disturbances are

ω	wind gust (ft/sec)
ν_q	pitch rate sensor noise
ν_α	angle of attack sensor noise
$\nu_A z$	normal accelerometer sensor noise

ν_{Ax} longitudinal accelerometer sensor noise

and the input is

δ elevon deflection angle (deg)

All disturbances are zero-mean uncorrelated white noise processes with unit spectral density. The port and starboard elevon is modeled as a slaved system because only longitudinal dynamics are considered for this simple example. The elevon actuator dynamics are not included. The system matrices are

$$A = \begin{bmatrix} -0.0674 & 0.0430 & -0.8886 & -0.5587 & 0.0430 \\ 0.0205 & -1.4666 & 16.5800 & -0.0299 & -1.4666 \\ 0.1377 & -1.6788 & -0.6819 & 0 & -1.6788 \\ 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1.1948 \end{bmatrix}$$

$$B_\omega = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1.57 \end{bmatrix}, \qquad B_\delta = \begin{bmatrix} -0.1672 \\ -1.5179 \\ -9.7842 \\ 0 \\ 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0.0591 & 0 & 1.0000 & 0 \\ 0.0139 & 1.0517 & 0.1485 & -0.0299 & 0 \\ -0.0677 & 0.0431 & 0.0171 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0.01 & 0 & 0 & 0 \\ 0 & 0.143 & 0 & 0 \\ 0 & 0 & 0.245 & 0 \\ 0 & 0 & 0 & 0.245 \end{bmatrix}$$

Now consider a fault detection system with two faults: a normal accelerometer sensor fault and an elevon fault. The normal accelerometer fault can be modeled as an additive term in the measurement equation

$$y = Cx + E_{Az}\mu_{Az} \qquad \text{where } E_{Az} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \tag{c.27}$$

and where μ_{Az} is an arbitrary time-varying real scalar. For the purpose of determining an associated detection space, the fault E_{Az} in (c.27) is equivalent to a two-dimensional fault F_{Az} (Douglas 1993)

$$\dot{x} = Ax + F_{Az}m_{Az} \qquad \text{with } F_{Az} = [F_{Az}^1, F_{Az}^2]$$

where the directions F_{Az}^1 and F_{Az}^2 are given by

$$\begin{aligned} E_{Az} &= CF_{Az}^1 \\ F_{Az}^2 &= AF_{Az}^1 \end{aligned}$$

so that

$$F_{Az} = \begin{bmatrix} 0 & 0.9986 \\ 0 & 0.0534 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \end{bmatrix}$$

The elevon fault is given simply as $F_\delta = B_\delta$. Since CF_{Az}^1, CF_{Az}^2 and CF_δ are all nonzero and since none of the triples (C, A, F_{Az}^1), (C, A, F_{Az}^2), (C, A, F_δ) have invariant zeros, the minimal unobservability subspaces for the faults are given by the fault directions themselves, that is, $\mathcal{T}_{Az}^{1*} = \operatorname{Span} F_{Az}^1$, $\mathcal{T}_{Az}^{2*} = \operatorname{Span} F_{Az}^2$ and $\mathcal{T}_\delta^* = \operatorname{Span} F_\delta$. The faults are mutually detectable so there are no constraints on the spectrum of the detection filter.

The first step toward finding a fault detection filter gain is to find $\hat{L}$ as in (c.17h). This gain forms an observer with the correct detection space structure but without regard to stability or any performance considerations.

$$\hat{L} = -\sum_{i=1}^{q} AW_i\hat{F}_i\tilde{H}_i$$

Considering the two-dimensional normal accelerometer sensor fault as a pair of output separable faults, the $\hat{F}_i$ are identity matrices and the W_i are just the fault directions themselves. To find the $\tilde{H}_i$, let $F = [F_{Az}, F_\delta]$ and form the left inverse of CF as $(CF)^{-\ell} = (F^TC^TCF)^{-1}F^TC^T$. Now take $\tilde{H}_{Az}$ as the first two rows of $(CF)^{-\ell}$ and $\tilde{H}_\delta$ as the third row. Finally $\hat{L} = -AF_{Az}\tilde{H}_{Az} - AF_\delta\tilde{H}_\delta$ and all components needed to apply Proposition C.7 are now given.

Application of Proposition C.7 involves solving a modified algebraic Riccati equation. One approach which has achieved practical success is to form a modified *differential* Riccati equation and to numerically integrate until a steady state is reached. An initial condition for the integration is chosen by solving the algebraic Riccati equation found by truncating the modifying quadratic term. Choosing an $\mathcal{H}_\infty$ bounding parameter $\gamma = 1.2$ results in a filter with eigenvalues $-29.4629, -1.6062, -0.4351, -0.0032$ and -1.1013.

Figure c.1 shows the maximum singular values in decibels of two fault detection filter transfer matrices. One is from the wind disturbance and sensor noise to the residual which isolates a normal accelerometer fault. The other is from the normal accelerometer sensor to the same residual. A third transfer matrix, one from the elevon deflection is zero, as it should be, and is not shown. Figure c.2 shows the maximum singular values of transfer matrices to the elevon fault residual. Here the transfer matrix from the normal accelerometer sensor is zero and is not shown. In both figures, the residual is scaled so that the DC gain of the disturbance component is 0 db. Both faults have been scaled by two to emphasize that fault detection in the presence of disturbances resolves to a threshold selection problem.

Note that in the case of the elevon fault, both the residual and the detection space are one-dimensional so the associated filter eigenvector is fixed. There is no way to increase the residual component due to the fault without at the same time increasing the component due to the noise.

This is not the case for the normal accelerometer residual since it is two dimensional. A fault enhancing residual direction is found from (c.26) as $q_i^T = [-0.126, -0.992]$. The singular value frequency responses for the improved residual are also shown in Figure c.1. Disturbance reduction is seen mainly at frequencies above 1 rad/sec. A modest increase in the fault signal is seen at all frequencies.

Figures c.3 and c.4 show residual histories where white noise is applied to the wind gust model and the sensors. Figure c.3 shows the normal accelerometer residual history when a 2 ft/sec^2 bias is added to the accelerometer signal after one second. Figure c.4 shows the elevon residual history when a 2 degree bias is added to the elevon deflection after one second. Clearly, in both cases, a hard fault is detectable with an appropriate threshold (Emami-Naeini et al. 1988).

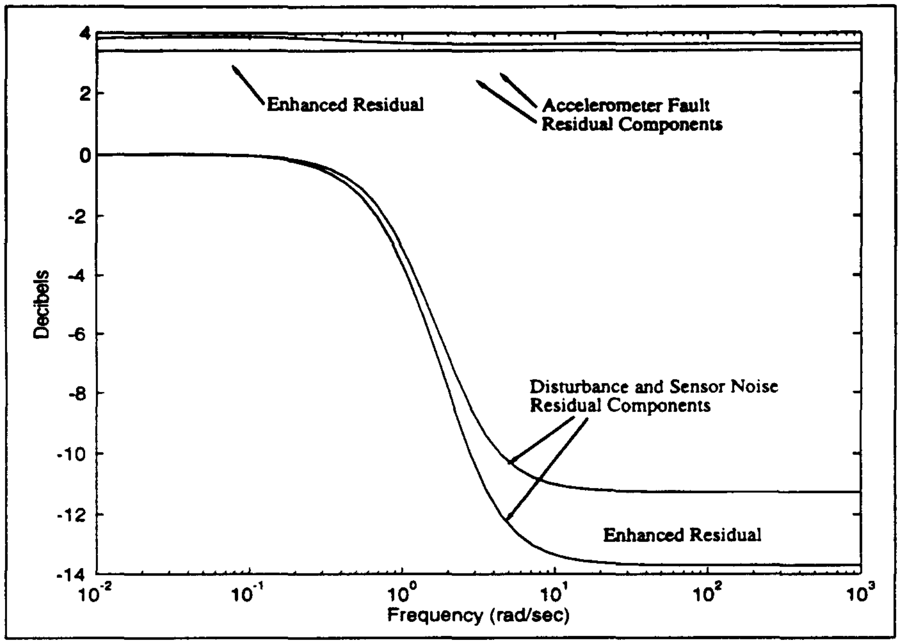

Figure c.1: Magnitude of transfer functions to the normal accelerometer fault isolation residual.

C.6 Conclusions

A stable and $\mathcal{H}_\infty$ bounded detection filter is found by solving a modified algebraic Riccati equation (c.18). This equation does not have an associated Hamiltonian and its properties are not well known; however, in (Veillette et al. 1992), a similar equation appears in the context of decentralized system control and there it is reported that a solution when it exists can usually be found by iterative, numerical means. Future work will focus on finding necessary and sufficient conditions for (c.18) to have a solution.

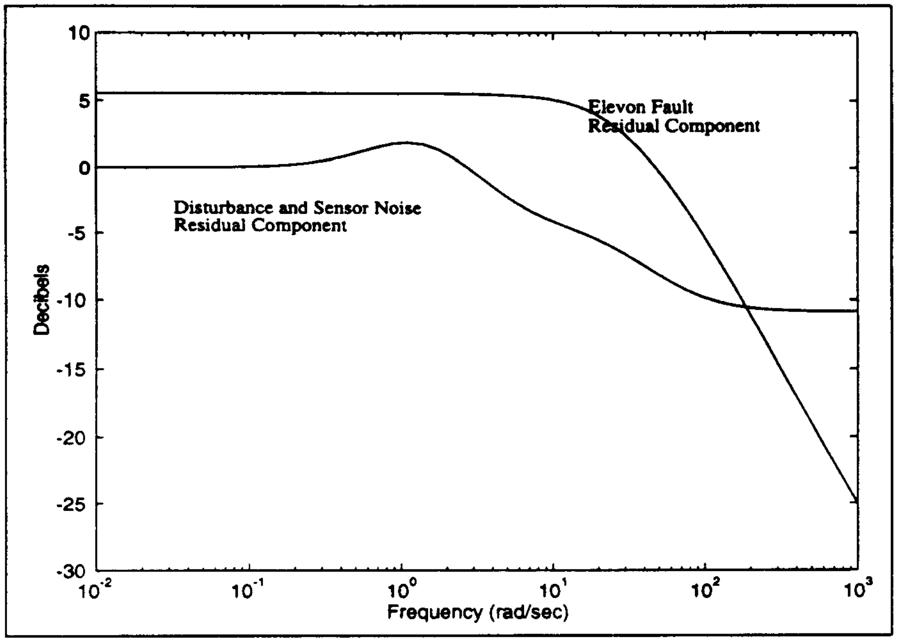

Figure c.2: Magnitude of transfer functions to the elevon fault isolation residual.

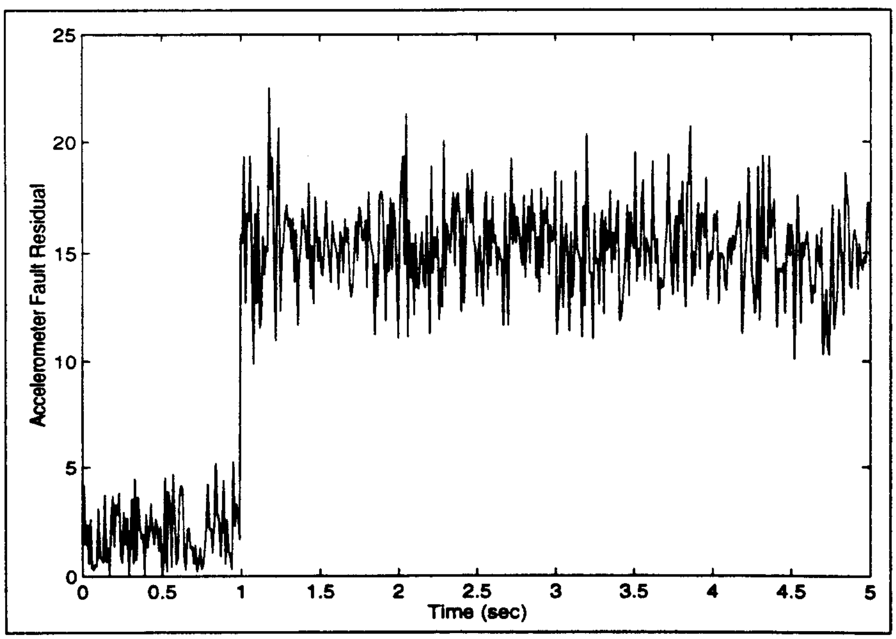

Figure c.3: Normal accelerometer fault isolation residual. $2\frac{\text{ft}}{\text{sec}^2}$ accelerometer fault occurs at t=1 sec.

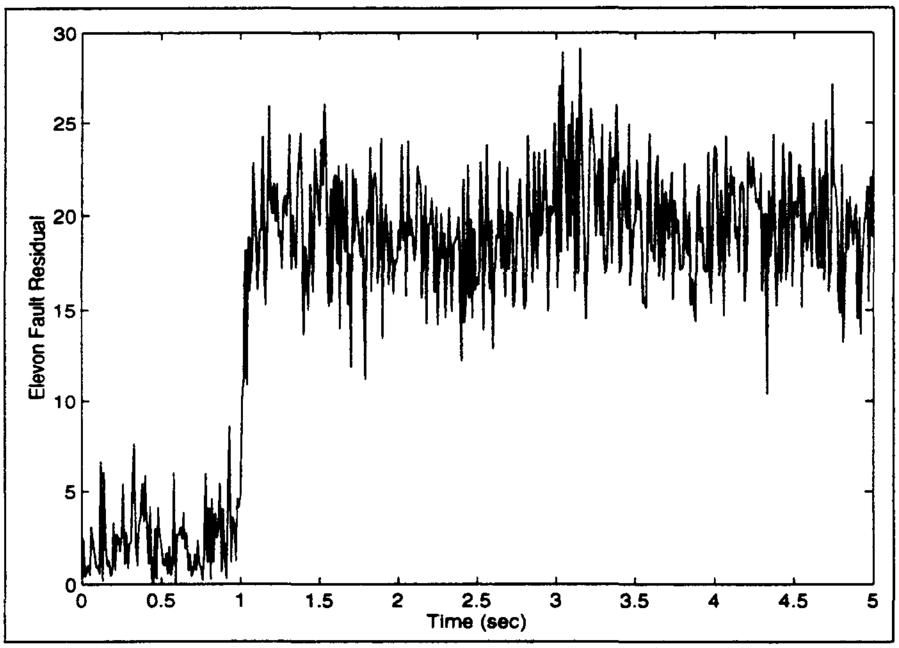

Figure c.4: Elevon fault isolation residual. 2 degree elevon fault occurs at t=1 sec.