Limitations of Binary Numbers

- **N-bit binary numbers**
  - range: $0 \sim 2^{n-1}$
  - accuracy: 1 i.e., smallest resolvable difference
  - precision: good for large values, bad for small values
    precision: total number of digits to express a value
    range: the difference between the largest and the
    smallest representable values
    dynamic range: ratio of largest and smallest
    representable (non-zero) values

- **Numerical applications often need**
  - larger range and/or larger dynamic range
  - maintain precision, independent of magnitude
  - fractions and real numbers

How do you satisfy some or all with the same N bits?
Fixed-Point Representation

- Let $b_{n-1}b_{n-2}...b_2b_1b_0$ represent an $n$-bit unsigned fixed-point number
  - must also specify the weight of $b_0$
    (suppose it’s $2^x$, $x$ could be negative)
  - its value is
    $$2^x \cdot \sum_{i=0}^{n-1} 2^i b_i$$
- a finite representation with range between 0 and $2^{n-x} - 2^x$ and with an accuracy of $2^x$
  - if $0 > x > -n+1$ then the representation includes a whole-number portion and a fractional portion, i.e., $b_{n-1}b_{n-2}...b_x..b_{-1}b_0$  
  - 2’s complement fixed-point representation can be derived similarly
  - arithmetic between identically formatted numbers is unchanged except for multiplication which requires re-scaling to position the binary point (just like in long-hand)

Choosing a Fixed Point Format

- Num. of bits determines dynamic range and precision
- Weight of $b_0$ determines accuracy
- Determine the largest magnitude (say $\sim 2^Y$) and the smallest non-zero magnitude (say $\sim 2^X$) you want to represent accurately during a computation
  - ratio $2^Y/2^X = \text{dynamic range}$
  - $(Y-X+1)$-bit fixed-point with $b_0$ weighing $2^x$
    notice: the largest magnitude by itself is unimportant
- Precision and dynamic range matters the most if
  - a small value is multiplied by a large value, e.g. $3.14 \times 10000$
  - comparable large values are subtracted, e.g. $1.0 \times 10^4 - 9999$
  - a small value is used to divide a large value, e.g. $10000/3.14$
Shortcomings of Fixed Point

- Example: let $A=2^{63}$, $B=2^{32}$ and $C=2^{32}-1$
  a program with \{A, B\} requires a 32-bit format for dynamic range
  a program with \{B, C\} requires a 33-bit format for accuracy
  a program with \{A, B, C\} requires a 64-bit format even if A and C are never operated together
  Dynamic range and accuracy are inflexibly coupled

- Cannot represent large and small values to the same precision
  - e.g. 8'b11111111 / 2 vs. 8'b00000011 / 2

- Compactness of dynamic range encoding
  - for N-bit unsigned, dynamic range $\approx 2^N$
  
  Compact?? 32-bit$\Rightarrow4\times10^9$ 64-bit$\Rightarrow16\times10^{18}$

Runtime Re-Scaling

- Range, dynamic range and accuracy requirements of a computation are not static throughout
- One could design fixed-point algorithms where the weight of $b_0$ are scaled as necessary
- A very simple example,
  - suppose we are summing $K$ same-format $N$-bit fixed-point numbers in a binary reduction tree
  - final sum could require $(\log_2 K+N)$ bits to not overflow
  - Alternatively, suppose $N$ bits of precision is sufficient, we can retain a $N$-bit format throughout and only adjust the "binary point" 1 position to the right after each level of reduction

- Applicable in many cases but impossible without deep knowledge of both the algorithm and the expected input values
Floating Point Format

- Scientific notation (e.g., 6.022\times10^{23}) in binary
  
  \((-1)^s \times (\text{significand})_{\text{N}} \times 2^{(\text{exponent})_{\text{M}}}
  
  \begin{array}{|c|c|c|}
  \hline
  s & \text{exponent} & \text{significand} \\
  \hline
  \end{array}

  \text{note**: sign-magnitude based scheme}

- A number of variants have been implemented
  
  - encoding size \approx 1 + N + M
  
  - bit allocation for exponent vs. significand
    
    range = 0 \sim 2^{M-1}
    
    dynamic range = 2^{2M}
    
    N \text{ significant (binary) digits (precise to 1 part in } 2^N)
    
    Note "accuracy" changes with magnitude but precision is fixed
  
  - special encodings

IEEE 754-1985

- Virtually universally adopted (especially on anything mass marketed)

- Standard specifies: Only 20 pages!?
  
  - number representations
  
  - precisely defined operations and behaviors
  
  - exceptional conditions and trapping behaviors
    
    seemingly every detail is deliberately chosen

- Goals
  
  - portability of numerical code
  
  - maximize numerical stability and accuracy
    
    for us mortals, use "double" and forget about it
  
  - get it right once and for all
  
  - doesn't always make it easy for HW implementation
**Formats**

- **4 formats**: single, extended-single, double, extend double

```
+---+---+---+
| s | e | f |
+---+---+---+
```

<table>
<thead>
<tr>
<th>Encoding</th>
<th>Single</th>
<th>Single extended</th>
<th>Double</th>
<th>Double extended</th>
</tr>
</thead>
<tbody>
<tr>
<td>32-bit</td>
<td>≥ 43-bit</td>
<td>64-bit</td>
<td>≥ 79-bit</td>
<td></td>
</tr>
<tr>
<td>24-bit</td>
<td>≥ 32-bit</td>
<td>53-bit</td>
<td>≥ 64</td>
<td></td>
</tr>
<tr>
<td>8-bit -126~127</td>
<td>≥ 11-bit</td>
<td>11-bit -1022~1023</td>
<td>≥ 15-bit</td>
<td></td>
</tr>
</tbody>
</table>

Note**: in normal form (i.e., 1.xxxxx), leading bit is always 1 and hence 1 bit of the significand is implicit.

- **Single precision**
- **Double precision**
  - Dynamic range: \(10^{-38} \sim 10^{38}\)
  - Dynamic range: \(10^{-307} \sim 10^{307}\)
  - Precision: \(\approx 1\) in \(10^7\)
  - Precision: \(\approx 1\) in \(10^{16}\)

**Values**

- In normal form, value = \((-1)^s \times 1.f \times 2^{e-bias}\)
  - Implicit leading 1 (how do you represent 0?)
  - Biased exponent format (single::127, double::1023)
    - Enable positive values to be compared (>,<) as integers
    - Note max and min values of \(e\) fall outside of allowable range!!

- if \(e=0\) then
  - if \(f=0\) then value = \(\pm 0\) (depend on sign)
  - else value = \((-1)^s \times 0.f \times 2^{1-bias}\) (denormal values)

- if \(e=111...111\) then
  - if \(f=0\) then value = \(\pm \infty\) (infinity)
  - else NaN (Not a Number)
**Denormal Number**

- The smallest "normal" number is $1.0 \times 2^{\text{bias}}$.
- Denormal allows a smooth approach toward 0 as precision gradually underflows:
  
  $$1.0 \times 2^{\text{bias}} / 2 = 0.1 \times 2^{\text{bias}}$$

- Alternative is to flush denormal directly to 0:
  - The difference is minute (~$10^{-307}$ for double)
  - Why does it matter?

  **suppose** $x = 1.1 \times 2^{\text{bias}}$ and $y = 1.0 \times 2^{\text{bias}}$
  - $x \neq y$ (clearly distinguishable using integer comparison)
  - with flush-to-0, $x - y = 0$ (contradicts $x \neq y$)
  - with denormal, $x - y$ behaves more predictably near the edge of the representable space.

**Special Values and Rules**

- **NaN**
  - generated by invalid operations, such as SQRT(-1)
  - computation continues even after NaN is encountered
  - rule: any operation on NaN operands in turn outputs NaN
  
  $$\Rightarrow$$ enable more streamlined coding where exceptions are checked once at the end

- **± \infty**
  - generated by overflow or 1/0
  - certain operations can continue on ±\infty values
    
    i.e., $1/\infty \Rightarrow 0$, $1+\infty \Rightarrow \infty$ but $\infty-\infty \Rightarrow \text{NaN}$, $\infty/\infty \Rightarrow \text{NaN}$

- **negative 0??**
  - $1/\infty \Rightarrow -0$, a negative value underflows to -0
  - if $x=y$ & $x!=-0$ then $x - y \Rightarrow +0$
  - but $(+0) + (+0) \Rightarrow +0$, $(0) - (-0) \Rightarrow -0$
FP Exceptions

- **Types**
  - Invalid: when an operation receives unacceptable operands
  - Divide by Zero: \(1 / 0\)
  - Overflow and Underflow: exponent too large or too small after operation
  - Inexact: when results are rounded

- **Effects**
  - Exceptions should not stop computation by default! Output NAN and keep going
  - Exceptions set flags that must be explicitly cleared by user
  - Trapping is an implementation choice
  - Certain information must be preserved for the trap handler if invoked ⇒ precise interrupt

FP Add (think scientific notation)

1. Compare the exponents of the two numbers. Shift the smaller number to the right until its exponent would match the larger exponent

2. Add the significands

3. Normalize the sum, either shifting right and incrementing the exponent or shifting left and decrementing the exponent

4. Round the significand to the appropriate number of bits

Is \(e > E_{\text{max}}\) or \(e < E_{\text{min}}\)?

Start

- \(9.999 \times 10^1\)
- \(+ 1.610 \times 10^{-1}\)

\(- 0.01610 \times 10^1\)

1.001510 \times 10^2

\(1.002 \times 10^2\)

*Okay in this case, but 9.9999, for example, would require re-normalization after rounding*

[Figure 3.16 from P&H, COPYRIGHT 2004 Elsevier. ALL RIGHTS RESERVED.]
Datapath

- Compare exponents
- Sign
- Exponent
- Significand

Small ALU

- Sign
- Exponent
- Significand

Big ALU

- Sign
- Exponent
- Significand

Shift right

Shift left or right

Round by increment or decrement

Rounding hardware

step 1

Add

Normalize

step 2

Max shift distance?

Round

implies an addition?

Rounding

- "Nearest" rounding
  - the default rounding mode
  - as if arithmetic is performed with infinite precision and the infinitely precise result is rounded to the closet representable value
  - if the infinitely precise result is exactly mid-way between 2 representable values then choose the one that has an "even" significand

- Also, directed rounding modes
  - toward 0
  - toward ∞
  - toward -∞
Round and Sticky

- When two significands are aligned for addition, some bits of the smaller value appears to not matter. Can we throw them away? Not quite!

Given the number \( b_{n-1}b_{n-2}\ldots b_2b_1b_0 \ldots \)

\[ b_{n-1}b_{n-2}\ldots b_2b_1b_0 r \ s_1s_2s_3 \ldots \]

- If \( r = 0 \) then \( b_{n-1}b_{n-2}\ldots b_2b_1b_0 \)
- If \( r = 1 \) and \( \text{sticky} = 1 \) then round up \( b_{n-1}b_{n-2}\ldots b_2b_1b_0 + 1 \)
- If \( r = 1 \) and \( \text{sticky} = 0 \) then round to even

Guard

- When subtracting 2 significands, one may lose the MSB, requires a left-shift to normalize, and must keep an extra "guard" bit just in case.

- What happens if one loses more than 1 MSB in subtraction?
  - Only possible if two subtracted values are similar in magnitude
  - At most 1 initial right-shift in the first place

\[ 1.000 \times 2^0 - 1.111 \times 2^{-3} \]

\[ 1.000 \times 2^0 - 0.0011(s=1) \times 2^0 \]

\[ 0.1100(s=1) \times 2^{-1} \]

1 guard, 1 round and 1 sticky are sufficient always!
FP Mult

1. Add the biased exponents of the two numbers, subtracting the bias from the sum to get the true biased exponent.

2. Multiply the significands.

3. Normalize the product if necessary, shifting it right and incrementing the exponent.

4. Round the significand to the appropriate number of bits.

5. Set the sign of the product to positive if the signs of the original operands are the same; if they differ make the sign negative.

step 1: don’t forget exponent is biased in IEEE float

10.212000 x10^5  step 2

1.0212000 x10^6  step 3

Is (e > E_{max}) or (e < E_{min})?

1.021 x10^6  step 4

no carry, looks good

sign_A ⊕ sign_B  step 5

FP Division by Newton Iteration

- Compute 1/b by finding the root of f(x)=1/x - b using Newton-Raphson
  - guess initial x_0
  - for x_i find root of a straight line going through (x_i, f(x_i)) with slope f'(x_i)
  - iterate using solved root as new guess x_{i+1}=x_i-f(x_i)/f'(x_i)

Accuracy improves as O(i^2)

- Fast division and sqrt if fast addition and multiplication are available (either as HW or SW techniques)

computing 1/3 using “double” and x_0=0.3 converges to 16 decimal places in 4 iter
Typical FP Latencies

- **FAdd:** 1 ~ 2 cycles, fully pipelined
- **FMult:** ~4 cycles, fully pipelined
- **FDiv:** 15~20 cycles, unpipelined

*Relative to 1-cycle integer add*

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von Neumann on Floating Point

5.3. Several of the digital computers being built or planned in this country and England are to contain a so-called "floating decimal point". This is a mechanism for expressing each word as a characteristic and a mantissa---e.g., 123.456 would be carried in the machine as (0.12345603), where the 3 is the exponent of 10 associated with the number. There appear to be two major purposes in a "floating" decimal point system both of which arise from the fact that the number of digits in a word is a constant, fixed by design considerations for each particular machine. The first of these purposes is to retain in a sum or product as many significant digits as possible and the second of these is to free the human operator from the burden of estimating and inserting into a problem "scale factors"---multiplicative constants which serve to keep numbers within the limits of the machine.

There is, of course, no denying the fact that human time is consumed in arranging for the introduction of suitable scale factors. We only argue that the time so consumed is a very small percentage of the total time we will spend in preparing an interesting problem for our machine. The fact advantage of the floating point is, we feel, somewhat illusory. In order to have such a floating point one must waste memory capacity which could otherwise be used for carrying more digits per word. It would therefore seem to us not at all clear whether the modest advantages of a floating binary point offset the loss of memory capacity and the increased complexity of the arithmetic and control circuits.

There are certainly some problems within the scope of our device which really require more than 2-4 precision. To handle such problems we wish to plan in terms of words whose lengths are some fixed integral multiple of 40, and program the machine in such a manner as to give the corresponding aggregates of 40 digit words the proper treatment. We must then consider an addition or multiplication as a complex operation programmed from a number of primitive additions or multiplications (cf. 10, Part II). There would seem to be considerable extra difficulties in the way of such a procedure in an instrument with a floating binary point.

The reader may remark upon our alternate spells of radicalism and conservatism in deciding upon various possible features for our mechanism. We hope, however, that he will agree, on closer inspection, that we are guided by a consistent and sound principle in judging the merits of any idea. We wish to incorporate into the machine—in the form of circuits—only such logical concepts as are either necessary to have a complete system or highly convenient because of the frequency with which they occur and influence they exert in the relevant mathematical situations.

Burks, Goldstein, von Neumann, Preliminary discussion of the logical design of an electronic computing instrument, 1946.
Intel Pentium FDIV Bug

If you believe "bug-free" processor designs exist in this world, try reading the Intel Core 2 Duo errata sheet (aka "Specification Updates" in polite circles) ftp://download.intel.com/design/mobile/SPECUPDT/31407906.pdf

Further Reading

- If you are interested about computer arithmetic, a great place to start is Appendix H of *Computer Architecture: A quantitative approach* by Hennessy and Patterson

- go to www.mkp.com/CA3 and follow link to "companion site"