Probabilistic techniques in a.c. load-flow analysis

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Abstract

The paper describes how the previously published formulations for an a.c. probabilistic load-flow analysis can be extended to ensure precise computation of expected values, and how these may be used to obtain more precise values of standard deviations and probability-density curves than was possible thereto. Both of the present formulations linearise the problem around the precise expected value, and one of them accounts for the coupling effect between active and reactive powers. It is shown that this coupling effect may have a significant impact on some of the voltages and reactive powers, but little effect on angles and active powers. A typical system is analysed and discussed to illustrate the increased depth of information that can be gained from these improved techniques.

List of symbols

B_{ik} = imaginary part of element ik of admittance matrix $= \dfrac{X_{ik}}{R_{ik}^2 + X_{ik}^2}$

$B_{i(sh)}$ = imaginary part of shunt admittance at node i

B'_{ik} = half of susceptance of line i–k

G_{ik} = real part of element ik of admittance matrix $= \dfrac{-R_{ik}}{R_{ik}^2 + X_{ik}^2}$

n = number of nodes

n_1 = number of PQ nodes

n_2 = number of PV nodes including slack busbar

P_i = injected active power at node i

P_{ik} = active power flow in line i–k

P_s = injected active power at slack node

Q_i = injected reactive power at node i

Q_{ik} = reactive power flow in line i–k

$Q_{i(sh)}$ = injected reactive power by shunt element at node i

R_{ik} = resistance of line i–k

t_{ik} = transformer tap ratio

V_i = voltage magnitude at node i

X_{ik} = reactance of line i–k

θ_i = angle at node i referred to slack node

θ_{ik} = difference in angles between nodes i and k

μ = expected value

σ = standard deviation

Subscripts and superscripts

g generation quantity

i, j, k, n node numbers

l load quantity

o expected value

s slack node

sh shunt element

$\hat{}$ represents inverted matrix and its elements

1 Introduction

Several papers[1–5] have been published recently that have shown how the load-flow problem can be modelled and solved probabilistically instead of deterministically. Some authors[4–5] have produced an a.c. model using a linearisation process and assuming normal distributions for all the input nodal quantities, while others[1–3] have produced a d.c. model that permits the nodal quantities to be specified by any reasonable and practical probability-density function. The latter model is more realistic from a system viewpoint, but only the angles and the active-power flows can be computed.

More recently a technique was published[6] that extended the previous d.c. model[1–3] that allowed the angles, voltages, active-power flows, reactive-power flows and injected reactive powers to be computed using input nodal quantities that could still be specified by various practical distributions. It was shown[6] that small errors occurred in the computed expected values which could be compensated for by shifting the computed probability-density curve so that its expected value coincided with the value deduced from a conventional deterministic analysis.

Since then, further studies have been made that have produced a different technique for modelling the same problem. This new technique uses the expected value obtained from a deterministic solution as the basis for computing the probability-density curves. This paper describes the new technique and compares results obtained from this analysis with those obtained by the previously published techniques.[6]

2 Basic concept of new technique

The difficulties concerning the solution of the load-flow equations probabilistically were discussed in the previous paper.[6] To overcome these difficulties it is necessary to linearise the load-flow equations. In the present technique, a linearisation process is made around the solution point of a conventional deterministic load-flow study; i.e. the expected value of a probabilistic analysis.

To illustrate the basic concept of this technique consider two random variables X and Y which, at some stage of computation, are multiplied to give a third random variable; i.e.

$$Z = XY \tag{1}$$

If the expected values X_0 and Y_0 of X and Y are known, then

$$X = X_0 + \Delta X \tag{2}$$

$$Y = Y_0 + \Delta Y \tag{3}$$

where ΔX and ΔY are random changes of X and Y about X_0 and Y_0. Therefore

$$\begin{aligned} Z &= (X_0 + \Delta X)(Y_0 + \Delta Y) \\ &\simeq X_0 Y_0 + X_0 \Delta Y + Y_0 \Delta X \quad \text{neglecting } \Delta X \Delta Y \\ &= X_0 Y + Y_0 X - X_0 Y_0 \end{aligned} \tag{4}$$

Therefore, if the random variations are small, the variable Z can be linearised once the expected values of X_0 and Y_0 have been deduced. This technique can be applied to the angles and voltages of a power system since their random variations are generally small.

Applying the concept of eqns. 1–4 to the voltages gives

$$V_i V_k = V_{i0} V_k + V_{k0} V_i - V_{i0} V_{k0} \tag{5}$$

$$V_i^2 = 2 V_{i0} V_i - V_{i0}^2 \tag{6}$$

Also using Maclaurin's series:

$$\sin\theta_{ik} \simeq \theta_{ik} - \frac{\theta_{ik}^3}{6} \qquad \cos\theta_{ik} \simeq 1 - \frac{\theta_{ik}^2}{2} \tag{7}$$

Applying the concept of eqns. 1–4 to eqn. 7 gives

$$\cos\theta_{ik} = a_{ik} + b_{ik}\theta_{ik} \qquad \sin\theta_{ik} = c_{ik} + d_{ik}\theta_{ik} \tag{8}$$

where

$$a_{ik} = 1 + \frac{\theta_{ik0}^2}{2} \qquad b_{ik} = -\theta_{ik0}$$

$$c_{ik} = \frac{\theta_{ik0}^3}{3} \qquad d_{ik} = 1 - \frac{\theta_{ik0}^2}{2}$$

The voltages and angles have thus been linearised around the solution point of their expected values. Using this technique, two new

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formulations of the a.c. probabilistic load-flow problem can be devised. These will be called formulations 3 and 4 to differentiate them from formulations 1 and 2 discussed in the previous paper.[6] Briefly, formulation 3 assumes the active and reactive powers to be decoupled while formulation 4 does not make this assumption.

3 New formulations of probabilistic load-flow equations

3.1 Basic load-flow equations

The basic load-flow equations used in all types of analysis are

$$P_i = V_i \sum_{k=1}^{n} V_k (G_{ik} \cos\theta_{ik} + B_{ik} \sin\theta_{ik}) \quad (9)$$

$$Q_i = V_i \sum_{k=1}^{n} V_k (G_{ik} \sin\theta_{ik} - B_{ik} \cos\theta_{ik}) \quad (10)$$

$$P_{ik} = -t_{ik} G_{ik} V_i^2 + V_i V_k (G_{ik} \cos\theta_{ik} + B_{ik} \sin\theta_{ik}) \quad (11)$$

$$Q_{ik} = t_{ik} B_{ik} V_i^2 - B'_{ik} V_i^2 + V_i V_k (G_{ik} \sin\theta_{ik} - B_{ik} \cos\theta_{ik}) \quad (12)$$

$$Q_{i(sh)} = V_i^2 B_{i(sh)} \quad (13)$$

In the present formulations, eqns. 5, 6 and 8 are used to linearise the load-flow equations eqns 9–13 as shown in the following Sections.

3.2 Formulation 3

In this formulation, the active and reactive powers are considered decoupled. Consequently, it is assumed that the voltages have no effect on the angles and active powers, and angles have no effect on the voltages and reactive powers.

From this assumption, the values of V_i and V_k in eqns. 9 and 11 can be made equal to V_{i0} and V_{k0}.

Substituting eqns. 8 into eqn. 9 gives

$$P_i = V_{i0} \sum_{k=1}^{n} V_{k0} (f_{ik}\theta_i - f_{ik}\theta_k + e_{ik}) \quad (14)$$

where

$$e_{ik} = a_{ik} G_{ik} + c_{ik} B_{ik}$$

and

$$f_{ik} = b_{ik} G_{ik} + d_{ik} B_{ik}$$

From eqn. 14, it can be shown (see Appendix 8.1) that

$$\theta_i = \sum_{j=1}^{n-1} \hat{Y}_{ij} P_j - \sum_{j=1}^{n-1} \hat{Y}_{ij} R_j \quad (i = 1 \ldots n-1) \quad (15)$$

$$P_{ik} = g_{ik} \sum_{j=1}^{n-1} (\hat{Y}_{ij} - \hat{Y}_{kj}) P_j - g_{ik} \sum_{j=1}^{n-1} (\hat{Y}_{ij} - \hat{Y}_{kj}) R_j + h_{ik} \quad (16)$$

and

$$P_s = \sum_{j=1}^{n-1} T_j P_j - \sum_{j=1}^{n-1} T_j R_j + R_s \quad (17)$$

where g_{ik}, h_{ik}, R_j, R_s, $\hat{Y}_{ij}$ and T_j are defined in Appendix 8.1.

Similarly, the value of θ_{ik} in eqns. 10 and 12 can be made equal to $\theta_{ik0} (= \theta_{i0} - \theta_{k0})$.

Substituting eqn. 5 in eqn. 10 gives

$$Q_i = \sum_{k=1}^{n} A_{ik} (V_{k0} V_i + V_{i0} V_k - V_{i0} V_{k0}) \quad (18)$$

where

$$A_{ik} = G_{ik} \sin\theta_{ik0} - B_{ik} \cos\theta_{ik0}$$

or

$$Q_i = \sum_{k=1}^{n} A'_{ik} V_k - W_i \quad (19)$$

where

$$A'_{ik} = A_{ik} V_{i0}$$

$$A'_{ii} = A_{ii} V_{i0} + \sum_{k=1}^{n} A_{ik} V_{k0}$$

$$W_i = V_{i0} \sum_{k=1}^{n} A_{ik} V_{k0}$$

Using eqn. 19, the voltages at all load busbars and the injected reactive powers at all generation nodes can be deduced as shown in Appendix 8.1

Substituting eqns. 5 and 6 into eqn. 12 gives

$$Q_{ik} = \alpha_{ik} V_i + \beta_{ik} V_k + \gamma_{ik} \quad (20)$$

where

$$\alpha_{ik} = 2V_{i0}(t_{ik} B_{ik} - B'_{ik}) + A_{ik} V_{k0}$$

$$\beta_{ik} = A_{ik} V_{i0}$$

$$\gamma_{ik} = -V_{i0}^2 (t_{ik} B_{ik} - B'_{ik}) - A_{ik} V_{i0} V_{k0}$$

Substituting eqn. 6 into eqn. 13 gives

$$Q_{i(sh)} = 2V_{i0} B_{i(sh)} V_i - V_{i0}^2 B_{i(sh)} \quad (21)$$

As shown in Appendix 8.1, the reactive powers can be deduced from eqns. 20 and 21.

3.3 Formulation 4

The main difference between formulation 4 and that described above is that, in the present case, the active and reactive powers are not assumed to be decoupled. Clearly, therefore, the assumptions made in Section 3.2 are no longer applicable, and more generalised forms of eqns. 14–21 must be derived. Only the general description of the concepts will be described since the detailed derivation remains similar to that of formulation 3.

In the present case, linearised forms of $V_i V_k \sin\theta_{ik}$ and $V_i V_k \cos\theta_{ik}$ must be derived. Using the concept defined by eqns. 1–4, these linearised forms are

$$\begin{aligned} V_i V_k \sin\theta_{ik} &= a'_{ik} + b'_{ik}\theta_{ik} + c'_{ik} V_i + d'_{ik} V_k \\ V_i V_k \cos\theta_{ik} &= a''_{ik} + b''_{ik}\theta_{ik} + c''_{ik} V_i + d''_{ik} V_k \end{aligned} \quad (22)$$

where

$$a' = 2V_{i0}V_{k0}\theta_{ik0}(\tfrac{1}{3}\theta_{ik0}^2 - 1) \qquad a'' = V_{i0}V_{k0}(\tfrac{3}{2}\theta_{ik0}^2 - 1)$$

$$b' = V_{i0}V_{k0}(1 - \tfrac{1}{2}\theta_{ik0}^2) \qquad b'' = -V_{i0}V_{k0}\theta_{ik0}$$

$$c' = V_{k0}(\theta_{ik0} - \tfrac{1}{6}\theta_{ik0}^3) \qquad c'' = V_{k0}(1 - \tfrac{1}{2}\theta_{ik0}^2)$$

$$d' = V_{i0}(\theta_{ik0} - \tfrac{1}{6}\theta_{ik0}^3) \qquad d'' = V_{i0}(1 - \tfrac{1}{2}\theta_{ik0}^2)$$

In Appendix 8.2 it is shown how eqn. 22 can be used to reform the load-flow equations eqns. 9 and 10 so that they may be solved using probabilistic techniques.

3.4 General comments

As discussed in the previous paper[6] it is necessary to linearise the nonlinear load-flow equations to apply probabilistic and convolution techniques.[3] A considerable number of different formulations have been devised and tested on a wide range of systems and data. From these formulations, those presented in this paper and the previous paper[6] appear to be the most promising. In the following Section, results obtained for the IEEE 14-busbar 20-line system using formulations 3 and 4 are presented and discussed. These results are compared with those published previously[6] using two different types of formulations.

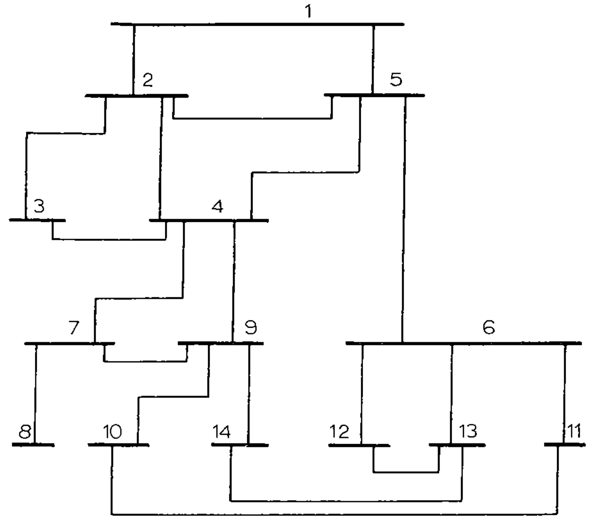


Fig. 1
Line diagram of IEEE 14-busbar system

4 Analysis of typical system

4.1 Results

The IEEE 14-busbar test system is shown in Fig. 1 and the system data used are shown in Tables 1 and 2. Using this system and data the expected values, standard deviations and probability-density curves of angles, voltages, active and reactive power flows, active and reactive line losses and injected reactive powers were computed. A selection of typical computed values are shown in Tables 3 and 4. With the present formulations the injected active power at the slack busbar (busbar 1) was found to have an expected value of 232·4 MW and a standard deviation of 17·94 MW (formulation 3) and 17·96 MW (formulation 4). This expected value could be determined from any deterministic load-flow analysis but, clearly, the standard deviation could not.

Table 1
LINE DATA FOR IEEE 14-BUSBAR SYSTEM

Busbar Sending	Busbar Receiving	Resistance	Reactance	Susceptance	Transformer tap
		p.u.	p.u.	p.u.	%
1	2	0·01938	0·05917	0·02640	–
1	5	0·05403	0·22304	0·02640	–
2	3	0·04699	0·19797	0·02190	–
2	4	0·05811	0·17632	0·01870	–
2	5	0·05695	0·17388	0·01700	–
3	4	0·06701	0·17103	0·01730	–
4	5	0·01335	0·04211	0·00640	–
4	7	–	0·20912	–	−2·2
4	9	–	0·55618	–	−3·1
5	6	–	0·25202	–	−6·8
6	11	0·09498	0·19890	–	–
6	12	0·12291	0·25581	–	–
6	13	0·06615	0·13027	–	–
7	8	–	0·17615	–	–
7	9	–	0·11001	–	–
9	10	0·03181	0·08450	–	–
9	14	0·12711	0·27038	–	–
10	11	0·08205	0·19207	–	–
12	13	0·22092	0·19988	–	–
13	14	0·17093	0·34802	–	–
9	9	–	−5·26000	–	–

Table 2
NODAL DATA USED

1 Normal distributions

Busbar Number	Busbar Type	Voltage	Active power μ	Active power σ	Reactive power μ	Reactive power σ
		p.u.	MW	%	MVAr	%
2	PV	1·045	−21·74	9·00	−12·70	9·2
3	PV	1·010	−94·20	10·00	−19·00	10·5
4	PQ	–	−47·80	11·00	3·90	9·7
5	PQ	–	−7·60	5·00	−1·60	5·0
6	PV	1·070	−11·20	6·00	−7·50	6·3
7	PQ	–	0·00	0·00	0·00	0·0
8	PV	1·090	0·00	0·00	0·00	0·0
10	PQ	–	−9·00	10·00	−5·80	10·0
11	PQ	–	−3·50	9·50	−1·80	9·5
12	PQ	–	−6·10	7·60	−1·60	8·6
13	PQ	–	−13·50	10·50	−5·80	9·5
14	PQ	–	−14·90	8·60	−5·00	8·6

2 Binomial distributions

Busbar Number	Busbar Type	Voltage	Unit rating	Forced outage rate	Number of units
		p.u.	MW		
1	SLACK	1·060	25·0	0·08	10
2	PV	1·045	22·0	0·09	2

3 Any discrete distribution

Busbar Number	Busbar Type	Voltage	Active power		Reactive power	
		p.u.	MW	probability	MVAr	probability
9	PQ	–	−13·4	0·10	−7·5	0·10
			−19·6	0·15	−11·0	0·15
			−30·2	0·30	−17·0	0·30
			−34·8	0·25	−19·6	0·25
			−37·3	0·20	−21·0	0·20

4.2 Discussion of results

4.2.1 Expected values and standard deviations

The expected values of all angles and voltages were computed using a conventional deterministic load-flow program from which all the other output quantities were computed using formulations 3 and 4. Consequently, all these expected values are very precise. Little discussion is therefore required concerning these values except to compare them with those obtained with formulations[6] 1 and 2.

As discussed in the previous paper,[6] the expected values obtained using formulations 1 and 2 are slightly in error due to the approximations made. Therefore, for greater precision, the techniques of formulations 3 and 4 are preferable. If a high degree of precision is not required, however, the reduced computational time associated with formulations 1 and 2 may justify their preferred use. This aspect will be discussed in more detail in Section 4.2.3. In the previous paper it was suggested, however, that the density curves obtained by formulations 1 and 2 could be shifted so that their expected values coincided with that given by a deterministic solution. This shifting technique is computationally very efficient and does not create any additional errors in the standard deviation or shape of the probability-density curve for a decoupled assumption. This is confirmed by the present results as discussed below.

Comparing the standard deviations of all output quantities obtained from formulations 1 and 2 with those obtained from formulation 3 shows an extremely close agreement. All three methods assumed that the active powers and reactive powers were decoupled although the linearisation techniques were considerably different. Therefore, provided the decoupling assumption is reasonably valid any of these three methods will give realistically precise values of standard deviation. As discussed above, this further justifies the previously published[6] viewpoint that the probability-density curves obtained from formulations 1 and 2 could be shifted so that their expected values are precise.

When the standard deviations of the various output quantities obtained from formulations 1, 2 and 3 are compared with those obtained from formulation 4, somewhat different observations are apparent. In the case of angles and active power flows, an extremely close agreement exists between all four sets of results (see Tables 3 and 4). In the case of voltages, reactive power flows and injected reactive powers, considerable differences can arise. In some instances the values, e.g. the voltages at busbars 7, and 9–14, and the reactive power flows in lines 6–12 and 10–11, are very similar. In others the

ANGLES, VOLTAGES AND INJECTED REACTIVE POWERS

Busbar	Angles, deg. μ	σ using formulation 3	4	1 & 2[6]	Voltages, p.u. μ	σ using formulation 3	4	2[6]
4	−10·31	0·71	0·69	0·66	1·0171	0·0009	0·0020	0·0009
5	−8·76	0·59	0·58	0·56	1·0187	0·0005	0·0016	0·0005
7	−13·36	0·99	0·98	0·96	1·0613	0·0025	0·0029	0·0027
9	−14·94	1·17	1·15	1·16	1·0557	0·0050	0·0052	0·0052
10	−15·09	1·11	1·10	1·10	1·0508	0·0042	0·0044	0·0044
11	−14·79	0·98	0·97	0·97	1·0568	0·0022	0·0023	0·0023
12	−15·07	0·87	0·88	0·88	1·0552	0·0005	0·0007	0·0005
13	−15·15	0·91	0·91	0·90	1·0503	0·0010	0·0012	0·0010
14	−16·03	1·08	1·06	1·06	1·0354	0·0033	0·0037	0·0034
					Injected reactive powers, MVAr μ	σ using formulation 3	4	2[6]
1	0·0	–	–	–	−13·31	0·24	3·10	0·23
2	−4·98	0·44	0·44	0·41	36·11	0·76	5·59	0·75
3	−12·73	0·99	1·00	0·92	7·29	0·43	4·12	0·44
6	−14·22	0·84	0·85	0·84	5·68	1·90	2·19	1·84
8	−13·36	0·99	0·98	0·96	17·78	1·55	1·77	1·52

Table 4
TYPICAL ACTIVE AND REACTIVE POWER FLOWS

line	Active power flows, MW μ	σ using formulation 3	4	1 & 2[6]	Reactive power flows, MVAr μ	σ using formulation 3	4	2[6]
1 − 5	75·47	4·85	4·79	4·35	5·62	0·24	0·51	0·23
5 − 1	−72·70	4·50	4·40		3·12	0·23	1·33	0·24
losses	2·77	0·35	0·35		8·74	0·01	1·45	0·00
3 − 4	−23·25	4·44	4·46	4·62	4·56	0·43	2·07	0·44
4 − 3	23·63	4·59	4·61		−5·38	0·43	1·75	0·44
losses	0·37	0·15	0·15		−0·82	0·01	0·38	0·00
4 − 5	−61·09	4·62	4·49	4·55	16·11	0·75	1·44	0·78
5 − 4	61·61	4·70	4·57		−15·15	0·75	1·26	0·78
losses	0·52	0·08	0·08		0·97	0·00	0·24	0·00
6 − 12	7·79	0·40	0·42	0·41	2·51	0·17	0·20	0·16
12 − 6	−7·72	0·39	0·41		−2·36	0·17	0·19	0·16
losses	0·07	0·01	0·01		0·15	0·00	0·01	0·00
10 − 11	−3·80	1·37	1·45	1·43	−1·66	0·93	1·01	0·94
11 − 10	3·81	1·38	1·45		1·69	0·94	1·01	0·94
losses	0·01	0·01	0·01		0·03	0·01	0·02	0·00
5 − 6	44·11	2·51	2·66	2·59	12·08	0·25	0·56	0·25
6 − 5	−44·11	2·51	2·66		−7·67	0·24	0·99	0·23
losses	0·00	0·0	0·0		4·41	0·01	0·49	0·02
7 − 9	28·06	3·65	3·57	3·53	5·76	2·34	2·34	2·28
9 − 7	−28·06	3·65	3·57		−4·96	2·32	2·28	2·29
losses	0·00	0·00	0·00		0·80	0·02	0·21	0·00

values, e.g. the voltages at busbars 4, 5, reactive power flows in lines 1–5, 3–4, 4–5, are very different. These results indicate that the effect of coupling causes little effect on angles and active powers but can cause considerable impact on voltages and reactive powers. It is evident therefore that, for greatest precision in the values of voltages and reactive powers, formulation 4 is the only alternative. As will be seen in Section 4.2.3, however, this increased precision is at the expense of increased computational time and it may be decided that a reasonable balance between efficiency and precision justifies the use of one of the other formulations.

4.2.2 Probability-density curves

Typical probability-density curves for voltage, active power flow, reactive power flow and injected reactive power are shown in Figs. 2–5. From these Figures, similar conclusions to those discussed above can be made.

It is evident from Fig. 2 that great similarity exists between the density curves of active power flow obtained from formulations 3 and 4. Similar curves were previously obtained using formulations 1 and 2. This clearly indicates that, even when coupled, voltages had little impact on the active power flows.

In contrast, it is seen from Figs. 3–5 that the curves for voltage and reactive powers were considerably different; those obtained from formulation 3 being very similar to those obtained[6] from formulations 1 and 2. It was found, however, that in several instances similar curves were obtained from all four formulations. This again confirms the discussion made in Section 4.2.1. From Figs. 3–5 it is evident that the curves obtained from formulation 4 are smoother and more continuous than those obtained from formulation 3. This reflects the number of independent random variables involved in the convolution.

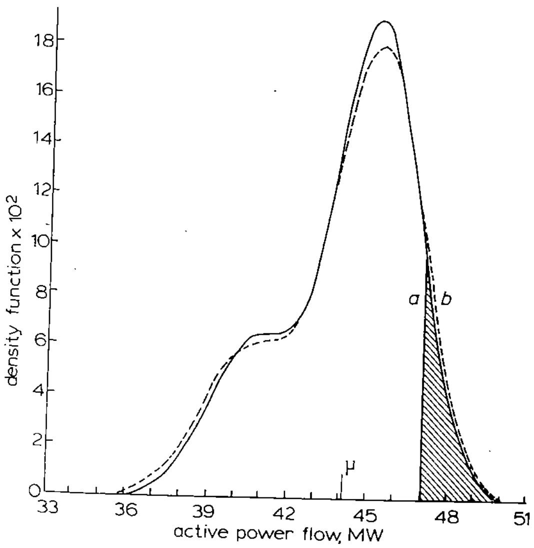


Fig. 2
Active power flow in line 5–6

a Formulation 3
b Formulation 4

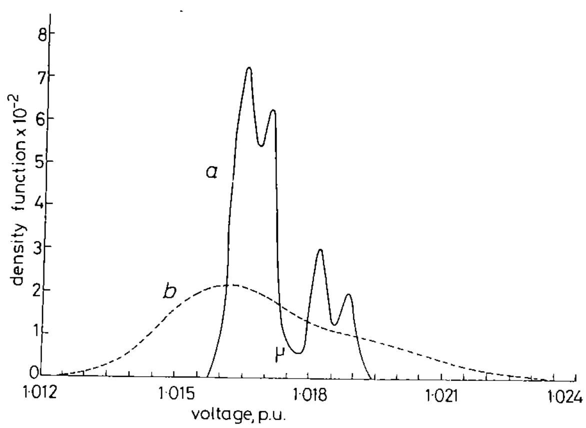


Fig. 3
Voltage at busbar 4

a Formulation 3
b Formulation 4

In formulation 3 the angles were not considered as random variables in computing voltages and reactive powers, and therefore the number of variables are less than in formulation 4. When these additional variables have a significant impact then clearly the convoluted density curve will become smoother since, in the limit, the convoluted curve should approach a normal distribution curve. It is evident from Fig. 4*b*, however, that, even with an increased number of variables, the density curve is still not normally distributed.

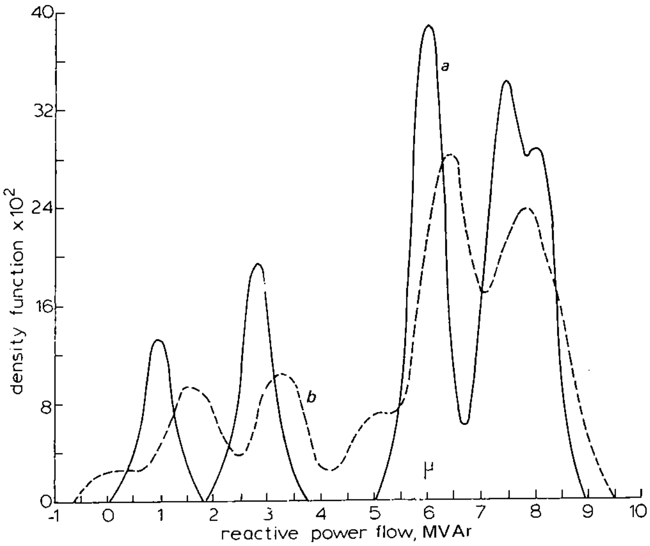
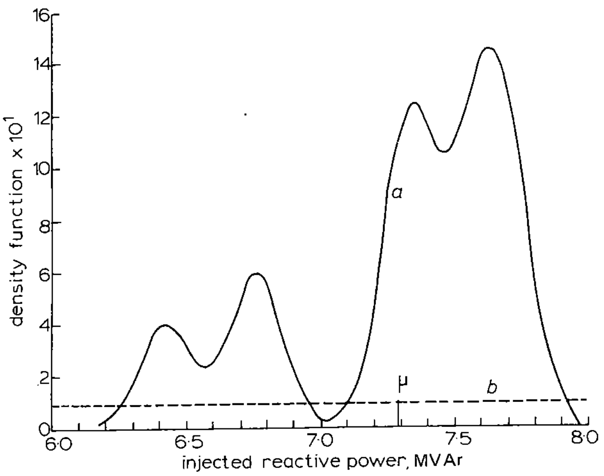


Fig. 4
Reactive power flow in line 7–9

a Formulation 3
b Formulation 4



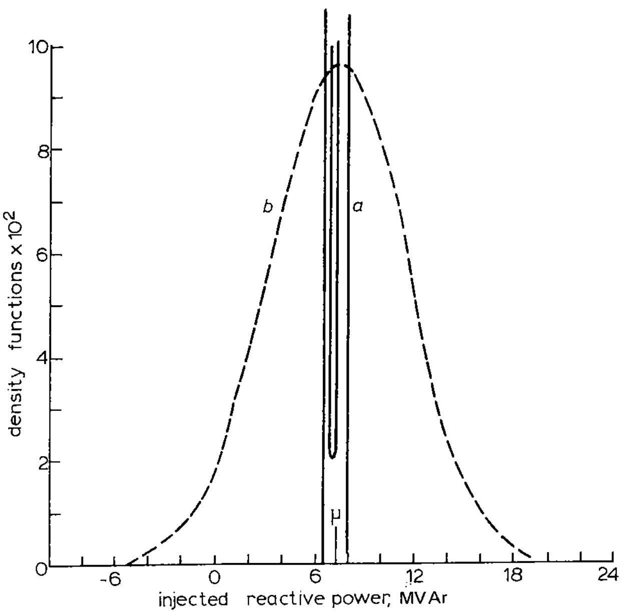


Fig. 5
Injected reactive power at busbar 3

a Formulation 3
b Formulation 4

Probability-density curves are the basis for deducing the probability or chances of the behaviour or performance of a system. These probabilities are derived by integrating the appropriate density curve to give the cumulative probability or probability-distribution function. In general, if $f(x)$ is the probability-density function then the probability of a system variable occurring between two limits X_1 and X_2 is

$$\text{prob}\ (X_1 \leqslant x \leqslant X_2) = \int_{X_1}^{X_2} f(x)\,\mathrm{d}x$$

With suitable values of X_1 and X_2, this concept can be applied to any of the system variables to give the probability of, for example, generation deficiency, line overload, busbar voltages outside of specified limits, violation of Q-limits, etc.

To illustrate the application of this concept, consider the probability of the active and reactive power flows in a line to be above a specified limit X_3, then

$$\text{prob}\ (-X_3 \geqslant x \geqslant +X_3) = \int_{-\infty}^{-X_3} f(x)\,\mathrm{d}x + \int_{+X_3}^{+\infty} f(x)\,\mathrm{d}x$$

$$= 1 - \int_{-X_3}^{+X_3} f(x)\,\mathrm{d}x$$

Applying this to the active and reactive power flows in line 5–6 gives the probabilities for exceeding various limits shown in Table 5. The probability of the active power flow exceeding 47 MW is also shown by the shaded area in Fig. 2*a*. In the case of formulation 2, Table 5 shows the probabilities obtained before and after the expected values given by this formulation have been shifted to coincide with those of formulations 3 and 4.

It can be seen from Table 5 that, in the case of the active power flow, all formulations give reasonably the same results provided the expected value given by formulation 2 is shifted; without this shifting, imprecision between the results is observed as would be expected. In the case of reactive power flows, the results of formulation 2 after shifting and formulation 3 are very similar, but both are significantly different from those of formulation 4. Again, both of these trends confirms the previous discussion concerning the precision of the various formulations.

Table 5
PROBABILITY OF POWER FLOWS EXCEEDING STATED LIMITS IN LINE 5–6

Limit	probability of exceeding limit using formulation 3	4	2*	2†
(*a*) Active power watts				
45	0·4300	0·4669	0·4583	0·2160
46	0·2568	0·2811	0·2482	0·0706
47	0·1169	0·1252	0·1087	0·0178
48	0·0286	0·0375	0·0325	0·0024
49	0·0054	0·0100	0·0060	–
50	0·0001	0·0009	0·0001	–
(*b*) Reactive power watts				
12·00	0·5331	0·5447	0·5306	–
12·25	0·2488	0·3777	0·2480	–
12·50	0·1037	0·2271	0·0942	–
12·75	0·0003	0·1362	0·0000	–
13·00	–	0·0706	–	–
14·00	–	0·0003	–	–

* after shifting expected value
† before shifting expected value

4.2.3 Computational aspects

As discussed in the previous Sections, the most precise formulation is formulation 4, since, with this method, the coupling effect between active and reactive powers is taken into account. However, it is the most time-consuming formulation to execute. To indicate these differences the time taken to compute all output parameters for the system discussed in this paper on a CDC 7600 computer was 3·9 s for formulation 1, 3·9 s for formulation 2, 5·6 s for formulation 3 and 6·4 s for formulation 4.

It is evident from these times that formulations 1 and 2 are clearly the least time consuming and, as expected by the complexity of the technique, formulation 4 is the longest. As discussed in Section 4.2.1 it is therefore necessary to decide whether greatest precision is required (i.e. formulation 4), a shorter thime is preferable with less precision in some of the computed values (i.e. formulation 3) or an even shorter time with the same imprecision as formulation 3 but without the deduction of active power line losses is acceptable (i.e. formulations 1 or 2). Unfortunately, there appears to be no reasonable method that can be used to predetermine whether formulation 4 is necessary for a given system and data. Therefore the authors recommend that formulation 4 is always used if all output parameters are required, unless precision and computational time are of greatest concern. If, however, only active powers are required then it does not seem necessary from present experience to use formulation 4.

5 Conclusions

This paper has described how the previously published formulations for an a.c. probabilistic load-flow analysis can be extended to ensure precise computation of expected values, and how this may be used to obtain more precise values of standard deviations and probability-density curves. It has also shown how the known coupling effect between active and reactive powers can be included.

From the detailed analysis the paper has indicated that the coupling aspect has no effect on the values of angles and active powers, but may have a significant impact on the values of voltages and reactive powers.

For the greatest precision in the computed values of voltage and reactive powers it can be concluded that formulation 4 is the preferred technique. However, if computing time is restricted it may be necessary to use one of the other less precise formulations. The shortest computing times occur with formulations 1 and 2. However, neither of these formulations permit computation of active power-line losses, and therefore the compromise would be formulation 3. In such cases it should be noted that imprecision may exist in some values of voltage and reactive power, although many of the computed results will be of sufficient accuracy for the purposes for which this type of analysis is intended.

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8 Appendixes

8.1 Formulation 3

(*a*) *Angles and active powers*

From eqn. 14

$$P_i = \sum_{k=0}^{n} Y_{ik}\theta_k + R_i \qquad (23)$$

where

$$R_i = V_{io}\sum_{k=1}^{n} V_{ko}e_{ik}$$

$$Y_{ik} = -V_{io}V_{ko}f_{ik}$$

$$Y_{ii} = V_{io}\sum_{k=1\neq i}^{n} V_{ko}f_{ik}$$

In matrix form and deleting the slack-node row and column gives

$$\boldsymbol{P} = \boldsymbol{Y\theta} + \boldsymbol{R}$$

$$\boldsymbol{\theta} = \boldsymbol{Y}^{-1}\boldsymbol{P} - \boldsymbol{Y}^{-1}\boldsymbol{R} = \hat{\boldsymbol{Y}}\boldsymbol{P} - \hat{\boldsymbol{Y}}\boldsymbol{R} \qquad (24)$$

or writing explicitly

$$\theta_i = \sum_{j=1}^{n-1}\hat{Y}_{ij}P_j - \sum_{j=1}^{n-1}\hat{Y}_{ij}R_j \qquad (i = 1, \ldots, n-1) \qquad (25)$$

From eqns. 8 and 11:

$$P_{ik} = g_{ik}\theta_i - g_{ik}\theta_k + h_{ik} \qquad (26)$$

where

$$g_{ik} = V_{io}V_{ko}f_{ik}$$

$$h_{ik} = -t_{ik}G_{ik}V_{io}^2 + V_{io}V_{ko}e_{ik}$$

Substituting eqn. 25 into eqn. 26 gives

$$P_{ik} = g_{ik}\sum_{j=1}^{n-1}(\hat{Y}_{ij} - \hat{Y}_{kj})P_j - g_{ik}\sum_{j=1}^{n-1}(\hat{Y}_{ij} - \hat{Y}_{kj})R_j + h_{ik} \qquad (27)$$

If i is a slack node, then $\hat{Y}_{ij} = 0$.

Similar equations for P_{ki} and active power losses $(P_{ik} + P_{ki})$ can be derived.

From eqn. 23, and since $\theta_s = 0$,

$$P_s = \sum_{i=1\neq s}^{n} Y_{si}\theta_i + R_s \qquad (28)$$

which in matrix form is:

$$P_s = \boldsymbol{K\theta} + R_s \qquad (29)$$

Substituting eqn. 24 into eqn. 29 gives

$$P_s = \boldsymbol{K\hat{Y}P} - \boldsymbol{K\hat{Y}R} + R_s$$

$$= \boldsymbol{TP} - \boldsymbol{TR} + R_s$$

where

$$\boldsymbol{T} = \boldsymbol{K\hat{Y}}$$

or in explicit form

$$P_s = \sum_{j=1}^{n-1} T_jP_j - \sum_{j=1}^{n-1} T_jR_j + R_s \qquad (30)$$

(*b*) *Voltages and reactive powers*

Writing eqn. 19 in partitioned matrix form gives

$$\begin{bmatrix} Q_l \\ Q_g \end{bmatrix} = \begin{bmatrix} M & L \\ N & J \end{bmatrix}\begin{bmatrix} V_l \\ V_g \end{bmatrix} - \begin{bmatrix} W_l \\ W_g \end{bmatrix} \qquad (31)$$

where Q_l has n_1 elements and Q_g has n_2 elements.

From eqn. 31:

$$Q_l = MV_l + LV_g - W_l$$

i.e.

$$V_l = \hat{M}Q_l + \hat{M}H \qquad (32)$$

where

$$H = W_l - LV_g$$

writing eqn. 32 explicitly gives

$$V_{i(l)} = \sum_{j=1}^{n_1}\hat{M}_{ij}Q_{j(l)} + \sum_{j=1}^{n_1}\hat{M}_{ij}H_j \qquad (i = 1 \ldots n_1) \qquad (33)$$

Also from eqn. 31

$$Q_g = NV_l + JV_g - W_g$$

Substituting for V_l (eqn. 32) gives

$$Q_g = DQ_l + E \qquad (34)$$

where

$$D = N\hat{M}$$

and

$$E = DH + JV_g - W_g$$

writing eqn. 34 explicitly gives

$$Q_{i(g)} = \sum_{j=1}^{n_1} D_{ij}Q_{j(l)} + E_i \qquad (i = 1 \ldots n_2) \qquad (35)$$

From eqns. 20 and 33

$$Q_{ik} = \sum_{j=1}^{n_1} (\alpha_{ik}\hat{M}_{ij} + \beta_{ik}\hat{M}_{kj})Q_{j(l)} + \sum_{j=1}^{n_1} (\alpha_{ik}\hat{M}_{ij} + \beta_{ik}\hat{M}_{kj})H_j + \gamma_{ik}$$

if both i and k are load busbars

$$Q_{ik} = \sum_{j=1}^{n_1} \beta_{ik}\hat{M}_{kj}Q_{j(l)} + \sum_{j=1}^{n_1} \beta_{ik}\hat{M}_{kj}H_j + \alpha_{ik}V_{io} + \gamma_{ik} \quad (36)$$

if i is a generator busbar

$$Q_{ik} = \sum_{j=1}^{n_1} \alpha_{ik}\hat{M}_{ij}Q_{j(l)} + \sum_{j=1}^{n_1} \alpha_{ik}\hat{M}_{ij}H_j + \beta_{ik}V_{ko} + \gamma_{ik}$$

if k is a generator busbar

Similar equations for Q_{ki} and the reactive power losses can be deduced. Finally, from eqns. 21 and 33

$$Q_{i(sh)} = 2V_{io}B_{i(sh)}\sum_{j=1}^{n_1}\hat{M}_{ij}Q_{j(l)} + 2V_{io}B_{i(sh)}\sum_{j=1}^{n_1}\hat{M}_{ij}H_j - V_{io}^2B_{i(sh)} \quad (37)$$

8.2 Formulation 4

From eqns. 9 and 22:

$$P_i = \sum_{k=1}^{n} f'_{ik}\theta_i - f'_{ik}\theta_k + g'_{ik}V_i + h'_{ik}V_k + e'_{ik} \quad (38)$$

where

$$e'_{ik} = a''_{ik}G_{ik} + a'_{ik}B_{ik}$$
$$f'_{ik} = b''_{ik}G_{ik} + b'_{ik}B_{ik}$$
$$g_{ik} = c''_{ik}G_{ik} + c'_{ik}B_{ik}$$
$$h'_{ik} = d''_{ik}G_{ik} + d'_{ik}B_{ik}$$

Eqn. 38 can be rewritten as

$$P_i = \sum_{k=1\neq i}^{n} (-f'_{ik}\theta_k) + \sum_{k=1\neq i}^{n} f'_{ik}\theta_i + \sum_{k=1\neq i}^{n} h'_{ik}V_k + (h'_{ii} + \sum_{k=1}^{n} g'_{ik})V_i + \sum_{k=1}^{n} e'_{ik} \quad (39)$$

An equation of similar form for Q_i can be deduced. The equations for P_i eqn. 39 and Q_i can be written in matrix form as

$$\begin{vmatrix} P \\ Q \end{vmatrix} = \begin{vmatrix} M' & L' \\ N' & J' \end{vmatrix} \begin{vmatrix} \boldsymbol{\theta} \\ V \end{vmatrix} + \begin{vmatrix} R' \\ H' \end{vmatrix} \quad (40)$$

where both P and Q have n elements. Eqn. 40 can be rearranged as

$$\begin{vmatrix} S_l \\ S_g \end{vmatrix} = \begin{vmatrix} M'' & L'' \\ N'' & J'' \end{vmatrix} \begin{vmatrix} X_l \\ X_g \end{vmatrix} + \begin{vmatrix} W_l \\ W_g \end{vmatrix} \quad (41)$$

where

$$S_l = \begin{vmatrix} P \\ Q_l \end{vmatrix} \qquad X_l = \begin{vmatrix} \boldsymbol{\theta} \\ V_l \end{vmatrix}$$

$$S_g = \begin{vmatrix} Q_g \\ P_s \end{vmatrix} \qquad X_g = \begin{vmatrix} V_g \\ \boldsymbol{\theta}_s \end{vmatrix}$$

and P and $\boldsymbol{\theta}$ include all nodes except the slack node, i.e. n-1 elements, Q_l and V_l include all load nodes (i.e. n_1 elements), Q_g and V_g include all generator nodes (i.e. n_2 elements) and P_s and $\boldsymbol{\theta}_s$ are for the slack node (i.e. one element).

Eqn. 41 can then be used to deduce the relevant equations for angles, voltages, active and reactive power flows, injected powers and line losses.