Set Theory: Cardinality of $\mathbb{R}$ and Uncountable Sets

pick up from José: Countable sets

$|\mathbb{Z}| = |\mathbb{N}| = |\mathbb{Q}| = \aleph_0$

Uncountable set

- infinite
- no one-to-one mapping to $\mathbb{N}$
- Example: $\mathbb{R}$

Theorem 1.7.1

$|P(X)| = 2^{|X|}$

Proof:

by induction:

1) $|\emptyset| = 0$, $|P(\emptyset)| = |\{\emptyset\}| = 1 = 2^0 = 2^{|\emptyset|}$

2) $|A_{n-1}| = n-1 \Rightarrow |P(A_{n-1})| = 2^{n-1}$

3) Let $A_{n-1} = \mathbb{Z}_{n-1}$

Show $|P(A_n)| = |2^{A_n}| = 2^{|A_n|}$:

$\mathbb{Z}_n = \mathbb{Z}_{n-1} \cup \{n-1\}$

$B_n = 2^{\mathbb{Z}_{n-1}} \cup \{n-1\}$, $|B_n| = 2^{n-1}$

$2^{\mathbb{Z}_n} = 2^{\mathbb{Z}_{n-1}} \cup B_n$

$|2^{\mathbb{Z}_n}| = 2^{n-1} + 2^{n-1} = 2^n$ ∎

Cantor's Theorem

$|P(\mathbb{N})| = 2^{|\mathbb{N}|} = 2^{\aleph_0}$ $\quad | \quad |P(A)| > A \quad \forall A$

Proof:

For $A \subseteq \mathbb{N}$, define indicator function

$$\mathbb{1}_A : \begin{cases} \mathbb{N} \to \{0, 1\} \\ \mathbb{1}_A(n) = \begin{cases} 1, & \text{if } n \in A \\ 0, & \text{else} \end{cases} \end{cases}$$

- For $A \supset \mathbb{N}$, $B \supset \mathbb{N}$, $A \neq B$: $\mathbb{1}_A(n) \neq \mathbb{1}_B(n)$

  $\Rightarrow$ 1-1 mapping between subsets of $\mathbb{N}$ and their characteristic functions

- Apply $\mathbb{1}_A(n)$ for $n = 0, 1, \dots$ $\Rightarrow$ semi-infinite binary sequence $\langle \alpha \rangle_n$

  $\Rightarrow$ 1-1 correspondence between $\mathbb{1}_A(n)$ and $\langle \alpha \rangle_n \in \{0,1\}^{\mathbb{N}}$

  $\Rightarrow$ 1-1 correspondence between $A \supset \mathbb{N}$ and $\langle \alpha \rangle_n \in \{0, 1\}^{\mathbb{N}}$

- $|\{0,1\}^{\mathbb{N}}| = 2^{|\mathbb{N}|}$

  $\Rightarrow$ $|P(\mathbb{N})| = 2^{|\mathbb{N}|}$ ■

Remark: identify sequences with $\langle \alpha \rangle_n = \begin{cases} 1 & \text{for } n > N \\ 0 & \text{for } n = N \end{cases}$
with $\langle \alpha \rangle_n = \begin{cases} 1 & \text{for } n = N \\ 0 & \text{for } n > N \end{cases}$

0 1 0 0 1 0 1 . . . . 1 . . .

0 1 0 0 1 1 0 0 . . . 0

"round up"

Remark: $P(A) = 2^A \cong \{f : A \to \{0, 1\}\}$

Example: $2^{\mathbb{N}}$ uncountable

The power set of $\mathbb{N}$ is uncountable $\quad 2^{\mathbb{N}}$

Diagonalization

Order all sequences (can be done if countable) $\alpha^i \in \{0,1\}^{\mathbb{N}}$

$\langle \alpha_0 \rangle_n = (a_{00} \; a_{01} \; a_{02} \cdots)$ construct $\langle s \rangle_n = \langle 1 \oplus a_{nn} \rangle_n$

$\langle \alpha_1 \rangle_n = (a_{10} \; \underline{a_{11}} \; a_{12} \cdots)$

$\langle \alpha_2 \rangle_n = (a_{20} \; a_{21} \; \underline{a_{22}} \cdots)$ $\Rightarrow$ differs from every $\langle \alpha_i \rangle_n$

$\Rightarrow$ not in enumeration

Example: Cardinality of $[0,1]$

any $r \in [0,1] \; = \; \alpha \in \{0,1\}^{\mathbb{N}}$

by $r = \sum_{i=1}^{\infty} \alpha_i 2^{-i}$, $\sum_{i=1}^{\infty} 2^{-i} = 1$

$\Rightarrow |\mathcal{P}(\mathbb{N})| = |[0,1]| \quad \Rightarrow |[0,1]| = 2^{\aleph_0}$

Example: $|[-1,1]|$

add sign bit : $\alpha_0 = 0 \Rightarrow r = \sum_{i=1}^{\infty} \alpha_i 2^{-i}$

$\alpha_0 = 1 \Rightarrow r = -\sum_{i=1}^{\infty} \alpha_i 2^{-i}$

$\Rightarrow |[-1,1]| = 2^{\aleph_0}$

Example: Cardinality of $\mathbb{R}$

define mapping: $q = \frac{r-1}{r+1} \quad \forall\, r \in \mathbb{R}$

observe:

$r \to -\infty \Rightarrow q \to 1$

$r \to \infty \Rightarrow q \to 1$

$r = 0 \Rightarrow q = -1$

$\Rightarrow$ 1–1 correspondence $\mathbb{R}$ vs. $[-1, 1)$

$\Rightarrow |\mathbb{R}| = |[-1,1)| = 2^{\aleph_0}$

Continuum Hypothesis

Cantor's theorem states

$|\mathbb{N}| < |2^{\mathbb{N}}|$, but leaves open: $\exists X : \aleph_0 < |X| < 2^{\aleph_0}$?

Special continuum hypothesis: or $\aleph_1 = 2^{\aleph_0} \cong 2^{\mathbb{N}} \cong \mathbb{R}$

$\nexists X : \aleph_0 < |X| < 2^{\aleph_0}$

Therefore:

$|\mathbb{N}|$ $|\mathbb{R}|, |\mathbb{R}^n|, |\mathbb{C}|, |C'|, |L^2| \ldots$ $\{f: \mathbb{R} \to \mathbb{R}\}$ $\{O : (\mathbb{R} \to \mathbb{R}) \to (\mathbb{R} \to \mathbb{R})\}$

$$0 < 1 < 2 < \ldots < \aleph_0 < \aleph_1 < \aleph_2 < \aleph_3 \ldots$$

$\aleph_1 = |\mathbb{R}| = |[0,1]| = |P(\mathbb{N})|$

$\aleph_2 = |P(\mathbb{R})| = |\{f : \mathbb{R} \to \mathbb{R}\}|$

General continuum hypothesis: $\nexists X : \aleph_i < |X| < \aleph_{i+1}$

Arithmetic with Cardinals

- Arithmetic on cardinals $\Leftrightarrow$ operations on sets

Relations, Equality

for cardinals $\mu = |X|$, $v = |Y|$ we get

$$X \subset Y \Leftrightarrow \mu < v$$
$$X \supseteq Y \Leftrightarrow \mu \geq v$$

- Examples: $\aleph_0 = |\mathbb{N}| < |\mathbb{R}| = \aleph_1$

$$|[a,b]| = |\mathbb{R}| = \aleph_1 \quad \forall\, a < b$$

Addition

$$|X| + |Y| = |X \cup Y| \quad \Leftrightarrow \quad \mu + v = |X \cup Y|$$

for finite cardinals: usual addition

for transfinite cardinals: $|X| + |Y| = \max\{|X|, |Y|\}$

$3 + \aleph_0 = \aleph_0$, $\aleph_0 + \aleph_0 = \aleph_0$, $\aleph_0 + \aleph_1 = \aleph_1$

Subtraction

- finite cardinals $\Rightarrow$ usual subtraction
- infinite: $v - \mu = \lambda \;:\Leftrightarrow\; \exists \lambda : \mu + \lambda = v \quad \forall \mu \leq v$

if $\mu < v$ then λ is unique

Example: $\aleph_0 + n = \aleph_0 \quad \forall n \in \mathbb{N} \Rightarrow \aleph_0 - \aleph_0 = n \quad \forall n \in \mathbb{N}$

$$\aleph_1 > \aleph_0 \Rightarrow \aleph_1 - \aleph_0 = \aleph_1$$

Multiplication

Defined through Cartesian product:

$$|X| \cdot |Y| = |X \times Y|$$

properties: $X = \emptyset \Rightarrow |X| = 0 \Rightarrow 0 \cdot |Y| = |Y| \cdot 0 = 0$

$|X| = 1 \Rightarrow |Y| \cdot 1 = 1 \cdot |Y| = |Y|$

both non-zero, one infinite $\Rightarrow |X| \cdot |Y| = \max\{|X|, |Y|\}$

Multiplication is associative and communitative, distributes w.r.t. +

$$|X| \cdot (|Y| + |Z|) = |X| \cdot |Y| + |X| \cdot |Z|$$
$$(|Y| + |Z|) \cdot |X| = |Y| \cdot |X| + |Z| \cdot |X|$$

Example: $\aleph_0 \cdot \aleph_0 = \max(\aleph_0, \aleph_0) = \aleph_0$

$\aleph_0 \cdot \aleph_1 = \max(\aleph_0, \aleph_1) = \aleph_1$

Division : $\nu / \mu = \lambda$

$\mu \neq 0$, ν infinite $\Rightarrow \exists \lambda : \mu \cdot \lambda = \nu \iff \mu \leq \nu$

λ is unique $\Leftarrow \mu < \nu$

Exponential : $|X|^{|Y|} = |X^Y|$ with $X^Y = \{f : Y \to X\}$

Usual identities hold: μ, ν, ε cardinals $\Rightarrow$

$0^{\mu} = 0$, $\mu^1 = \mu$, $\mu^{\nu + \varepsilon} = \mu^{\nu} \cdot \mu^{\varepsilon}$

$\mu^{\nu \cdot \varepsilon} = (\mu^{\nu})^{\varepsilon}$

$(\mu \cdot \nu)^{\varepsilon} = \mu^{\varepsilon} \cdot \nu^{\varepsilon}$

Cardinal Arithmetic Example

$$n + 2^{\aleph_0} = 2^{\aleph_0} \qquad \forall n \in \mathbb{N}$$

obvious: $n + 2^{\aleph_0} \geq 2^{\aleph_0}$

Second inequality:

$$n + 2^{\aleph_0} \underset{n \leq \aleph_0}{\leq} \aleph_0 + 2^{\aleph_0} \underset{\aleph_0 \leq 2^{\aleph_0}}{\leq} 2^{\aleph_0} + 2^{\aleph_0} = 2 \cdot 2^{\aleph_0} = 2^{\aleph_0}$$

Remarks

- We used Zermelo-Fraenkel (ZF) Axiomatic Set Theory
- E.g. "Set of all Sets" leads to contradiction, since the power set is strictly greater than the original set
- Also need to assume the "Axiom of Choice"
- Continuum Hypothesis is independent of standard axioms of set theory (can neither be proven/disproven in ZF)