**Complex Numbers**

**Definition:**
\[ \mathbb{C} = \{x + iy : x, y \in \mathbb{R}, \ i = \sqrt{-1}\} \]

\( (\mathbb{C}, +, \cdot) \) is a field, 2D vector space over \( \mathbb{R} \), basis \( 1, i \).

We extend the reals by \( i \), the solution to \( z^2 + 1 = 0 \).

\[ i = \sqrt{-1} \]

- Over \( \mathbb{C} \), \( z^2 = 1 \) for all \( z \in \mathbb{C} \).

- Over \( \mathbb{C} \), \( z^2 = 0 \) for all \( z \in \mathbb{C} \) can be solved.

- \( \mathbb{C} \) is closed w.r.t. \( \sqrt{\cdot} \).

- Physics/Math: \( \sqrt{-1} = i \), \( \sqrt{-1} = j \).

**Real/Imaginary part**

\[ x = \text{Re}(z), \ y = \text{Im}(z) \quad \Rightarrow \quad z = x + iy \quad \mathbb{R}, \mathbb{I} \]

**Rectangular form:** \( x + jy \)

- \( 12 + 3j \) is not in vector form.

**Diagram:**

- A complex number \( z \) is represented as a point in a 2D plane with real part \( x \) and imaginary part \( y \).

- The length of \( z \) is \( |z| \).

- The argument of \( z \) is \( \theta \).

- The vector form of \( z \) is \( z = x + jy \).
Polar representation

\[ z = x + jy \Rightarrow (r, \theta) \]
\[ r = |z| \]
\[ \theta = \arg z \quad \text{"argument" or "phase"} \]
\[ \text{However: } \arg z = \Arg z + 2k\pi = \{\theta_0, \theta + 2\pi, \theta + 4\pi, \ldots\} \]
\[ \uparrow \quad \text{PV (arg z)}: [-\pi, \pi), \left[-\pi, \frac{\pi}{2}\right), \left[\frac{\pi}{2}, \pi\right) \]
\[ \arg(\cdot) \text{ is a "multi-valued function", countable many values} \]
\[ \text{Toward: folding the phase (unwinding it) is a common challenge in signal processing} \]

\[ \text{Go to (3) Examples lens} \]

Polar \quad \rightarrow \quad \text{Rectangular}

\[ \text{for } z = x + jy: \]
\[ r = |z| = \sqrt{x^2 + y^2} \geq 0 \]
\[ \theta = \arctan \frac{y}{x} \]

\[ \text{for } z = (r, \theta): \]
\[ x = r \cos \theta \]
\[ y = r \sin \theta \]
Polar Form Examples

\[ \sqrt{2} + j \sqrt{2} = (2, 45^\circ) = (2, \frac{\pi}{4}) = (2, \frac{9\pi}{4}) \]

\[ \sqrt{2} - j \sqrt{2} = (2, -45^\circ) = (2, \frac{7\pi}{4}) \]

Rectangular Examples

\[ -2 + j \]

\[ -3 - 2j \]

\[ 3 + 2j \]

\[ -1 - j \]
Arg and PV of \( \tan^{-1} \)

- \( \tan^{-1}(\cdot) \) is inverse of \( \tan(\cdot) \)
- multi-valued function with countable many numbers
- \( \tan^{-1}(x) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \)

\( \Rightarrow \) need to look at sign of \( x \) and \( y \) to know quadrant

we use \( \tan^{-1}(\frac{y}{x}) = \text{Arg} \ 7 \in [-\pi, \pi) \)

\[ \text{PV} \]

\[ \text{Arg} \ 7 = \text{arg} \ 7 \mod 2\pi = \text{arg} \ 7 + 2\pi \left\lfloor -\frac{1}{2} - \frac{\text{arg} \ 2}{2\pi} \right\rfloor \]

\( \Gamma\) ceiling function \( (\text{smallest integer greater than number}) \)

\[ \Gamma \left\lfloor m < x \right\rfloor = m \in \mathbb{Z} \quad m \geq x \]

\( \Gamma(-3) = -3, \Gamma(-3.5) = -3, \Gamma(4.25) = 5 \)
Exponential representation

- periodicity of $z^2$ "wraps" the argument around the unit circle

- Euler Formula:
  $e^{i\theta} = \cos \theta + j \sin \theta$
  $|e^{i\theta}| = \sqrt{\cos^2 \theta + \sin^2 \theta} = 1$

- $z = r e^{i\theta}$
- careful: $r \geq 0$
  if not: $z = 1/2 \sqrt{(\theta - \pi)}$

Exponential $\rightarrow$ Polar

$|z| = |r e^{i\theta}| = \sqrt{r^2 \cos^2 \theta + r^2 \sin^2 \theta} = 1 \theta$

arg $z = \theta + 2\pi k$, $k \in \mathbb{Z}$, $\theta \geq 0$

if $r < 0$:
  arg $z = \theta + \pi (2k + 1)$, $k \in \mathbb{Z}$

Polar and Exponential representation are closely related: $z = r e^{i\theta}$
**Exponential ↔ Rectangular**

\[ z = x + jy \]

\[ \Rightarrow \quad z = \sqrt{x^2 + y^2} \quad e^{j \tan^{-1} \frac{y}{x}} \]

**Examples**

\[ 1 + j = \sqrt{2} \quad e^{j \frac{\pi}{4}} \]

\[-3 = -3 \quad e^{j \pi} \]

\[ j = e^{j \frac{\pi}{2}} \]
Arithmetic with complex numbers

Equality

\[ z_1 = z_2 \implies \text{Re } z_1 = \text{Re } z_2 \land \text{Im } z_1 = \text{Im } z_2 \]

\[ z_1, z_2 \in \mathbb{C} \]

Example:

\[ 1 + j = \frac{\sqrt{2}}{2} \cos \frac{\pi}{4} + \frac{\sqrt{2}}{2} j \sin \frac{\pi}{4} \]

Complex Conjugation

For \( z = x + jy \):

\[ z^* = x - jy \quad \text{rect} \]

\[ z = r e^{j\theta} \implies z^* = r e^{-j\theta} \quad \text{exp.} \]

\[ z = (r, \theta) \implies z^* = (r, -\theta) \quad \text{Polar} \]

Remark: conjugating a complex number in polar/exp. form:

\[ \text{Arg } z \implies -\text{Arg } z \]

\[ -\text{arg } z = -|\text{Arg } z + 2k\pi| = \begin{cases} -\text{Arg } z + 2k\pi, & \text{Arg } z > -\pi \\ \text{Arg } z + 2k\pi, & \text{Arg } z = -\pi \end{cases} \]

Remark:

\[ z^* = z \text{ for } z \in \mathbb{R} \]

\[ z^* = -z \text{ for } z \in j\mathbb{R} \]
Addition

Addition only works in rectangular format:

\[ z_1 = x_1 + iy_1, \quad z_2 = x_2 + iy_2 \Rightarrow z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2) \]

If in another format, first convert:

\[ z_1 = e^{i\pi} = -1, \quad z_2 = e^{i\frac{\pi}{2}} = i \Rightarrow z_1 + z_2 = -1 + i \]

C is a 2D vector space over \( \mathbb{R} \) with basis \( \{1, i\} \)

\( \Rightarrow \) addition is vector addition.
Multiplication

Multiplication can be done in rect format but is easier in exp format.

\[ z_1 = x_1 + jy_1, \quad \Rightarrow \quad z_1 \cdot z_2 = (x_1 + jy_1) \cdot (x_2 + jy_2) = \]
\[ = x_1x_2 + jy_1x_2 + jy_1x_2 + j^2 y_1y_2 = \]
\[ = (x_1x_2 - y_1y_2) + j(x_1y_2 + x_2y_1) \]

standard assoc, dist, comm rules apply as \( C \) is a field and \( R \subseteq C \)

Exp format:

\[ z_1 = r_1 e^{j\theta_1}, \quad \Rightarrow \quad z_1 \cdot z_2 = r_1 e^{j\theta_1} \cdot r_2 e^{j\theta_2} = r_1 r_2 e^{j(\theta_1 + \theta_2)} \]

Remark:

Multiplication is "stretch + rotate."
- If \( z_2 \in R \Rightarrow \) only stretch
- If \( |z_2| = 1 \Rightarrow \) only rotate

Note that \( z_1 \) rotated \( z_2 \) and \( z_2 \) rotates \( z_1 \)

Therefore very useful for BCC
$$\left(3 + j \right) \cdot \left(2 + j \right) = -2 + 6j$$