First Order ODEs

Scope
- Differential & Difference
- Ordinary, i.e. only one variable (usually time)
- constant coefficients
- Linear

Out of Scope
- PDEs (more than 1 variable)
- varying coefficients
- non linear

Differential equation: continuous function \( f(t) \)

\[ f'(t) = \dot{f} \]

\[ f(t) = F(t, \dot{f}) \]

\[ x, \text{IVP} \rightarrow t \]

Difference equation: discrete time

\[ f(n) = F(n, f(n-1)) \]

\[ 0 \quad 1 \quad 2 \quad 3 \]
\[ 2 \frac{dx(t)}{dt} + 4x(t) = 6e^{-4t}, \quad t > 0, \quad x(0) = 22 \]

- **t** ... independent variable
- defined over \( \mathbb{R} \)
- ordinary
- 1st order
- time interval and initial condition
- canonical form: \[ \frac{dx(t)}{dt} + \text{constant} \]

**Solving an ODE**

- function that satisfies the equation
- goes through initial value (IVP)

**Question:** existence 

- unique non-1D structural question

For 1st order ODEs:
- solutions exist, but not unique
- IVPs: unique
Recipe: 4 steps: (given by general theory)

1) Homogeneous solution
2) Particular solution
3) General solution
4) IVP solution

1) Homogeneous solution

\[ \frac{dx_h(t)}{dt} + 2x_h(t) = 0 \quad \forall \ t \geq 0 \]

\[ \text{LHS} = 0 / \]

\[ x_h \to \text{homogeneous} \]

Guessing method (Ansatz):

\[ x_h(t) = a e^{\lambda t} \quad \forall \ t \geq 0 \]

(we know the shape of the solution)

⇒ substitute in into ODE:

\[ \alpha \lambda e^{\lambda t} + 2a e^{\lambda t} = 0 \]

\[ a e^{\lambda t} (\lambda + 2) = 0 \quad \Rightarrow \quad \lambda = 0 \quad \text{or} \quad \lambda = -2 \]

⇒ \[ x_h(t) = a e^{-2t} \quad \forall \ t \geq 0 \]
2) Particular Solution

\[ x_p(t) \text{ is one solution that makes ODE an identity, } \]
- does not matter which one (seems arbitrary)
- needed to get all solutions of ODE

\[ \frac{dx_p(t)}{dt} + 2x_p(t) = -3e^{-4t} \]

Guessing of solution:

\[ x_p(t) = \beta e^{-4t}, \quad t \geq 0 \]

\[-4\beta e^{-4t} + 2\beta e^{-4t} = -3e^{-4t}\]

\[ \Rightarrow \beta = -\frac{3}{2} \quad x_p(t) = -\frac{3}{2}e^{-4t}, \quad t \geq 0 \]

One should verify that \( x_p(t) \) indeed is a solution to the ODE.

3) General Solution

Theory: \[ x_g(t) = x_h(t) + x_p(t) \]

\[ x_g(t) = e^{-2t} - \frac{3}{2}e^{-4t}, \quad t \geq 0 \]

Check: substitute in LHS of ODE, derive RHS.
4) LUP

\[ x_g(t) = a e^{-2t} \bigg|_{t=0} - \frac{3}{2} e^{-4t} \bigg|_{t=0} = a - \frac{3}{2} \]

\[ \Rightarrow a = \frac{47}{2} \]

\[ \Rightarrow x(t) = \frac{47}{2} e^{-2t} - \frac{3}{2} e^{-4t}, \quad t \geq 0 \]

---

**Linearity w.r.t. initial condition**

Consider an ODE LUP with \( f(t) = 0 \).

For initial condition

\[ x_0 = x_1 x_{\text{I}} + x_2 x_{\text{II}} \]

the general solution is

\[ x_g(t) = x_1 x_{\text{I}}(t) + x_2 x_{\text{II}}(t) \]

with

\[ x_{\text{I}}(t) \quad \text{general zero input solution for i.c. } x_{\text{I}} \]

\[ x_{\text{II}}(t) \quad \text{general zero input solution for i.c. } x_{\text{II}} \]
Linearly w.r.t. forcing term (particular solution)

For forcing term

\[ f(t) = a_1 f_1(t) + a_2 f_2(t) \]

A particular solution is given by

\[ x_p(t) = a_1 x_1(t) + a_2 x_2(t) \]

with

- \( x_1(t) \) is particular solution for \( f_1(t) \)
- \( x_2(t) \) is particular solution for \( f_2(t) \)
General Solution Approach

Always the 4 steps

- $x_h$
- $x_p$
- $g$
- $x_g(0) = x(0)$

Important ideas:
- Separation of variables
- Foucault Method
- Variation of constants
- Linearity of $x_p$ allows for var. const. solutions

Solve the homogeneous: Separation of variable

$$\frac{dx(t)}{dt} = f(t), \quad x(0) = x_0 \quad (\text{Special Case})$$

\[ t \int_0^t \frac{dx(t)}{dt} dt = \int_0^t f(t) dt \]
\[ x(t) = x_0 + \int_0^t f(t) dt \]

Linear: $x$

- both homogeneous and particular solutions have interesting linearity properties
Variation of Constants

How to Find \( x_p(t) \)

Idea: \( x_n(t) \) is already found (separation of variables)

Answer: \( x_p(t) = c(t)x_n(t) \)

\[
\dot{x}_p(t) = \dot{x}_n(t)c(t) + x_n(t)c'(t)
\]

Substitute into ODE, simplify

\[
\Rightarrow \quad x_n(t)c'(t) = f(t)
\]

\[
\Rightarrow \quad \frac{dc(t)}{dt} = \frac{f(t)}{x_n(t)}
\]

Separation of variables

\[
c(t) = c_0 + \int_0^t x_n'(\tau)f(\tau)\,d\tau
\]

\[
x_p(t) = x_n(t)c(t)
\]

\[
\Rightarrow \quad \text{Closed form solution for } x + \theta_0 x = f, \quad x(0) = x_0
\]

\[
x(t) = e^{-\theta_0 t}x_0 + \int_0^t e^{-\theta_0(t-\tau)}f(\tau)\,d\tau \quad \to \quad \text{as } t \to \infty
\]

Stability

Behavior at \( t \to \infty \) depends on \( \lambda = -\theta_0 \)

- \( \text{Re } \lambda < 0 \) \( \Rightarrow \) stable
- \( \text{Re } \lambda > 0 \) \( \Rightarrow \) unstable
- \( \text{Re } \lambda = 0 \) \( \Rightarrow \) bounded
Solution formula for LTI:
\[ x(t) = e^{-\theta_0 t} x_0 + \int_0^t e^{-\theta_0 (t-\tau)} f(\tau) d\tau \]

Zero input response

For \( f(t) \equiv 0, \ t \geq 0 \)
\[ x(t) \text{ reduces to } h_{zi}(t) = e^{-\theta_0 t} x_0 \]

Zero state response

For \( x_0 = 0 \)
\[ x(t) \text{ reduces to } h_{zs}(t) = \int_0^t e^{-\theta_0 (t-\tau)} f(\tau) d\tau \]

Asymptotic Behavior

\[ \lim_{t \to \infty} x(t) \]

Zero input response \( x_{zi}(t) \)

Stability:
\[ \lim_{t \to \infty} x_{zi}(t) = e^{-\theta_0 t} x_0 \]
\[ \lim_{t \to \infty} x_{zi}(t) = \lim_{t \to \infty} e^{-\theta_0 t} x_0 = \int_0^{\infty} x_0 \delta(\tau) d\tau = \delta(x_0) \]

For \( \theta_0 > 0 \)
\[ \theta_0 = 0 \]
\[ \theta_0 < 0 \]