# Track Initialization from Incomplete Measurements

Christian R. Berger\*, Martina Daun<sup>†</sup> and Wolfgang Koch<sup>†</sup>

\*Department of Electrical and Computer Engineering, University of Connecticut, Storrs, Connecticut 06269, USA Email: crberger@engr.uconn.edu

<sup>†</sup>Sensor Networks and Data Fusion Group, FKIE, FGAN e.V., 53343 Wachtberg, Germany Email: daun@fgan.de, w.koch@fgan.de

Abstract—Target tracking from incomplete measurements of distinct sensors in a sensor network is a task of data fusion, present in a lot of applications. Difficulties in tracking using Extended Kalman filters lead to unstable behavior, mainly caused by difficult initialization. Instead of using numerical batchestimators, we offer an analytical approach to initialize the filter from a minimum number of observations. Additionally, we provide the possibility to estimate only sub-sets of parameters, and to reliably model resulting added uncertainties by the covariance matrix.

The approach will be studied in two practical examples: 3D track initialization using bearings-only measurements and using slant-range and azimuth only. Numerical results will include performance and consistency analysis via Monte-Carlo simulations and comparison to the Cramer-Rao lower bound.

#### I. INTRODUCTION

Target tracking from incomplete measurements, like bearings only or range only, is a topic which has been investigated thoroughly, e.g., target motion analysis and related questions of observability [1]. The results utilize derivatives of standard Extended Kalman filters, typical of tracking targets using measurements in polar or spherical coordinates, while modeling their movement in Cartesian coordinates [2], [3].

Especially in the case of incomplete measurements, the initialization of the extended Kalman filter with an initial state estimate and corresponding covariance is crucial for its performance, otherwise the filter can easily become unstable. In the case of incomplete measurements this can not be accomplished by direct inversion of the measurement function. Multiple measurements will have to be combined for initialization, which calls for a sensible data fusion.

Typically, numerical batch estimators are used to find a Maximum Likelihood (ML) estimate [4]. Although these estimators offer close to optimal performance, in the sense of achieving the Cramer-Rao lower bound (CRLB) in estimation accuracy, they need a large number of measurements for 'benign' numerical behavior. We offer instead an analytical approach using a minimum number of measurements, to return an initial estimate and a corresponding covariance.

By making statistical assumptions about some components of the state vector, we can initialize these state elements with their mean and covariance, and thereby find an initial estimate even in cases when observability of the full state vector is not given. When splitting the state vector into two parts, one which is estimated and the second which is initialized through statistical assumptions, the covariance of the latter is a design choice, but we will still have to derive the cross-correlation between the two. More importantly, the added uncertainty in the covariance of the estimated parameters will also be modeled. This is caused by the loss of information due to not estimating part of the state vector and instead interpreting them as additional perturbation.

After deriving a general approach, we will apply this to different scenarios, mostly using incomplete spherical measurements typical for radar/sonar applications. The focus will be on position initialization from bearings-only measurements (Sect. III), which was also used in [3], from which this work was strongly inspired by. As an additional application scenario we will also present: position initialization from two slant-range and azimuth measurements. Then, both scenarios are numerically evaluated via Monte-Carlo simulation. Focus will be on absolute performance like estimation error and comparisons to the corresponding CRLBs. Consistency of our initialization will be scrutinized to check if our estimators are unbiased and if the covariance precisely characterizes the estimation error.

This work has the following structure: After this introduction, we will describe the system model and derive our initialization scheme in Section II. As mentioned before Section III will cover the bearings-only scenario, which is followed by the scenario of range and azimuth (Sect. IV). We will discuss our numerical simulations in Section V and Section VI concludes this paper.

# II. DESCRIPTION OF SYSTEM MODEL AND MAP ESTIMATION

#### A. System Model

Let x be the state vector of the target with dimension  $\eta_x$ , which is modeled in Cartesian coordinates using, e.g., a second order motion model

$$\mathbf{x}(n+1) = F(n)\mathbf{x}(n) + \boldsymbol{\nu}(n) \tag{1}$$
$$\mathbf{x} = \begin{pmatrix} \mathbf{r} \\ \dot{\mathbf{r}} \end{pmatrix} F(n) = \begin{pmatrix} I & t_{n+1} - t_n \\ 0 & I \end{pmatrix}$$

with x being the state vector, containing position vector r and velocity vector  $\dot{\mathbf{r}}$ ;  $\boldsymbol{\nu}$  is the process noise and F the state

propagation matrix, all of dimension  $\eta_x$ . The measurement function h is generally non-linear, dependant on the observer position  $\mathbf{x}_s$  and not invertible,

$$\mathbf{z}(n) = h(\mathbf{x}(n) - \mathbf{x}_s(n)) + \mathbf{w}(n).$$
(2)

The zero mean, Gaussian measurement noise **w** with covariance **R** and the observation  $\mathbf{z} \in M$  (*M* is the space of the measurements) are of dimension  $\eta_{\mathbf{z}}$ . We partition the state vector into two sub-vectors  $\mathbf{x}_o$  and  $\bar{\mathbf{x}}_o$ , one which is initially estimated by a function of the first measurements and the other which is initialized by appropriate modeling assumptions.

W.l.o.g, we can choose to reorder the state vector to achieve,

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_o \\ \bar{\mathbf{x}}_o \end{bmatrix} \tag{3}$$

Then we formally express  $\mathbf{x}$  as

$$\mathbf{x} = K\mathbf{x}_o + \bar{K}\bar{\mathbf{x}}_o$$
 with  $K = \begin{bmatrix} I\\0 \end{bmatrix}, \ \bar{K} = \begin{bmatrix} 0\\I \end{bmatrix}$ . (4)

Using k different measurements we try to find a function t with  $t: M^k \to \mathbb{R}^{\eta_{\mathbf{x}_o}}$  which fulfills the following condition:

if 
$$\bar{\mathbf{x}}_o = E[\bar{\mathbf{x}}_o]$$
 then  $t \circ h(\mathbf{x}) = \mathbf{x}_o$ , (5)

i.e., if our assumptions on  $\bar{\mathbf{x}}_o$  hold, then t gives the correct value  $\mathbf{x}_o$ . In the case of linear functions, this would be equivalent to an unbiased estimator in terms of the additional perturbation  $\bar{\mathbf{x}}_o$ . We have to combine at least k measurements so that  $k\eta_{\mathbf{z}} \geq \eta_{\mathbf{x}_o}$ . To simplify notation, we will use the notation  $t(\mathbf{z})$  also when referring to  $t(\mathbf{z}^k)$  and use  $h(\mathbf{x})$  even for mapping to  $\mathbf{z}^k$ . Since the function t is generally not readily available, defining it in a sensible way will be one of the main tasks of this work.

Generally if we have  $k\eta_z > \eta_{x_o}$ , there is no solution to  $z = h(Kx_o + \bar{K}\bar{x}_o)$  due to the measurement errors. Instead of picking an  $x_o$  which reproduces all measurements as close as possible in the ML sense, we drop n arbitrary parts of the measurement to achieve  $k\eta_z - n = \eta_{x_o}$ . Even though we are faced with non-linear equations, this guarantees that there will be either none or countable many solutions.

By systematically dropping different parts of the measurements we produce several,  $\binom{k\eta_z}{n}$ , estimates and pick one using an optimality criterion, e.g., the trace of the estimated covariance matrix. If one combination of measurements renders no solution, it is excluded (a solution to the case of ambiguity will be given in one example Sect. III-C). Afterwards each of the *n* remaining datum is merged using Kalman filtering which, if the problem would be purely linear/Gaussian, would return the same final result for any combination of measurements. Since our problem is inherently non-linear, this way we can initialize from the measurements with the best geometry.

# B. MAP Estimate using Extended Kalman Filter Linearization

Let the likelihood function  $p(\mathbf{z}|\mathbf{x})$  be given by a normal distribution  $\mathcal{N}(\mathbf{z}; h(\mathbf{x}), \mathbf{R})$ , with a known measurement covariance matrix  $\mathbf{R}$ . The probability density of  $t(\mathbf{z})$  given  $\mathbf{x}$  can be approximated by linearizing  $t(\mathbf{z})$ :

$$p(t(\mathbf{z})|\mathbf{x}) = \mathcal{N}\left(t(\mathbf{z}), t[h(\mathbf{x})], \frac{\partial t}{\partial \mathbf{z}} \mathbf{R} \frac{\partial t'}{\partial \mathbf{z}}\right).$$
(6)

Looking at (5), we can easily see that by definition  $t \circ h$  can be linearized around  $E[\bar{\mathbf{x}}_o]$  to

$$t \circ h(\mathbf{x}) = \mathbf{x}_o + \frac{\partial (t \circ h)}{\partial \mathbf{x}} \bar{K}(\bar{\mathbf{x}}_o - E[\bar{\mathbf{x}}_o])$$
  
=  $\mathbf{x}_o + G(\bar{\mathbf{x}}_o - E[\bar{\mathbf{x}}_o])$  (7)

accordingly, we have

$$p(t(\mathbf{z})|\mathbf{x}) = \mathcal{N}\left(t(\mathbf{z}); \mathbf{x}_o + G(\bar{\mathbf{x}}_o - E[\bar{\mathbf{x}}_o]), \frac{\partial t}{\partial \mathbf{z}} \mathbf{R} \frac{\partial t'}{\partial \mathbf{z}'}\right)$$
(8)

and due to linearization we can switch  $t(\mathbf{z})$  and  $\mathbf{x}_o$ ,

$$p(\mathbf{x}_o | \mathbf{z}, \bar{\mathbf{x}}_o) = \mathcal{N}\left(\mathbf{x}_o; t(\mathbf{z}) - G(\bar{\mathbf{x}}_o - E[\bar{\mathbf{x}}_o]), \frac{\partial t}{\partial \mathbf{z}} \mathbf{R} \frac{\partial t'}{\partial \mathbf{z}'}\right)$$
(9)

where conditioning on z is the same as conditioning on t(z). If we now substitute  $\mathbf{x} = K\mathbf{x}_o + \bar{K}\bar{\mathbf{x}}_o$ 

$$E[\mathbf{x}|\mathbf{z}, \bar{\mathbf{x}}_o] = Kt(\mathbf{z}) - KG(\bar{\mathbf{x}}_o - E[\bar{\mathbf{x}}_o]) + \bar{K}\bar{\mathbf{x}}_o$$
  
=  $Kt(\mathbf{z}) + KGE[\bar{\mathbf{x}}_o] + (\bar{K} - KG)\bar{\mathbf{x}}_o$  (10)

$$\operatorname{Cov}[\mathbf{x}|\mathbf{z}, \bar{\mathbf{x}}_o] = K \frac{\partial t}{\partial \mathbf{z}} \mathbf{R} \frac{\partial t'}{\partial \mathbf{z}} K'$$
(11)

we can approximate x as Gaussian with the above parameters as (12). To get rid of the conditioning on  $\bar{\mathbf{x}}_o$  we go along the lines of Bayes total probability theorem for continuous random variables, averaging over  $\bar{\mathbf{x}}_o$  we get,

$$p(\mathbf{x}|\mathbf{z}) = \int p(\mathbf{x}|\mathbf{z}, \bar{\mathbf{x}}_o) p(\bar{\mathbf{x}}_o) \, d\bar{\mathbf{x}}_o.$$
(13)

Changing the conditioning and integrating the final pdf is given in (14), which results in the MAP (maximum a priori) estimator of x

$$\hat{\mathbf{x}} = E[\mathbf{x}|\mathbf{z}] = Kt(\mathbf{z}) + \bar{K}E[\bar{\mathbf{x}}_o]$$
(15)

with covariance

$$\mathbf{P} = \operatorname{Cov}[\mathbf{x}|\mathbf{z}]$$
  
=  $K \frac{\partial t}{\partial \mathbf{z}} \mathbf{R} \frac{\partial t}{\partial \mathbf{z}}' K' + (\bar{K} - KG)P(\bar{K} - KG)'.$  (16)

$$p(\mathbf{x}|\mathbf{z}, \bar{\mathbf{x}}_o) = \mathcal{N}\left(\mathbf{x}; Kt(\mathbf{z}) + KGE[\bar{\mathbf{x}}_o] + (\bar{K} - KG)\bar{\mathbf{x}}_o, K\frac{\partial t}{\partial \mathbf{z}}\mathbf{R}\frac{\partial t'}{\partial \mathbf{z}}K'\right)$$
(12)

$$p(\mathbf{x}|\mathbf{z}) = \mathcal{N}\left(\mathbf{x}; Kt(\mathbf{z}) + \bar{K}E[\bar{\mathbf{x}}_o], K\frac{\partial t}{\partial \mathbf{z}}\mathbf{R}\frac{\partial t'}{\partial \mathbf{z}}'K' + (\bar{K} - KG)P(\bar{K} - KG)'\right)$$
(14)

#### C. MAP Estimate using the Unscented Transform

To calculate  $p(\mathbf{x}|\mathbf{z})$  in (14) we can replace the use of linearization typical for the Extended Kalman filter, by using the Unscented Transform [5]. What we basically did to calculate  $p(\mathbf{x}|\mathbf{z})$  in the previous case, was to linearize the functional relationships between  $\mathbf{z}, \bar{\mathbf{x}}_o$  and  $\mathbf{x}_o$  which we had in  $t(\mathbf{z})$  and  $t \circ h(\mathbf{x})$ .

Instead we can directly derive a functional relationship of the measurements  $\mathbf{z}$ , the parameters  $\bar{\mathbf{x}}_o$  which are initialized by modeling assumptions and  $\mathbf{x}_o$  the estimated parameters. Using  $\mathbf{z} = h(\mathbf{x}_o, \bar{\mathbf{x}}_o) + \mathbf{w}$  we will solve for  $\mathbf{x}_o$ , i.e.,  $\mathbf{x}_o = g(\mathbf{z} - \mathbf{w}, \bar{\mathbf{x}}_o)$ , where the function g is in direct relationship to  $t(\mathbf{z})$ , since

$$g(\mathbf{z}, E[\bar{\mathbf{x}}_o]) = t(\mathbf{z}). \tag{17}$$

Once we have this, usually non-linear, functional relationship, we can use the Unscented Transform to derive  $p(\mathbf{x}_o|\mathbf{z})$ from  $p(\mathbf{z} - \mathbf{w}, \bar{\mathbf{x}}_o|\mathbf{z})$ , using the functional relationship in (17) to map the influence of the measurement noise and the parameters modeled as random on the estimate of the other parameters.

To find the probability density of  $p(\mathbf{z} - \mathbf{w}, \bar{\mathbf{x}}_o | \mathbf{z})$  we can use that  $p(\mathbf{z} - \mathbf{w} | \mathbf{z})$  has obviously mean  $\mathbf{z}$  and covariance  $\mathbf{R}$ and  $p(\bar{\mathbf{x}}_o | \mathbf{z}, \mathbf{z} - \mathbf{w})$  is by modelling assumption  $\mathcal{N}(\bar{\mathbf{x}}_o; 0, P)$ , since these parameters are assumed to be independent of the measurements. Due to conditional independency we get

$$p(\mathbf{z} - \mathbf{w}, \bar{\mathbf{x}}_o | \mathbf{z}) = p(\mathbf{z} - \mathbf{w} | \mathbf{z}) p(\bar{\mathbf{x}}_o | \mathbf{z}, \mathbf{z} - \mathbf{w})$$
(18)

which we can transform into  $p(\mathbf{x}|\mathbf{z})$ , using

$$\mathbf{x} = Kg(\mathbf{z} - \mathbf{w}, \bar{\mathbf{x}}_o) + \bar{K}\bar{\mathbf{x}}_o.$$
 (19)

#### D. Calculating the CRLB Using Prior Information

Since we consider non-linear measurements and additive white Gaussian noise, the Cramer-Rao lower bound (CRLB) can be derived in a standard way. The general calculation of the Fisher information matrix  $J_0$  as in [2], can be replaced by the more specialized formula,

$$\mathbf{J}_{0} = E\left\{\left[\nabla_{\mathbf{x}}\log\Lambda(\mathbf{x})\right]\left[\nabla_{\mathbf{x}}\log\Lambda(\mathbf{x})\right]'\right\} \\
= \frac{\partial h}{\partial \mathbf{x}}\operatorname{Cov}(\mathbf{w})^{-1}\frac{\partial h'}{\partial \mathbf{x}},$$
(20)

where  $\Lambda(\mathbf{x}) = p(\mathbf{z}|\mathbf{x})$  is the likelihood function. Poorly, for a minimum number of measurements, the matrix  $\mathbf{J}_0$  will usually not be invertible. This reflects that we can not estimate the full state vector  $\mathbf{x}$  without additional assumptions. As information is additive, these additional assumptions, in the form of a prior distribution on  $\bar{\mathbf{x}}_o$ , can be added to the Fisher information matrix [6],

$$\mathbf{J} = \mathbf{J}_0 + \mathbf{J}_P \tag{21}$$

where  $\mathbf{J}_P$  is the Fisher information of the prior. Assuming a Gaussian prior on  $\bar{\mathbf{x}}_o$ , this will take the following form,

$$\mathbf{J}_P = \begin{bmatrix} 0 & 0\\ 0 & P^{-1} \end{bmatrix}.$$
 (22)



Fig. 1. Bearings-only measurements scenario

## III. TRACK INITIALIZATION FROM BEARINGS-ONLY MEASUREMENTS

## A. Scenario Description

In this scenario only the spherical coordinates azimuth and elevation are measured, see Fig.(1). Initializing from k measurements is possible, if the measurements are taken at different positions  $\mathbf{x}_s(n)$  [1], either from different sensors or a moving observer. For brevity without explicit dependency on n,  $\mathbf{x} = [x, y, z, \dot{x}, \dot{y}, \dot{z}]'$  and  $\mathbf{z} = [\phi, \theta]'$ , the measurement equations are the following,

$$\phi = \arctan\left(\frac{y - y_s}{x - x_s}\right) \tag{23}$$

$$\theta = \arctan\left(\frac{z - z_s}{\sqrt{(x - x_s)^2 + (y - y_s)^2}}\right)$$
(24)

To initialize the position only  $\mathbf{x}_o = [x, y, z]'$ , it is sufficient to have two measurements k = 2, which gives us

$$k\eta_{\mathbf{z}} = 4 \ge \eta_{\mathbf{x}_o} = 3. \tag{25}$$

Those two measurements should be taken from two distinct arbitrary points  $\mathbf{x}_{s,i}$ , i = 1, 2 with distance

$$|\mathbf{x}_{s,1} - \mathbf{x}_{s,2}| = d. \tag{26}$$

W.l.o.g., we can assume them to be on the x-axis at -d/2 and d/2. Using

$$h(\mathbf{x}(n+1) - \mathbf{x}_s(n+1)) = h(F(n)\mathbf{x}(n) - \mathbf{x}_s(n+1))$$
(27)

and defining  $T = t_{n+1} - t_n$  as the time difference between the measurements, the equations can be expressed as,

$$\phi_{1} = \arctan\left(\frac{y}{x+d/2}\right)$$

$$\theta_{1} = \arctan\left(\frac{z}{\sqrt{(x+d/2)^{2}+(y)^{2}}}\right)$$

$$\phi_{2} = \arctan\left(\frac{y+jT}{x+\dot{x}T-d/2}\right)$$

$$\theta_{2} = \arctan\left(\frac{z+\dot{z}T}{\sqrt{(x+\dot{x}T-d/2)^{2}+(y+\dot{y}T)^{2}}}\right)$$
(29)

depending only on  $\mathbf{x}(n)$  and assuming constant velocity. In the following we will refer to  $\mathbf{z} - \mathbf{w} = [\phi_1, \theta_1, \phi_2, \theta_2]'$  as the stacked vector of the true measurements to simplify notation.

Since we will not estimate the velocity, accordingly  $\bar{\mathbf{x}}_o = [\dot{x}, \dot{y}, \dot{z}]'$ , we will have to make statistical assumptions about it. The velocity will be assumed to be zero-mean Gaussian distributed  $\mathcal{N}(\bar{\mathbf{x}}_o; 0, P)$ , which is reasonable since there is no preferred direction.

Since we have  $k\eta_{\mathbf{z}} = 4 > \eta_{\mathbf{x}_o} = 3$ , we are left with one degree of freedom (n = 1). Accordingly we can choose any three elements of  $[\phi_1, \theta_1, \phi_2, \theta_2]'$  which will give us four different inverse functions. Now if we solve (2) for  $\mathbf{x}_o$ , we get

$$\mathbf{x}_o = t(\mathbf{z} - \mathbf{w}) + f(\bar{\mathbf{x}}_o, \mathbf{z} - \mathbf{w})$$
(30)

where we choose f such that  $f(\bar{\mathbf{x}}_o, \mathbf{z}-\mathbf{w}) = 0$  for  $\bar{\mathbf{x}}_o = E[\bar{\mathbf{x}}_o]$ . This approach will be consequently applied to all four possible combinations of three measurements.

# B. Two Azimuths and One Elevation

Using  $\{\phi_1, \phi_2, \theta_1\}$  of (28)-(29) and solving for  $\mathbf{x}_o$  we first solve for,

$$x = \frac{\dot{x}T\tan\phi_2 - \dot{y}T - d/2(\tan\phi_1 + \tan\phi_2)}{\tan\phi_1 - \tan\phi_2}$$
  
=  $d\frac{\sin\phi_1\cos\phi_2}{\sin(\phi_2 - \phi_1)} + \frac{d}{2} - T\cos\phi_1\frac{\dot{x}\sin\phi_2 - \dot{y}\cos\phi_2}{\sin(\phi_2 - \phi_1)}$  (31)

and

$$y = \frac{(\dot{x}T - d)\tan\phi_1 \tan\phi_2 - (\dot{y}T)\tan\phi_1}{\tan\phi_1 - \tan\phi_2} = d\frac{\sin\phi_1 \sin\phi_2}{\sin(\phi_2 - \phi_1)} - T\sin\phi_1 \frac{\dot{x}\sin\phi_2 - \dot{y}\cos\phi_2}{\sin(\phi_2 - \phi_1)}.$$
 (32)

From which we now calculate z as,

$$z = \tan(\theta_1) \sqrt{(x+d/2)^2 + y^2} = \tan(\theta_1) \left| \frac{d\sin\phi_2}{\sin(\phi_2 - \phi_1)} - T \frac{\dot{x}\sin\phi_2 - \dot{y}\cos\phi_2}{\sin(\phi_2 - \phi_1)} \right|.$$
(33)

The first inverse function  $t_1(\mathbf{z} - \mathbf{w})$  is accordingly

$$t_1(\mathbf{z} - \mathbf{w}) = \begin{pmatrix} d\frac{\sin \phi_1 \cos \phi_2}{\sin(\phi_2 - \phi_1)} + \frac{d}{2} \\ d\frac{\sin \phi_1 \sin \phi_2}{\sin(\phi_2 - \phi_1)} \\ \tan(\theta_1) \left| \frac{d\sin \phi_2}{\sin(\phi_2 - \phi_1)} \right| \end{pmatrix}$$
(34)

where we used  $\phi_1$ ,  $\theta_1$  and  $\phi_2$  to estimate  $\mathbf{x}_o(n)$  and  $\mathbf{x}$  as  $Kt_1(\mathbf{z} - \mathbf{w}) + \bar{K}E[\bar{\mathbf{x}}_o]$ .

To calculate the covariance matrix according to (16) we need the linearization  $\frac{\partial t_1}{\partial z - w}$  and  $\frac{\partial t_1 \circ h}{\partial x}$ , which can be found in App. I.

Alternatively we can use the Unscented Transform (UT) for which we need to find a function g which maps from  $\bar{\mathbf{x}}_o$  and  $\mathbf{z} - \mathbf{w}$  to  $\mathbf{x}_o$ . This function  $g_1$  is already available in (31)-(33), i.e.  $\mathbf{x}_o = g_1(\mathbf{z} - \mathbf{w}, \bar{\mathbf{x}}_o)$ . So we are directly able to use the UT.

The estimation function  $t_2$  for  $\{\phi_1, \phi_2, \theta_2\}$  can be calculated by changing  $\phi_1 \leftrightarrow \phi_2$ ,  $\theta_1 \leftrightarrow \theta_2$  and  $d \leftrightarrow -d$  due to the symmetry of the scenario. For the covariance we also substitute  $T \leftrightarrow -T$ .

#### C. One Azimuth and Two Elevations

As the second possibility to calculate  $\mathbf{x}_o(n)$ , we can use  $\{\phi_1, \theta_1, \theta_2\}$  of (28)-(29). Substituting (see Fig. 1)

$$x = \rho_1 \cos \phi_1 - d/2 \tag{35}$$

$$y = \rho_1 \sin \phi_1 \tag{36}$$

$$z = \rho_1 \tan \theta_1 \tag{37}$$

with the ground range from sensor one  $\rho_1$ , we get

$$(\rho_1 \tan \theta_1 + \dot{z}T)^2 = \\ \tan^2 \theta_2 \left( (\rho_1 \cos \phi_1 - d + \dot{x}T)^2 + (\rho_1 \sin \phi_1 + \dot{y}T)^2 \right) \quad (38)$$

This is a quadratic equation in  $\rho_1$  and using  $q = \frac{\tan \theta_2}{\tan \theta_1}$  for simpler notation,

$$\rho_1^2(1-q^2) + 2\rho_1(\cot\theta_1\dot{z}T + q^2(\cos\phi_1(d-\dot{x}T) - \sin\phi_1\dot{y}T)) + \cot^2\theta_1\dot{z}^2T^2 - q^2\left((d-\dot{x}T)^2 + \dot{y}^2T^2\right) = 0.$$
(39)

Setting T = 0 (equivalent to  $\bar{\mathbf{x}}_o = E[\bar{\mathbf{x}}_o] = 0$ ) we can calculate the inverse function by solving first for  $\rho_1$ 

$$\rho_1^2(1-q^2) + 2d\rho_1 q^2 \cos\phi_1 - q^2 d^2 = 0 \tag{40}$$

For q = 1 we get

$$\rho_1 = \frac{d}{2\cos\phi_1} \tag{41}$$

and else

$$\rho_1 = d \left( -\frac{q^2 \cos \phi_1}{1 - q^2} \pm \left| \frac{q}{1 - q^2} \right| \sqrt{1 - q^2 \sin^2 \phi_1} \right) \quad (42)$$

To solve this ambiguity we use the second azimuth  $\phi_2$ . For geometrical reasons, if  $q^2 < 1$ , there is only one positive solution. Otherwise, if  $|\phi_2| < \pi/2$  we choose the positive root and negative else.

Again, the linearizations needed to calculate the covariance can be found in App. I

To calculate the covariance matrix for  $t_3$  using the Unscented Transform (UT), we have to solve (39) for  $\rho_1$ , without setting T = 0. This still means solving a quadratic equation, but is notational more cumbersome

$$\rho_1^2(1-q^2) + 2\rho_1(\cot\theta_1\dot{z}T + q^2(\cos\phi_1(d-\dot{x}T) - \sin\phi_1\dot{y}T)) + \cot^2\theta_1\dot{z}^2T^2 - q^2\left((d-\dot{x}T)^2 + \dot{y}^2T^2\right) = 0. \quad (43)$$

With 
$$Q = -\frac{\cot \theta_1 \dot{z}T + q^2(\cos \phi_1(d - \dot{x}T) - \sin \phi_1 \dot{y}T)}{(1 - q^2)}$$
, we calculate

$$\rho_1 = Q \pm \sqrt{Q^2 - \frac{\cot^2 \theta_1 \dot{z}^2 T^2 - q^2 \left( (d - \dot{x}T)^2 + \dot{y}^2 T^2 \right)}{(1 - q^2)}}.$$
(44)

Inserting this into (35)-(37) yields  $g_2$ .

The last estimation functions  $t_4$  for  $\{\phi_2, \theta_1, \theta_2\}$  can be calculated through the function  $t_3(\mathbf{z} - \mathbf{w})$  by changing  $\phi_1 \leftrightarrow \phi_2$ ,  $\theta_1 \leftrightarrow \theta_2$  and  $d \leftrightarrow -d$ . And for the covariance we also substituting  $T \leftrightarrow -T$ .

#### IV. TRACK INITIALIZATION FROM RANGE AND AZIMUTH

#### A. Description

Traditional active radars give range measurements and sometimes only partial bearings. This is usually not a problem if the setup can be approximated by a 2-dimensional interpretation. Even so, the increased uncertainty should be incorporated in the covariance, which is a good reasoning to apply our approach.

Again we will use k = 2 measurements to estimate the position and make statistical assumptions about the velocity. The measurement model for range and azimuth is as follows,

$$r = \sqrt{(x - x_s)^2 + (y - y_s)^2 + (z - z_s)^2}$$
(45)

and  $\phi$  defined as in (23). As in the section on bearings only (Sect. III), the measurements are taken d apart with time difference T. Since we have  $k\eta_z > \eta_{x_o}$ , we will be able to choose four different functions  $t_i$ .

### B. Two Azimuths and One Range

Using  $\{r_1, \phi_1, \phi_2\}$  we get the same results for x, y as in the previous section, see (31),(32). Solving for z we get

$$z = \sqrt{r_1^2 - (x + d/2)^2 - y^2}$$
$$= \sqrt{r_1^2 - \left(d\frac{\sin\phi_2}{\sin(\phi_2 - \phi_1)} - T\frac{\dot{x}\sin\phi_2 - \dot{y}\cos\phi_2}{\sin(\phi_2 - \phi_1)}\right)^2} \quad (46)$$

and accordingly  $t_1(\mathbf{z} - \mathbf{w})$  varies from (34) only in the last component.

The linearizations necessary to compute the covariance can be found in App II.

To use the Unscented Transform (UT) we need g. In this case the function is readily available in (31), (32) and (46), which leads to immediate applicability.

The function  $t_2$  can be generated from  $t_1$ , by replacing  $r_1 \leftrightarrow r_2$ ,  $\phi_1 \leftrightarrow \phi_2$  and  $d \leftrightarrow -d$  as the setup is symmetric again. To calculate the covariance also  $T \leftrightarrow -T$  has to be exchanged.

# C. One Azimuth and Two Ranges

To calculate  $t_2(\mathbf{z} - \mathbf{w})$ , we use  $\{r_1, r_2, \phi_1\}$ . First we use  $r_1, r_2$  which is geometrically speaking the intersection of two spheres. Solving for x and taking the expectation over  $\dot{x}, \dot{y}, \dot{z}$ , we get,

$$x = \frac{r_1^2 - r_2^2 + (2\sigma_v^2 + \sigma_h^2)T^2}{2d}.$$
 (47)

The solution for x is unambiguous since the intersection is a circle normal to and centered on the line connecting the centers of the spheres, which coincides with the x-axis. To solve for y, we use the definition of  $\phi$ ,

$$y = (x + d/2) \tan \phi_1$$
  
=  $\frac{r_1^2 - r_2^2 + (2\sigma_v^2 + \sigma_h^2)T^2 + d^2}{2d} \tan \phi_1$  (48)

and similarly the definition of r to solve for z,

$$z = \sqrt{r_1^2 - (x + d/2)^2 - y^2}$$
  
=  $\sqrt{r_1^2 - \left(\frac{r_1^2 - r_2^2 + (2\sigma_v^2 + \sigma_h^2)T^2 + d^2}{2d}\right)^2 (1 + \tan^2 \phi_1)},$  (49)

where we disregard the ambiguity towards  $\pm z$ , since we can assume positive z. Finally the complete function is,

$$t_3(\mathbf{z} - \mathbf{w}) = \begin{pmatrix} \rho_1 \cos \phi_1 - d/2 \\ \rho_1 \sin \phi_1 \\ \sqrt{r_1^2 - \rho_1^2} \end{pmatrix}$$
(50)

with

$$\rho_1 = \frac{r_1^2 - r_2^2 + (2\sigma_v^2 + \sigma_h^2)T^2 + d^2}{2d\cos\phi_1}.$$
(51)

We can see that if the z component turns complex, there is no solution. This usually happens if the radii don't render an intersection of the two spheres or the azimuth is off too far, due to measurement errors or the unknown speeds.

The linearizations to calculate the covariance can be found in App. II.

The function  $t_4$  can be generated from  $t_3$  again, by replacing  $r_1 \leftrightarrow r_2$ ,  $\phi_1 \leftrightarrow \phi_2$  and  $d \leftrightarrow -d$  due to the symmetric setup. To calculate the covariance also  $T \leftrightarrow -T$  has to be exchanged.

## V. NUMERICAL RESULTS

#### A. Bearings-Only Measurements

As described in Section III, the sensors are located at  $\pm d/2$ , where we choose d = 10km for our numerical example. We plot results for an symmetric x/y half-plane of 40km by 20km and pick a constant height for each simulation. The two sets of measurements needed in this scenario are taken with an arbitrary time difference T which is usually in the range of a few seconds and the target is assumed to move with constant speed within this time interval. The speeds are random with  $\dot{x}, \dot{y}, \dot{z}$  assumed independent and  $\dot{x}, \dot{y}$ 



Fig. 2. Root-mean-square position error (RMSPOS) for an medium target height z = 4km and synchronous measurements T = 0s.



Fig. 3. Root-mean-square position error (RMSPOS) for target height z = 4km and asynchronous measurements T = 2.



Fig. 4. Cramer-Rao lower bound (CRLB) for target height of z = 4km and asynchronous measurements T = 2s.

with the same  $\sigma_v = 100$  m/s and  $\dot{z}$  with  $\sigma_h = 10$  m/s. Other parameters are the measurement noises,  $\sigma_{\phi}$ ,  $\sigma_{\theta}$ , which are both  $2 \times 10^{-3}$  rad  $\approx 0.1^{\circ}$  and the number of Monte-Carlo runs  $N = 10^3$ . The covariance are generated using the Extended Kalman filter linearization.

For T = 0 the measurements are not dependent on the unknown speed. The root-mean-square position estimation error (RMSPOS) derived by Monte-Carlo simulation for a height of z = 4km is plotted in Fig. 2. The RMSPOS is minimal in close proximity to the sensors; it increases with distance to the sensors, considerably less so on the y-axis, when the base spanned by sensors is orthogonal to the bearings.

When plotting the RMSPOS for non-zero T (see Fig. 3), the average error is noticeably higher, due to the influence of the unknown velocity. As a comparison we plot the CRLB for these values in Fig. 4. It can be observed that the estimation error meets the lower bound very well, so the increased estimation error variance is inherent to the problem setup.

The consistency is evaluated using the normalized esti-



Fig. 5. Normalized estimation error squared (NEES) for target height z = 4km and asynchronous measurements T = 2s. The 95% acceptance region is [5, 787; 6, 216], the covariance is about 10% too pessimistic.

mation error squared (NEES), which is calculated using the full estimation error vector. Even though  $\bar{\mathbf{x}}_o$  is not actually estimated, we include it in the NEES calculation to check the consistency of the cross-correlation. Accordingly the NEES should have the distribution of a chi-square distribution with  $\eta_{\mathbf{x}}N = 6 \times 10^3$  degrees of freedom. The NEES is plotted in Fig. 5; it is slightly below the 95% acceptance region, which makes our estimated covariance pessimistic. On top of the sensor positions the estimates are inconsistent, which is a problem of calculating the covariance with zero ground range.

For larger heights, e.g., z = 8km the error is more even. It is higher around the sensors, since the minimum distance is limited by the increased height. Also it rises less on the *x*-axis, since it has now higher elevation - which improves the performance initializing from one azimuth and two elevation.

Decreasing the height to z = 1km, the errors increase; especially on the x-axis, outside the sensors. This is a problem inherent to the estimation geometry; on the x-axis in general  $t_1, t_2$  don't work, since intersecting two glancing azimuth angles makes the estimation accuracy tend towards zero. For low heights,  $t_3, t_4$  work badly on the x-axis outside the sensors, since now the measured elevation angles glance.

#### B. Range and Azimuth Measuements

The setup for position initialization from two measurements is basically the same as in Section V-A. The sensors are on the x-axis with distance d = 10km, we simulate over a halfplane, the assumed distribution of the velocity is the same, noise variance is  $\sigma_{\phi} \approx 0.1^{\circ}$  and  $\sigma_r = 65$ m. We used the Unscented Transform (UT) to calculate the covariance.

We will look at a larger height of z = 8km and time offset of T = 2s. The RMSPOS is plotted in Fig. 6; for the chosen range standard deviation of  $\sigma_r = 65$ m and same azimuth precision, the estimation error is generally higher compared to the bearings-only scenario. Especially for positions far from the sensors the errors increase much stronger. As a comparison we plot the CRLB in Fig. 7. We observe that the stronger



Fig. 6. Root-mean-square position error (RMSPOS) for larger target height z = 8km and asynchronoeous (T = 2s) measurements of range and azimuth.



Fig. 7. Cramer-Rao lower bound for target height z = 8km and asynchroneous (T = 2s) measurements of range and azimuth.

increase is inherent to the setup as we can not go beneath the bound, but in corners where the errors increase strongly, our estimation errors do not achieve the CRLB.

Fig. 8 shows the consistency. It is less pessimistic, but there are regions where the very large errors lead to inconsistencies, coinciding with where the estimation error doesn't achieve the CRLB.

For lower heights, e.g., z = 4km, the errors increase, in particular in the regions of high errors. Again this is inherent to the probelm geometry can not be improved upon, which we validify via CRLB analysis. Therefore in this scenario our approach has only limited applicability. It works well for short ranges and high elevations, while for large range and low elevations a 2D approximations is justified.

# C. Implementing the Covariance Matrix

Linearization and Unscented Transform are both methods to handle non-linerarities. Sometimes none of these methods is able to generate a suited covariance matrix. Generally for big measurement errors and/or strong abberation of the modeled parameter from the expected parameter, we notice



Fig. 8. Normalized estimation error squared (NEES) for target height z = 8km and asynchroneous (T = 2s) of range and azimuth.

enlarging regions which become inconsistent. The Unscented Transform seems to work slightly better with this problems than linearization, which can be observed in the case of bearings only measurements with larger measurement errors  $\sigma_{\phi} = \sigma_{\theta} = 1^{o}$  and in the case of range and azimuth measurements.

# VI. CONCLUSION

We provided a general approach for initialization, applicable to a large range of estimation problems using incomplete measurements. The approach allows modeling some parameters statistically to reduce the dimension of the estimation problem. This way we can initialize also in scenarios where observability would otherwise not be given. A covariance matrix has been derived analytically to describe the estimation errors. The covariance accounts explicitly for uncertainties related to reducing the estimation problem which is our main contribution. We included a detailed discussion and numerical analysis of examples using Cramer-Rao lower bounds and Monte Carlo Simulation. The examples included 3D position initialization scenarios of bearings-only and slant-range and azimuth.

## APPENDIX I

# LINEARIZATIONS FOR BEARINGS-ONLY SCENARIO

A. Two Azimuth, One Elevation

Linearizing (34) we get

$$\frac{\partial t_1}{\partial \mathbf{z} - \mathbf{w}} = \begin{pmatrix}
d \frac{\sin \phi_2 \cos \phi_2}{\sin^2(\phi_2 - \phi_1)} & 0 & -d \frac{\sin \phi_1 \cos \phi_1}{\sin^2(\phi_2 - \phi_1)} & 0 \\
d \frac{\sin^2(\phi_2)}{\sin^2(\phi_2 - \phi_1)} & 0 & -d \frac{\sin^2(\phi_1)}{\sin^2(\phi_2 - \phi_1)} & 0 \\
\frac{\partial \rho_1}{\partial \phi_1} \tan \theta_1 & \rho_1 / \cos^2 \theta_1 & \frac{\partial \rho_1}{\partial \phi_2} \tan \theta_1 & 0
\end{pmatrix} (52)$$

where 
$$\rho_1 = \sqrt{(x + \frac{d}{2})^2 + y^2}$$
 and  
 $\frac{\partial \rho_1}{\partial \phi_i} = \frac{1}{\rho_1} \left( \frac{\partial x}{\partial \phi_i} (x + d/2) + \frac{\partial y}{\partial \phi_i} y \right) , i = 1, 2$ 

Instead of taking the derivative of h towards  $\mathbf{x}$  to apply the chain rule, we can directly take the derivative of (31)-(33). Since  $G = \frac{\partial(t \circ h)}{\partial \bar{\mathbf{x}}_o}$ , we only need to take the derivative towards  $\bar{\mathbf{x}}_o$  which the equations are already linear in. It is easy to see that

$$\frac{\partial t_1 \circ h}{\partial \bar{\mathbf{x}}_o} = -\frac{T}{\sin(\phi_2 - \phi_1)} \times \begin{pmatrix} \cos \phi_1 \\ \sin \phi_1 \\ \tan \theta_1 \end{pmatrix} \left( \begin{array}{c} \sin \phi_2 & -\cos \phi_2 & 0 \end{array} \right)$$
(53)

which gives everything necessary to calculate the covariance according to (16).

#### B. One Azimuth, Two Elevation

We calculate the linearization  $\frac{\partial t_3}{\partial \mathbf{z} - \mathbf{w}}$  and  $\frac{\partial t_3 \circ h}{\partial \mathbf{x}}$ . (35)-(37), which all depend on  $\rho_1$  from (40), have the linearization in  $\mathbf{z} - \mathbf{w}$  given in (54), with the following partial derivatives

$$\frac{\partial \rho_1}{\partial \phi_1} = \frac{dq^2 \sin \phi_1 \rho_1}{dq^2 \cos \phi_1 + (1 - q^2)\rho_1} \tag{55}$$

$$\frac{\partial \rho_1}{\partial \theta_i} = \frac{\rho_1^2 - 2d\cos\phi_1\rho_1 + d^2}{dq^2\cos\phi_1 + (1 - q^2)\rho_1} q \frac{\partial q}{\partial \theta_i} \ i = 1, 2$$
(56)

$$\frac{\partial q}{\partial \theta_1} = -\frac{\tan \theta_2}{\sin^2 \theta_1}, \quad \frac{\partial q}{\partial \theta_2} = \frac{1}{\tan \theta_1 \cos^2 \theta_2}.$$
(57)

To linearize  $t_3 \circ h$  in  $\bar{\mathbf{x}}_o$  we will linearize (39) around T = 0. We find

$$\frac{\partial \rho_1}{\partial T} = -\frac{\rho_1 \cot \theta_1 \dot{z} - q^2 \rho_1 (\cos \phi_1 - d) \dot{x} - q^2 \rho_1 \sin \phi_1 \dot{y}}{\rho_1 (1 - q^2) + (q^2 \cos \phi_1 d)}$$
(58)

which is linear in  $\dot{x}, \dot{y}$  and  $\dot{z}$ . For small T we can approximate  $\rho_1$  by  $\rho_1 = \rho_{1|_{T=0}} + T \frac{\partial \rho_1}{\partial T}$ . Defining

$$N = \rho_1 (1 - q^2) + (q^2 \cos \phi_1 d)$$
(59)

and using (35)-(37), the result is

$$\frac{\partial t_3 \circ h}{\partial \bar{\mathbf{x}}_o} = -T \begin{pmatrix} \cos \phi_1 \\ \sin \phi_1 \\ \tan \theta_1 \end{pmatrix} \times \left( -\frac{q^2 \rho_0 (\cos \phi_0 - d)}{N} - \frac{q^2 \rho_0 \sin \phi_1}{N} - \frac{\rho_0 \cot \theta_1}{N} \right)$$
(60)

where  $\rho_0 = \rho_{1|_{T=0}}$  is the solution to (40).

#### APPENDIX II

#### LINEARIZATIONS FOR RANGE AND AZIMUTH SCENARIO

#### A. Two Azimuth, One Range

To calculate the covariance matrix, we already have most derivatives from the previous section, but still need to linearize (46). The result for  $\frac{\partial t_1}{\partial \mathbf{z} - \mathbf{w}}$  is

$$\frac{\partial t_1}{\partial \mathbf{z} - \mathbf{w}} = \begin{pmatrix} d \frac{\sin \phi_2 \cos \phi_2}{\sin^2(\phi_2 - \phi_1)} & 0 & -d \frac{\sin \phi_1 \cos \phi_1}{\sin^2(\phi_2 - \phi_1)} & 0 \\ d \frac{\sin^2(\phi_2)}{\sin^2(\phi_2 - \phi_1)} & 0 & -d \frac{\sin^2(\phi_1)}{\sin^2(\phi_2 - \phi_1)} & 0 \\ -\frac{d^2}{z} \frac{\sin^2(\phi_2) \cos(\phi_2 - \phi_1)}{\sin^3(\phi_2 - \phi_1)} & \frac{r_1}{z} & -\frac{d^2}{z} \frac{\sin \phi_1 \sin \phi_2}{\sin^3(\phi_2 - \phi_1)} & 0 \end{pmatrix}$$
(61)

where for z we have to use our estimate. For  $\frac{\partial t_1 \circ h}{\partial \bar{\mathbf{x}}_1}$ 

$$\frac{\partial t_1 \circ h}{\partial \bar{\mathbf{x}}_o} = \frac{T}{\sin(\phi_2 - \phi_1)} \times \begin{pmatrix} \cos \phi_1 \\ \sin \phi_1 \\ -\frac{d \sin \phi_2}{z \sin(\phi_2 - \phi_1)} \end{pmatrix} (\sin \phi_2 - \cos \phi_2 - 0) \quad (62)$$

## B. One Azimutn, Two Range

For the  $\{r_1, r_2, \phi_1\}$  case the derivatives to calculate the covariance are,

$$\frac{\partial t_3}{\partial \mathbf{z} - \mathbf{w}} = \begin{pmatrix} 0 & \frac{r_1}{d} & 0 & -\frac{r_2}{d} \\ \frac{\rho_1}{\cos \phi_1} & \frac{r_1}{d} \tan \phi_1 & 0 & -\frac{r_2}{d} \tan \phi_1 \\ \frac{\rho_1^2 \tan \phi_1}{z \cos^2 \phi_1} & \frac{r_1}{z_0} \left( 1 - \frac{\rho_1}{2d \cos \phi_1} \right) & 0 & \frac{r_2}{z_0} \frac{\rho_1}{2d \cos \phi_1} \end{pmatrix} (63)$$

$$\frac{\partial t_3 \circ h}{\partial \bar{\mathbf{x}}_o} = \frac{T}{2d} \begin{pmatrix} 1\\ \tan \phi_1\\ -\frac{x+d/2}{z}(1+\tan^2 \phi_1) \end{pmatrix} \times \begin{pmatrix} d-2x & -2y & -2z \end{pmatrix}$$
(64)

where for  $\rho$  and x, y, z the estimated values have to be used.

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$$\frac{\partial t_3}{\partial \mathbf{z} - \mathbf{w}} = \begin{pmatrix} \frac{\partial \rho_1}{\partial \phi_1} \cos \phi_1 - \rho_1 \sin \phi_1 & \frac{\partial \rho_1}{\partial \theta_1} \cos \phi_1 & 0 & \frac{\partial \rho_1}{\partial \theta_2} \cos \phi_1 \\ \frac{\partial \rho_1}{\partial \phi_1} \sin \phi_1 + \rho_1 \cos \phi_1 & \frac{\partial \rho_1}{\partial \theta_1} \sin \phi_1 & 0 & \frac{\partial \rho_1}{\partial \theta_2} \sin \phi_1 \\ \frac{\partial \rho_1}{\partial \phi_1} \tan \theta_1 & \frac{\partial \rho_1}{\partial \theta_1} \tan \theta_1 + \frac{\rho_1}{\cos^2 \theta_1} & 0 & \frac{\partial \rho_1}{\partial \theta_2} \tan \theta_1 \end{pmatrix}$$
(54)