FICTITIOUS PLAY IN POTENTIAL GAMES

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Abstract. This work studies the convergence properties of continuous-time fictitious play in potential games. It is shown that in almost every potential game and for almost every initial condition, fictitious play converges to a pure-strategy Nash equilibrium. We focus our study on the class of regular potential games; i.e., the set of potential games in which all Nash equilibria are regular. As byproducts of the proof of our main result we show that (i) a regular mixed-strategy equilibrium of a potential game can only be reached by a fictitious play process from a set of initial conditions with Lebesgue measure zero, and (ii) in regular potential games, solutions of fictitious play are unique for almost all initial conditions.

Key words. Game theory, Learning, Potential games, Fictitious play, Multi-agent systems, Best-response dynamics

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1. Introduction. A Nash equilibrium (NE) is a solution concept for multi-player games in which no player can unilaterally improve their personal utility. Formally, a Nash equilibrium is defined as a fixed point of the best response mapping—that is, a strategy $x^*$ is said to be a NE if

$$x^* \in \text{BR}(x^*),$$

where BR denotes the best response mapping (see Section 2 for a formal definition).

A question of fundamental interest is, given the opportunity to interact, how might a group of players adaptively learn to play a NE strategy over time? In response, it is natural to consider the dynamical system induced by the best response mapping itself:

$$\dot{x} \in \text{BR}(x) - x.$$ (1)

By definition, the set of NE coincide with the equilibrium points of these dynamics. Historically, these dynamics are known as fictitious play (FP). After a time change and a discretization, the dynamics (1) yield the familiar (discrete-time) FP process first introduced by Brown in [5] (see, e.g., [4,11,17]).

FP is known to converge to the set of NE in several important classes of games [4, 11, 23, 25, 26], though not all games [14, 27]. Of particular interest to this work are results showing that FP converges to the set of NE in a class of games known as potential games [3, 24, 25].

In a potential game, there exists an underlying potential function which all players implicitly seek to maximize. Such games are fundamentally cooperative in nature (all players benefit by maximizing the potential), and have many important applications in engineering and economics [19, 24]. Along with so called harmonic games (which are fundamentally adversarial in nature), potential games may be seen as one of the basic building blocks of general $N$-player games [6].

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1When we say that FP converges to some set, we mean that solutions of (1) converge to that set.
The set of NE may be subdivided into pure-strategy (deterministic) NE and mixed-strategy (probabilistic) NE. Mixed-strategy NE can be problematic for a number of reasons [14]. In engineering applications involving potential games, mixed-strategy NE can be undesirable since they are nondeterministic, have suboptimal expected utility, and do not always have clear physical meaning [2]. Consequently, in applications, preference is generally given to algorithms that are guaranteed to converge to a pure-strategy NE [2,18,20,21].

For FP learning dynamics, mixed-strategy NE are problematic at an even more fundamental level (both in and out of potential games), since they can cause non-uniqueness of solutions and make it impossible to establish general convergence rate estimates [11].

It has been speculated that, for FP learning dynamics, the difficulties arising due to mixed equilibria rarely occur in practice. For example, in [2] it is noted that “it is generally believed that convergence of [FP] to a mixed (but not pure) Nash equilibrium happens rarely when [players’] utilities are not equivalent to a zero-sum game”. Despite such speculation, there are currently no rigorous results showing that FP behaves in this manner (e.g., only reaching mixed equilibria from a null set of initial conditions) in any particular class of games.

An important benefit of potential games is that they guarantee the existence of pure-strategy NE. Nevertheless, it is well known that FP can converge to a mixed-strategy equilibrium in such games. In fact, this deficiency was first noted in the paper where it was originally proven that FP converges to NE in potential games (see [25], Remark (2)).

In this paper, we refine the convergence result for FP in potential games in an attempt to redeem it somewhat in this regard and address the issues noted above. The following theorem is our main result.

**Theorem 1.** In almost all potential games, and for almost all initial conditions, FP converges to a pure-strategy NE.

In particular, we show that in any regular potential game, FP converges to a pure-strategy NE from almost every initial condition. The notion of a regular game was introduced by Harsanyi [12]. In a companion paper [28], we show that almost all potential games are regular.

Three important byproducts of the proof of Theorem 1 are (i) in a regular potential game, FP can only reach the set of mixed equilibria in finite time from a null set of initial conditions (see Proposition 15), (ii) in a regular potential game, if FP converges to the set of mixed-strategy equilibria, then it does so in finite time (see Proposition 22), and (iii) except for a null set of initial conditions, solutions of FP are unique in any regular potential game (see Lemma 17 and Remark 21).

**1.1. Related Work.** A related result for two-player games has been shown in [15], where it was demonstrated that FP almost never converges cyclically to a mixed-strategy equilibrium in which both players use more than two pure-strategies. In this paper we focus on $N$-player potential games and we show that, in such games, convergence to mixed equilibria is generically impossible.

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Footnote 2: As another example, in [13] it is noted that “for bimatrix games it seems this difficulty (finite-time convergence to mixed equilibria and non-uniqueness of solutions) occurs only for a null set of initial conditions. But this needs a proof!”.

Footnote 3: We note that this partially resolves a conjecture made in [13], that, for regular mixed equilibria in bimatrix games, these issues can only occur from a null set of initial conditions. See [13], footnote 8 and surrounding discussion.
Continuous-time best-response dynamics similar to those we consider here have been studied in various works, including [4, 11, 13, 22]. These papers study a variety of convergence properties for FP and replicator dynamics in a wide class of games, but do not consider the question of generic convergence to pure-strategy equilibria.

As noted earlier, the problem of characterizing the rate of convergence in FP, as studied in [11], is closely related to our work here. The mere existence of mixed-strategy equilibria presents a fundamental barrier to establishing precise convergence rate estimates in FP. This issue is partially resolved by the present work since we show that mixed-strategy NE can only be reached in finite time from a null set of initial conditions (see Proposition 15.) We intend to further address the issue of characterizing the rate of convergence of FP in potential games in a future work.

1.2. Proof Strategy. The basic strategy is to leverage two noteworthy properties satisfied in almost every potential game:

1. The FP vector field cannot concentrate mass in finite time, meaning it cannot map a set of positive (Lebesgue) measure to a set of zero measure in finite time (see Section 5.1).

2. In a neighborhood of an interior Nash equilibrium (i.e., a completely mixed equilibrium), the magnitude of the time derivative of the potential along paths grows linearly in the distance to the Nash equilibrium, while the value of the potential varies only quadratically; that is,

\[
\frac{d}{dt}U(x(t)) \geq d(x(t), x^*) \geq \sqrt{|U(x(t)) - U(x^*)|},
\]

where \(U\) denotes the potential function and \(x^*\) is the equilibrium point.

Using Markov’s inequality, property 2 immediately implies that if a path converges to an interior NE then it must do so in finite time (see Section 5.2). Hence, properties 1 and 2 together imply that the set of points from which FP converges to an interior NE must have Lebesgue measure zero.

In order to handle mixed NE that are not in the interior of the strategy space (i.e., incompletely mixed equilibria), we consider a projection that maps incompletely mixed equilibria to the interior of the strategy space of a lower dimensional game. Using the techniques described above we are then able to handle completely and incompletely mixed equilibria in a unified manner.

In particular, we see that the set of points from which FP converges to the set of mixed-strategy NE has Lebesgue measure zero. Since any FP path must converge to a NE [4], this implies Theorem 1.

Properties 1 and 2 hold as long the equilibrium \(x^*\) is regular. In a companion paper we show that, in almost all potential games, all equilibria are regular (see Theorem 4 below).

The remainder of the paper is organized as follows. Section 2 sets up notation. Section 3 discusses regular potential games. Section 4 establishes the two key inequalities used to prove Theorem 1, and Section 5 gives the proof of Theorem 1.

1.3. Comparison with Classical Techniques. Given a classical ODE, one can prove that an equilibrium point may only be reached from a set of measure zero by studying the linearized dynamics at the equilibrium point. Assuming all eigenvalues associated with the linearized system are non-zero, the dimension of the
stable manifold (i.e., the set of initial conditions from which the equilibrium can be reached) is equal to the dimension of the stable eigenspace of the linearized system [8]. Hence, to prove that an equilibrium can only be reached from a set of measure zero, it is sufficient to prove that at least one eigenvalue of the linearized system lies in the right half plane.

In FP, the vector field is discontinuous—hence, it is not possible to linearize around an equilibrium point, and such classical techniques cannot be directly applied. However, the gradient field of the potential function may be seen as an approximation of the FP vector field (see Lemma 8). Unlike the FP vector field, the gradient field of the potential function can be linearized. In a non-degenerate game, any completely mixed-strategy NE is a non-degenerate saddle point of the potential function. Hence, at least one eigenvalue of the linearized gradient system must lie in the right-half plane. This implies that, for the gradient dynamics of the potential function, the stable manifold associated with an equilibrium point has dimension at most $\kappa - 1$, where $\kappa$ is the dimension of the strategy space.

Since the FP vector field approximates the gradient field of the potential function, intuition suggests that for FP dynamics, each mixed equilibrium should also admit a similar low-dimensional stable manifold.

While this provides an intuitive explanation for why one might expect Theorem 1 to hold, we did not use any such linearization arguments in the proof of this result. We found that studying the rate of potential production near mixed equilibria (e.g., as discussed in the “proof strategy” section above) led to shorter and simpler proofs.

2. Preliminaries.

2.1. Notation. A game in normal form is represented by the tuple $
 \Gamma := (N, (Y_i, u_i)_{i=1,\ldots,N}), \n$ where $N \in \{2, 3, \ldots\}$ denotes the number of players, $Y_i = \{y_1^i, \ldots, y_{K_i}^i\}$ denotes the set of pure strategies (or actions) available to player $i$, with cardinality $K_i := |Y_i|$, and $u_i : \prod_{j=1}^{N} Y_j \to \mathbb{R}$ denotes the utility function of player $i$. Denote by $Y := \prod_{i=1}^{N} Y_i$ the set of joint pure strategies, and let $K := \prod_{i=1}^{N} K_i$ denote the number of joint pure strategies.

For a finite set $S$, let $\triangle(S)$ denote the set of probability distributions over $S$. For $i = 1, \ldots, N$, let $\Delta_i := \triangle(Y_i)$ denote the set of mixed-strategies available to player $i$. Let $\Delta := \prod_{i=1}^{N} \Delta_i$ denote the set of joint mixed strategies.\footnote{It is implicitly assumed that players’ mixed strategies are independent; i.e., players do not coordinate.} Let $\Delta_{-i} := \prod_{j \in \{1,\ldots,N\} \setminus \{i\}} \Delta_j$. When convenient, given a mixed strategy $\sigma = (\sigma_1, \ldots, \sigma_N) \in \Delta$, we use the notation $\sigma_{-i}$ to denote the tuple $(\sigma_j)_{j \neq i}$. Given a mixed strategy $\sigma \in \Delta$, the expected utility of player $i$ is given by

\[
U_i(\sigma_1, \ldots, \sigma_N) = \sum_{y \in Y} u_i(y) \sigma_1(y_1) \cdots \sigma_N(y_N).
\]

For $\sigma_{-i} \in \Delta_{-i}$, the best response of player $i$ is given by the set-valued function $\text{BR}_i : \Delta_{-i} \rightrightarrows \Delta_i$,

\[
\text{BR}_i(\sigma_{-i}) := \arg \max_{\sigma_i \in \Delta_i} U_i(\sigma_i, \sigma_{-i}),
\]

and for $\sigma \in \Delta$ the joint best response is given by the set valued function $\text{BR} : \Delta \rightrightarrows \Delta$

\[
\text{BR}(\sigma) := \text{BR}_1(\sigma_{-1}) \times \cdots \times \text{BR}_N(\sigma_{-N}).
\]
A strategy $\sigma \in \Delta$ is said to be a Nash equilibrium (NE) if $\sigma \in \text{BR}_i(\sigma)$. For convenience, we sometimes refer to a Nash equilibrium simply as an equilibrium.

We say that $\Gamma$ is a potential game [24] if there exists a function $u : Y \rightarrow \mathbb{R}$ such that $u_i(y'_i, y_{-i}) - u_i(y''_i, y_{-i}) = u(y'_i, y_{-i}) - u(y''_i, y_{-i})$ for all $y_{-i} \in Y_{-i}$ and $y'_i, y''_i \in Y_i$, for all $i = 1, \ldots, N$.

Let $U : \Delta \rightarrow \mathbb{R}$ be the multilinear extension of $u$ defined by

$$U(\sigma_1, \ldots, \sigma_N) = \sum_{y \in Y} u(y) \sigma_1(y_1) \cdots \sigma_N(y_N).$$

The function $U$ may be seen as giving the expected value of $u$ under the mixed strategy $\sigma$. We refer to $U$ as the potential function and to $u$ as the pure form of the potential function.

Using the definitions of $U_i$ and $U$ it is straightforward to verify that

$$\text{BR}_i(\sigma_{-i}) := \arg \max_{\sigma_i \in \Delta_i} U_i(\sigma_i, \sigma_{-i}) = \arg \max_{\sigma_i \in \Delta_i} U(\sigma_i, \sigma_{-i}).$$

Thus, in order to compute the best response set we only require knowledge of the potential function $U$, not necessarily the individual utility functions $(U_i)_{i=1,\ldots,N}$.

By way of notation, given a pure strategy $y_i \in Y_i$ and a mixed strategy $\sigma_{-i} \in \Delta_{-i}$, we will write $U(y_i, \sigma_{-i})$ to indicate the value of $U$ when player $i$ uses a mixed strategy placing all weight on the $y_i$ and the remaining players use the strategy $\sigma_{-i} \in \Delta_{-i}$.

Given a $\sigma_i \in \Delta_i$, let $\sigma^j_i$ denote value of the $k$-th entry in $\sigma_i$, so that $\sigma_i = (\sigma^j_i)_{k=1}$. Since the potential function is linear in each $\sigma_i$, if we fix any $i = 1, \ldots, N$ we may express it as

$$U(\sigma) = \sum_{k=1}^{K_i} \sigma^k_i U(y^k_i, \sigma_{-i}).$$

In order to study learning dynamics without being (directly) encumbered by the hyperplane constraint inherent in $\Delta_i$ we define

$$X_i := \{x_i \in \mathbb{R}^{K_i-1} : 0 \leq x^k_i \leq 1 \text{ for } k = 1, \ldots, K_i - 1, \text{ and } \sum_{k=1}^{K_i-1} x^k_i \leq 1\},$$

where we use the convention that $x_i^k$ denotes the $k$-th entry in $x_i$ so that $x_i = (x_i^k)_{k=1}^{K_i-1}$.

Given $x_i \in X_i$ define the bijective mapping $T_i : X_i \rightarrow \Delta_i$ as $T_i(x_i) = \sigma_i$ for the unique $\sigma_i \in \Delta_i$ such that $\sigma^k_i = x^k_i$ for $k = 2,\ldots,K_i$ and $\sigma^1_i = 1 - \sum_{k=1}^{K_i-1} x^k_i$. For $k = 1,\ldots,K_i$ let $T^k_i$ be the $k$-th component map of $T_i$ so that $T_i = (T^k_i)_{k=1}^{K_i}$.

Let $X := X_1 \times \cdots \times X_N$ and let $T : X \rightarrow \Delta$ be the bijection given by $T = T_1 \times \cdots \times T_N$. In an abuse of terminology, we sometimes refer to $X$ as the mixed-strategy space of $\Gamma$. When convenient, given an $x \in X$ we use the notation $x_{-i}$ to denote the tuple $(x_j)_{j \neq i}$. Letting $X_{-i} := \prod_{j \neq i} X_j$, we define $T_{-i} : X_{-i} \rightarrow \Delta_{-i}$ as $T_{-i} := (T_j)_{j \neq i}$. Let

$$\kappa := \sum_{i=1}^{N} (|Y_i| - 1)$$

denote the dimension of $X$, and note that $\kappa \neq K$, where $K$, defined earlier, is the cardinality of the joint pure strategy set $Y$. 

Throughout the paper we often find it convenient to work in $X$ rather than $\Delta$. In order to keep the notation as simple as possible we overload the definitions of some symbols when the meaning can be clearly derived from the context. In particular, let $BR_i : X_{-i} \Rightarrow X_i$ be defined by $BR_i(x_{-i}) := \{ x_i \in X_i : BR_i(\sigma_{-i}) = \sigma_i, \sigma_i \in \Delta_i, \sigma_{-i} \in \Delta_{-i}, \sigma_i = T_i(x_i), \sigma_{-i} = T_{-i}(x_{-i}) \}$. Similarly, given an $x \in X$ we abuse notation and write $U(x)$ instead of $U(T(x))$.

Given a pure strategy $y_i \in Y_i$, we will write $U(y_i, x_{-i})$ to indicate the value of $U$ when player $i$ uses a mixed strategy placing all weight on the $y_i$ and the remaining players use the strategy $x_{-i} \in X_{-i}$. Similarly, we will say $y_i^k \in BR_i(x_{-i})$ if there exists an $x_i \in BR_i(x_{-i})$ such that $T_i(x_i)$ places weight one on $y_i^k$.

Applying the definition of $T_i$ to (3) we see that $U(x)$ may also be expressed as

$$U(x) = \sum_{k=1}^{K_i} x_i^k U(y_i^{k+1}, x_{-i}) + \left( 1 - \sum_{k=1}^{K_i} x_i^k \right) U(y_i^1, x_{-i}).$$

for any $i = 1, \ldots, N$.

We use the following nomenclature to refer to strategies in $X$.

**Definition 2.** (i) A strategy $x \in X$ is said to be pure if $T(x)$ places all its mass on a single action tuple $y \in Y$.

(ii) A strategy $x \in X$ is said to be completely mixed if $x$ is in the interior of $X$.

(iii) In all other cases, a strategy $x \in X$ is said to be incompletely mixed.

A fictitious play process is defined as follows.

**Definition 3.** An absolutely-continuous mapping $x : \mathbb{R} \rightarrow X$ is said to be a fictitious play process with initial condition $x_0 \in X$ if $x(0) = x_0$ and (1) holds for almost all $t \in \mathbb{R}$.

Other notation as used throughout the paper is as follows.

- $\mathbb{N} := \{1, 2, \ldots\}$.
- $\nabla_x U(x) := (\frac{\partial U}{\partial x_i}(x))_{i=1}^{K_i}$ gives the gradient of $U$ with respect to the strategy of player $i$ only. $\nabla U(x) := (\frac{\partial U}{\partial x_i}(x))_{i=1,\ldots,N}$ gives the full gradient of $U$.
- Suppose $m, n, p \in \mathbb{N}$, $F_i : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$, for $i = 1, \ldots, p$. Suppose further that $F : \mathbb{R}^m \times \mathbb{R}^n \rightarrow \mathbb{R}$ is given by $F(w, z) = (F_i(w, z))_{i=1,\ldots,p}$. Then the operator $D_w F$ gives the Jacobian of $F$ with respect to the components of $w = (w_k)_{k=1,\ldots,m}$; that is

$$D_w F(w, z) = \begin{bmatrix}
\frac{\partial F_1(w, z)}{\partial w_1} & \ldots & \frac{\partial F_1(w, z)}{\partial w_m} \\
\vdots & \ddots & \vdots \\
\frac{\partial F_p(w, z)}{\partial w_1} & \ldots & \frac{\partial F_p(w, z)}{\partial w_m}
\end{bmatrix}.$$
2.2. Almost All Potential Games. The set of potential games in which there are $N$ players, each having an action space of size $K_i$, is isomorphic to $\mathbb{R}^{K^N}$, where $K_p := (\sum_{i=1}^{N} K_i) + N + K - 1$, [28]. We say that a property holds for almost all potential games if the subset of potential games where the property does not hold has $\mathcal{L}^{K^N}$-measure zero.

3. Regular Potential Games. The notion of a regular equilibrium was introduced by Harsanyi [12]. Regular equilibria possess a variety of desirable robustness properties [29].

Being a rather stringent refinement concept, not all games possess regular equilibria. However, “most” games do. A game is said to be regular if all equilibria in the game are regular. Harsanyi [12] showed that almost all $N$-player games are regular.

The set of potential games forms a low dimensional (Lebesgue-measure-zero) subspace within the space of all games. Since the set of potential games is itself a measure zero set, Harsanyi’s regularity result is inconclusive about the prevalence of regular equilibria within this class of games. In a companion paper [28], we study this issue and show the following theorem.

Theorem 4 ([28], Theorem 1). Almost every potential game is regular.

In this paper we will study the behavior of FP in regular potential games. The purpose of this restriction is twofold. First, there are degenerate potential games in which FP does not converge for almost all initial conditions. Restricting attention to regular potential games ensures that the game is not degenerate in this sense. Second, analysis of the behavior of FP is easier near equilibria that are regular. Regularity permits us to characterize the fundamental properties of the potential function $U$ without needing to look at anything higher than second order terms in the Taylor series expansion of $U$. This substantially simplifies the analysis.

If $x^*$ is a regular equilibrium of a potential game, then the derivatives of potential function can be shown to satisfy two non-degeneracy conditions at $x^*$. The first condition deals with the gradient of the potential function at $x^*$ and is referred to as the first-order condition; the second condition deals with the Hessian of the potential function at $x^*$ and is referred to as the second-order condition. These conditions, introduced in Sections 3.1–3.2 below, will be crucial in the subsequent analysis.

3.1. First-Order Degeneracy. Let $\Gamma$ be a potential game. Following Harsanyi [12], we will define the carrier set of an element $x \in X$, a natural modification of a support set to the present context. For $x_i \in X_i$ let

$$carr_i(x_i) := \text{spt}(T_i(x_i)) \subseteq Y_i$$

and for $x = (x_1, \ldots, x_N) \in X$ let $carr(x) := carr_1(x_1) \cup \cdots \cup carr_N(x_N)$.

Let $C = C_1 \cup \cdots \cup C_N$, where for each $i = 1, \ldots, N$, $C_i$ is a nonempty subset of $Y_i$. We say that $C$ is the carrier for $x = (x_1, \ldots, x_N) \in X$ if $C_i = carr_i(x_i)$ for $i = 1, \ldots, N$ (or equivalently, if $C = carr(x)$).

Let $\gamma_i := |C_i|$ and assume that the strategy set $Y_i$, is reordered so that $C_i = \{y_i^1, \ldots, y_i^{\gamma_i}\}$. Under this ordering, the first $\gamma_i - 1$ components of any strategy $x_i$ with $carr_i(x_i) = C_i$ are free (not constrained to zero by $C_i$) and the remaining components of $x_i$ are constrained to zero. That is $(x_i^k)_{k=1}^{\gamma_i - 1}$ is free under $C_i$ and $(x_i^k)_{k=\gamma_i}^{K_i} = 0$. The set of strategies $\{x \in X : carr(x) = C\}$ is precisely the interior of the face of $X$ given by

$$\Omega_C := \{x \in X : x_i^k = 0, \ k = \gamma_i, \ldots, K_i - 1, \ i = 1, \ldots, N\}.$$
Let \( x^* \) be an equilibrium with carrier \( C \). We say that \( x^* \) is \textit{first-order degenerate} if there exists a pair \((i,k), i = 1, \ldots, N, k = \gamma_i, \ldots, K_i - 1\) such that \( \frac{\partial U(x^*)}{\partial x_i^k} = 0 \), and we say \( x^* \) is \textit{first-order non-degenerate} otherwise.

**Remark 5.** We note that using the multi-linearity of \( U \), it is straightforward to verify that an equilibrium is first order non-degenerate if and only if it is quasi-strong, as introduced by Harsanyi [12]. In particular, an equilibrium \( x^* \) is first-order degenerate if and only if \( \text{carr}_i(x^*_i) \subseteq \text{BR}_i(x^*_{-i}) \) for some \( i = 1, \ldots, N \). We prefer to use the term first order non-degenerate since it emphasizes that we are concerned with the gradient of the potential function and it keeps nomenclature consistent with the notion of second-order non-degeneracy, introduced next.

### 3.2. Second-Order Degeneracy

Let \( C \) be some carrier set. Let \( \bar{N} := |\{i = 1, \ldots, N : \gamma_i \geq 2\}| \), and assume that the player set is ordered so that \( \gamma_i \geq 2 \) for \( i = 1, \ldots, \bar{N} \). Under this ordering, for strategies with \( \text{carr}(x) = C \), the first \( \bar{N} \) players use mixed strategies and the remaining players use pure strategies. Assume that \( \bar{N} \geq 1 \) so that any \( x \) with carrier \( C \) is a mixed (not pure) strategy.

Let the Hessian of \( U \) taken with respect to \( C \) be given by

\[
\tilde{H}(x) := \left( \frac{\partial^2 U(x)}{\partial x_i^k \partial x_j^\ell} \right)_{i,j=1,\ldots,\bar{N} \atop k=1,\ldots,\gamma_i-1 \atop \ell=1,\ldots,\gamma_j-1}.
\]

Note that this definition of the Hessian restricts attention to the components of \( x \) that are free under \( C \). We say an equilibrium \( x^* \in X \) is \textit{second-order degenerate} if the Hessian \( \tilde{H}(x^*) \) taken with respect to \( \text{carr}(x^*) \) is singular, and we say \( x^* \) is \textit{second-order non-degenerate} otherwise.

**Remark 6.** Note that both forms of degeneracy are concerned with the interaction of the potential function and the “face” of the strategy space containing the equilibrium \( x^* \). If \( x^* \) touches one or more constraints, then first-order non-degeneracy ensures that the gradient of the potential function is nonzero normal to the face \( \Omega_{\text{carr}(x^*)} \), defined in (6). Second-order non-degeneracy ensures that, restricting \( U \) to the face \( \Omega_{\text{carr}(x^*)} \), the Hessian of \( U \big|_{\Omega_{\text{carr}(x^*)}} \) is non-singular. If \( x^* \) is contained within the interior of \( X \), then the first-order condition becomes moot and the second-order condition reduces to the standard definition of a non-degenerate critical point.

Throughout the paper we will study regular potential games. The following lemma from [28] shows that, in any regular potential game, all equilibria are first and second-order non-degenerate.

**Lemma 7** ([28], Lemma 12). Let \( \Gamma \) be a potential game. An equilibrium \( x^* \) is regular if and only if it is both first and second-order non-degenerate.

### 4. Potential Production Inequalities

In this section we prove two key inequalities ((12) and (13)) that are the backbone of our proof of Theorem 1.

We note that in proving Theorem 1 there is a fundamental dichotomy between studying completely mixed equilibria and incompletely mixed equilibria. Completely mixed equilibria lie in the interior of the strategy space. At these points the gradient of the potential function is zero and the Hessian is non-singular; local analysis of the dynamics is relatively easy. On the other hand, incompletely mixed equilibria necessarily lie on the boundary of \( X \) and the potential function may have a nonzero gradient.
at these points. Analysis of the dynamics around these points is fundamentally more delicate.

In order to handle incompletely mixed equilibria we construct a nonlinear projection whose range is a lower dimensional game in which the image of the equilibrium under consideration is completely mixed. This allows us to handle both types of mixed equilibria in a unified manner.

4.1. Projection to a Lower-Dimensional Game. Let \( x^* \) be a mixed equilibrium.\(^6\) Let \( C_i = \text{carr}_i(x^*_i) \), where \( x^*_i \) is the player-\( i \) component of \( x^* \), let \( C = C_1 \cup \cdots \cup C_N = \text{carr}(x^*) \), and assume that \( Y_i \) is ordered so that \( \{y^1_i, \ldots, y^N_i\} = C_i \). Let \( \gamma_i = |C_i|, \) let \( \hat{N} := \{|i \in \{1, \ldots, N\} : \gamma_i \geq 2\} \), and assume that the player set is ordered so that \( \gamma_i \geq 2 \) for \( i = 1, \ldots, \hat{N} \). Since \( x^* \) is assumed to be a mixed-strategy equilibrium, we have \( \hat{N} \geq 1 \).

Given an \( x \in X \), we will frequently use the decomposition \( x = (x_p, x_m) \), where \( x_m := (x^k_i)_{i=1,\ldots,\hat{N}}, \ k = 1, \ldots, \gamma_i - 1 \) and \( x_p \) contains the remaining components of \( x \).\(^7\) Let \( \gamma := \sum_{i=1}^{\hat{N}} (\gamma_i - 1) \). Recalling that \( \kappa \) is the dimension of \( X \) (see (4)), note that for \( x \in X \) we have \( x \in \mathbb{R}^\kappa, \ x_m \in \mathbb{R}^\gamma, \) and \( x_p \in \mathbb{R}^{\kappa-\gamma} \).

The set of joint pure strategies \( Y \) may be expressed as an ordered set \( Y = \{y^1, \ldots, y^K\} \) where each element \( y^\tau \in Y, \ \tau \in \{1, \ldots, K\} \) is an \( N \)-tuple of strategies. For each pure strategy \( y^\tau \in Y, \ \tau = 1, \ldots, K, \) let \( u^\tau \) denote the pure-strategy potential associated with playing \( y^\tau \); that is, \( u^\tau := u(y^\tau) \), where \( u \) is the pure form of the potential function defined in Section 2. A vector of potential coefficients \( u = (u^\tau)_{\tau=1}^K \) is an element of \( \mathbb{R}^K \).

Given a vector of potential coefficients \( u \in \mathbb{R}^K \) and a strategy \( x \in X \), let\(^8\)

\[
F^k_i(x,u) := \frac{\partial U(x)}{\partial x^k_i},
\]

for \( i = 1, \ldots, \hat{N}, \ k = 1, \ldots, \gamma_i - 1 \), and let

\[
F(x,u) := \left( F^k_i(x,u) \right)_{i=1,\ldots,\hat{N}, \ k = 1, \ldots, \gamma_i - 1} = \left( \frac{\partial U(x)}{\partial x^k_i} \right)_{i=1,\ldots,\hat{N}, \ k = 1, \ldots, \gamma_i - 1}.
\]

Differentiating (5) we see that at the equilibrium \( x^* \) we have \( \frac{\partial U(x^*)}{\partial x^k_i} = 0 \) for \( i = 1, \ldots, \hat{N}, \ k = 1, \ldots, \gamma_i - 1 \) (see Lemma 26 in appendix), or equivalently,

\[
F(x^*,u) = F(x^*_p, x^*_m, u) = 0.
\]

By Definition 2, the (mixed) equilibrium \( x^* \) is completely mixed if \( \gamma = \kappa \), and is incompletely mixed otherwise. Suppose \( \gamma < \kappa \) so that \( x^* \) is incompletely mixed. Let \( J(x) := D_{x_m} F(x_p, x_m, u) \) and note that by definition we have \( J(x^*) = H(x^*) \).

Since \( \Gamma \) is assumed to be a non-degenerate game, \( J(x^*) \) is invertible. By the implicit function theorem, there exists a function \( g : \mathcal{D}(g) \rightarrow \mathbb{R}^\gamma \) such that \( F(x_p, g(x_p), u) \)

\(^5\)We note that in games that are first-order non-degenerate, the gradient is always non-zero at incompletely mixed equilibria.

\(^6\)We note that \( x^* \) is assumed to be fixed throughout the section and many of the subsequently defined terms are implicitly dependent on \( x^* \).

\(^7\)The subscript in \( x_m \) is suggestive of “mixed-strategy components” and the subscript in \( x_p \) is suggestive of “pure-strategy components”.

\(^8\)We note that the functions \( F^k_i \) and \( F \) defined here are identical to those defined in (12) and (13) of [28], and used extensively throughout [28].
The graph of \( g \) is given by

\[
\text{Graph}(g) := \{ x \in X : x = (x_p, x_m), x_p \in \mathcal{D}(g), x_m = g(x_p) \}.
\]

Note that \( \text{Graph}(g) \) is a smooth manifold with Hausdorff dimension \( (\kappa - \gamma) \) [9]. An intuitive interpretation of \( \text{Graph}(g) \) is given in Remark 11.

If \( \Gamma \) is a non-degenerate potential game then, using the multilinearity of \( U \), we see that \( \gamma \geq 2 \) (see Lemma 23 in appendix). This implies that

\[
\text{Graph}(g) \text{ has Hausdorff dimension at most } (\kappa - 2).
\]

Let \( \Omega := \Omega_C \), where \( \Omega_C \) is defined in (6), denote the face of \( X \) containing \( x^* \). Define the mapping \( \tilde{\varphi} : \mathcal{D}(\tilde{\varphi}) \to \Omega \), with domain \( \mathcal{D}(\tilde{\varphi}) := \{ x = (x_p, x_m) \in X : x_p \in \mathcal{D}(g) \} \), as follows. If \( x^* \) is completely mixed then let \( \tilde{\varphi}(x) := x \) be the identity. Otherwise, let

\[
\tilde{\varphi}(x) := x^* + (x - (x_p, g(x_p))).
\]

Let \( \tilde{\varphi}^k_i(x) \) be the \((i, k)\)-th coordinate map of \( \tilde{\varphi} \), so that \( \tilde{\varphi} = (\tilde{\varphi}^k_i)_{i=1,\ldots,N} \). Following the definitions, it is simple to verify that for \( x \in \mathcal{D}(\tilde{\varphi}) \) we have \( \tilde{\varphi}^k_i(x) = 0 \) for all \((i, k)\) with \( k \geq \gamma_i \), and hence \( \tilde{\varphi} \) indeed maps into \( \Omega \).

Let \( \tilde{X}_i := \{ \tilde{x}_i \in \mathbb{R}^{\gamma_i-1} : \tilde{x}_i^k \geq 0, k = 1, \ldots, \gamma_i - 1, \sum_{k=1}^{\gamma_i-1} \tilde{x}_i^k \leq 1 \}, i = 1, \ldots, \tilde{N} \), and let \( \tilde{X} := \tilde{X}_1 \times \cdots \times \tilde{X}_{\tilde{N}} \). Let \( \varphi : \mathcal{D}(\varphi) \to \tilde{X} \) with domain \( \mathcal{D}(\varphi) = \mathcal{D}(\tilde{\varphi}) \subset X \) be given by

\[
\varphi := (\tilde{\varphi}^k_i)_{i=1,\ldots,\tilde{N}, \ k=1,\ldots,\gamma_i-1}.
\]

Note that \( \varphi \) contains the components of \( \tilde{\varphi} \) not constrained to zero. As we will see in the following section, \( \varphi \) may be interpreted as a projection into a lower dimensional game in which \( \varphi(x^*) \) is a completely mixed equilibrium.

**4.2. Inequalities.** Let \( \tilde{U} : \tilde{X} \to \mathbb{R} \) be given by

\[
\tilde{U}(\hat{x}) := U(x^*_p, \hat{x}),
\]

where \( x^* = (x^*_p, x^*_m) \) is the mixed equilibrium fixed in the beginning of the section. Let \( \Gamma \) be a potential game with player set \{1, \ldots, \tilde{N}\}, mixed-strategy space \( \tilde{X}_i \), \( i = 1, \ldots, \tilde{N} \), and potential function \( \tilde{U} \). By construction, \( \varphi(x^*) \) is a completely mixed equilibrium of \( \Gamma \). Moreover, by the definition of a non-degenerate equilibrium, the Hessian of \( \tilde{U} \) is invertible at \( \varphi(x^*) \).

We are interested in studying the projection \( \varphi(x(t)) \) of a FP process into the lower dimensional game \( \Gamma \).\(^9\) We wish to show that the following two inequalities hold: (i) For \( x \) in a neighborhood of \( x^* \)

\[
|\tilde{U}(\varphi(x^*)) - \tilde{U}(\varphi(x))| \leq c_1 d^2(\varphi(x), \varphi(x^*)),
\]

\(^9\)In the lower dimensional game \( \Gamma \), the dynamics of the projected process are not precisely fictitious play dynamics. However, the behave *nearly* like fictitious play dynamics, which is what allows us to establish these inequalities.
for some constant $c_1 > 0$.

(ii) Suppose $(x(t))_{t \geq 0}$ is a FP process. For $x(t)$ residing in a neighborhood of $x^*$

$$
\frac{d}{dt} \hat{U}(P(x(t))) \geq c_2 d(P(x(t)), P(x^*)),
$$

for some constant $c_2 > 0$.

The first inequality follows from Taylor’s theorem and the fact that $\nabla \hat{U}(P(x^*)) = 0$. The following section is devoted to proving (13). \[10\]

4.3. Proving the Differential Inequality. We begin with Lemma 8 which shows—roughly speaking—that within the interior of the action space, the FP vector field approximates the gradient field of the potential function.

The following definitions are useful in the lemma. For $B \subseteq X$, let $P_{X_i}(B) := \{x_i \in X_i: (x_i, x_{-i}) \in B \text{ for some } x_{-i} \in X_{-i}\}$ be the projection of $B$ onto $X_i$. Given an $x_i \in X_i$, let

$$
d(x_i, \partial X_i) := \min\{x_i^1, \ldots, x_i^{K_i-1}, 1 - \sum_{k=1}^{K_i-1} x_i^k\}
$$

denote the distance from $x_i$ to the boundary of $X_i$. Let

$$
d(P_{X_i}(B), \partial X_i) := \inf_{x_i \in P_{X_i}(B)} d(x_i, \partial X_i)
$$
denote the distance between the set $P_{X_i}(B)$ and the boundary of $X_i$.

Since we will eventually be interested in studying a lower-dimensional game derived from $\Gamma$, in the lemma we consider an alternative game $\hat{\Gamma}$ of arbitrary size.

LEMMA 8. Let $\hat{\Gamma}$ be a potential game with player set $\{1, \ldots, \hat{N}\}$, action sets $\hat{Y}_i$, $i = 1, \ldots, \hat{N}$, with cardinality $K_i := |Y_i|$, and potential function $\hat{U}$. Let $\hat{X} = \hat{X}_1 \times \cdots \times \hat{X}_\hat{N}$ denote the mixed strategy space.

Let $B \subset \hat{X}$ and fix $i \in \{1, \ldots, \hat{N}\}$. Then for all $x \in B$ there holds

$$
z_i \cdot \nabla_x \hat{U}(x) \geq c \|\nabla_x \hat{U}(x)\|_1, \quad \forall z_i \in BR_i(x_{-i}) - x_i
$$

where the constant $c$ is given by $c = d(P_{X_i}(B), \partial \hat{X}_i)$.

Proof. Let $x \in B$. If $\|\nabla_x \hat{U}(x)\|_1 = 0$, then $\nabla_x \hat{U}(x) = 0$, and the inequality is trivially satisfied. Suppose from now on that $\|\nabla_x \hat{U}(x)\|_1 > 0$.

Without loss of generality, assume that $Y_i$ is ordered so that

$$
y_i^1 \in BR_i(x_{-i}).
$$

Differentiating (5) we find that\[11\]

$$
\frac{\partial \hat{U}(x)}{\partial x_i^{k+1}} = \hat{U}(y_i^{k+1}, x_{-i}) - \hat{U}(y_i^k, x_{-i}).
$$

\[10\]We note that when we write these inequalities, we mean they are satisfied in an integrated sense (e.g., as used in (41)–(42)). In this section, we treat all of these as pointwise inequalities. A rigorous argument could be constructed using the chain rule in Sobolev spaces (see, for example, [16]).

\[11\]Note that the domain of (expected) potential function $\hat{U}$ may be trivially extended to an open neighborhood around $\hat{X}$ (see Section 2). Using this extension we see that the derivative is well defined for $x$ lying on the boundary of $\hat{X}$. 

Together with (15), this implies that for \( k = 1, \ldots, \tilde{K} \) we have
\[
y^{k+1}_i \in \text{BR}_i(x_{-i}) \iff \frac{\partial \hat{U}(x)}{\partial x^k_i} = 0.
\]

Using the multilinearity of \( \hat{U} \) we see that if \( \xi_i \in \text{BR}_i(x_{-i}) \) and \( \xi_i^k > 0 \) then \( y^{k+1}_i \in \text{BR}_i(x_{-i}) \). But, by (17) this implies that if \( \xi_i \in \text{BR}_i(x_{-i}) \) and \( \xi_i^k > 0 \) then \( \frac{\partial \hat{U}(x)}{\partial x^k_i} = 0 \).

Noting that any \( \xi_i \in \text{BR}_i(x_{-i}) \) is necessarily coordinatewise nonnegative, this gives
\[
(\xi_i - x_i) \cdot \nabla_{x_i} \hat{U}(x) = \sum_{k=1}^{\tilde{K} - 1} \xi_i^k \frac{\partial \hat{U}(x)}{\partial x^k_i} = 0, \quad \xi_i \in \text{BR}_i(x_{-i})
\]

Since we assume \( x \in B \), we have \( x^k_i \geq d(P_{X_i}(B), \partial \tilde{X}_i) \), for all \( k = 1, \ldots, \tilde{K} - 1 \). Since we assume \( y^k_i \in \text{BR}_i(x_{-i}) \), from (16) we get that \( \frac{\partial \hat{U}(x)}{\partial x^k_i} \leq 0 \) for all \( k = 1, \ldots, \tilde{K} - 1 \).

Substituting into (18), this gives
\[
(\xi_i - x_i) \cdot \nabla_{x_i} \hat{U}(x) \geq d(P_{X_i}(B), \partial \tilde{X}_i) \sum_{k=1}^{\tilde{K} - 1} \left( \frac{\partial \hat{U}(x)}{\partial x^k_i} \right), \quad \xi_i \in \text{BR}_i(x_{-i}).
\]

But since \( \frac{\partial \hat{U}(x)}{\partial x^k_i} \leq 0 \) for all \( k \) we have \( \sum_{k=1}^{\tilde{K} - 1} \left( - \frac{\partial \hat{U}(x)}{\partial x^k_i} \right) = \| \nabla_{x_i} \hat{U}(x) \|_1 \), and hence
\[
(\xi_i - x_i) \cdot \nabla_{x_i} \hat{U}(x) \geq d(P_{X_i}(B), \partial \tilde{X}_i) \| \nabla_{x_i} \hat{U}(x) \|_1, \quad \xi_i \in \text{BR}_i(x_{-i}),
\]
which is the desired result. \( \square \)

**Remark 9.** Since the space \( X_i \) in Lemma 8 is finite dimensional, given any norm \( \| \cdot \| \), there exists a constant \( \tilde{c} > 0 \) such that
\[
z_i \cdot \nabla_{x_i} U(x) \geq \tilde{c} \| \nabla_{x_i} U(x) \|, \quad \forall z_i \in \text{BR}_i(x_{-i}) - x_i
\]
with \( c = \tilde{c}d(P_{X_i}(B), \partial X_i) \).

For each \( x = (x_p, x_m) \in X \) near to \( x^* \), the following lemma allows us to define an additional lower dimensional game \( \Gamma_{x_p} \) associated with \( x_p \) in which the best response set is closely related to the best response set for the original game \( \Gamma \). The lemma is a straightforward consequence of the definition of the best response correspondence and the continuity of \( \nabla U \).

**Lemma 10.** For \( x \) in a neighborhood of \( x^* \), the best response set satisfies
\[
\text{BR}_i(x_{-i}) \subseteq \text{BR}_i(x^*_{-i}), \quad \forall i = 1, \ldots, \tilde{N}.
\]

Given any \( x = (x_p, x_m) \in X \) we define \( \tilde{U}_{x_p} : \tilde{X} \to \mathbb{R} \) and \( \tilde{\text{BR}}_{x_p,i} : \tilde{X}_{-i} \Rightarrow \tilde{X}_i \) as follows. For \( \tilde{x} \in \tilde{X} \) let
\[
\tilde{U}_{x_p}(\tilde{x}) := U(x_p, \tilde{x}),
\]
and for \( \tilde{x}_{-i} \in \tilde{X}_{-i} \) let
\[
\tilde{\text{BR}}_{x_p,i}(\tilde{x}_{-i}) := \arg \max_{\tilde{x}_i \in \tilde{X}_i} \tilde{U}_{x_p}(\tilde{x}_i, \tilde{x}_{-i})
\]
Let $\Gamma_{x_p}$ be the potential game with player set $\{1, \ldots, N\}$, mixed strategy space $X$ and potential function $\hat{U}_{x_p}$. Note that since $U$ is continuous and $X$ is compact, $\hat{U}_{x_p}$ converges uniformly to $\hat{U}_{x_p}^* = \hat{U}$ as $x_p \rightarrow x_p^*$. In this sense the game $\Gamma_{x_p}$ can be seen as converging to $\hat{\Gamma}^*$ as $x_p \rightarrow x_p^*$.

**Remark 11.** The function $g$ defined in Section 4.1 admits the following interpretation. Suppose we fix some $x_p = (x_p^k)_{i=1,\ldots,N, k=\gamma_i,\ldots,K_i−1}$. Then $g(x_p)$ is a completely mixed Nash equilibrium of $\Gamma_{x_p}$. Moreover, if we let $x_p \rightarrow x_p^*$, then the corresponding equilibrium of the reduced game $\Gamma_{x_p}$ converges to $x^*$, i.e., $(x_p, g(x_p)) \rightarrow (x_p^*, g(x_p^*)) = x^*$, precisely along Graph($g$).

**Remark 12.** Suppose $x^*$ is a first-order non-degenerate equilibrium. Using the multilinearity of $\Gamma$ seen as converging to $\hat{\Gamma}$ as $x \rightarrow \hat{x}$ to the potential function $\hat{\Gamma}$ and potential function $\hat{\Gamma}$ under the projection $P$.

The following lemma extends the result of Lemma 8 so it applies in a useful way to the potential function $U$ under the projection $P$.

**Lemma 13.** There exists a constant $c > 0$ such that for all $x = (x_p, x_m)$ in a neighborhood of $x^*$ and all $\eta \in \mathbb{R}^{\gamma_i−1}$ with $||\eta||$ sufficiently small we have

$$
(z_i + \eta) \cdot \nabla_x \hat{U}(P(x)) \geq c ||\nabla_x \hat{U}(P(x))||
$$

for all $z_i \in \mathbb{R}_{x_p} \cdot (\{x_m\}_{m=1}^N) - [x_m]_i$, where $[x_m]_i := (x_m^k)_{k=\gamma_i,\ldots,K_i−1}$ refers to the player-$i$ component of $x_m$ and $[x_m]_{m=1}^N$ contains the components of $x_m$ corresponding to the remaining players.

**Proof.** By construction, the projection $P(x^*)$ maps $x^*$ into the interior of $\hat{X}$. Choose $c > 0$ such that the ball $B(P(x^*), \epsilon) \subset \hat{X}$ is separated by a positive distance from the boundary of $\hat{X}$. Applying Lemma 8 (and Remark 9) to the game $\Gamma_{x_p}$ we see that there exists a constant $c > 0$ such that for any $x = (x_p, x_m)$ in $X$ with $x_m \in B(P(x^*), \epsilon)$ there holds

$$
(z_i \cdot \nabla_x \hat{U}_{x_p}(x_m)) \geq 4c ||\nabla_x \hat{U}_{x_p}(x_m)||
$$

for all $z_i \in \mathbb{R}_{x_p} \cdot (\{x_m\}_{m=1}^N) - [x_m]_i$. Note that the constant in Lemma 8 is only dependent on the distance from the set $B$ (in this case, $B(P(x^*), \epsilon)$) to the boundary of the strategy space (in this case, $X$), and is independent of the particular potential function under consideration—this permits the choice of $c > 0$ in (22) that holds uniformly for all $x_p$.

By the continuity of $U$ and $P$ we have

$$
\left| \frac{\nabla \hat{U}_{x_p}(P(x)) - \nabla \hat{U}(P(x))}{\|\nabla \hat{U}_{x_p}(P(x))\|} \right| < \frac{2c}{\sqrt{2}}
$$

and $P(x) \in B(x^*, \epsilon)$, for all $x$ in a sufficiently small neighborhood of $x^*$ and $x \notin $ Graph($g$). Note that diam $\hat{X}_i := \max_{x_i, x_i' \in \hat{X}_i} ||x_i - x_i'|| = \sqrt{2}$, and hence, $||z_i|| \leq \sqrt{2}$ for any $z_i \in \mathbb{R}_{x_p} \cdot (\{x_m\}_{m=1}^N) - [x_m]_i$, $x_m \in \hat{X}$. Thus, (22) and (23) give

$$
z_i \cdot \nabla \hat{U}(P(x)) \geq 2c ||\nabla \hat{U}(P(x))||,
$$
for all $z_i \in \overline{BR}_{x_{-i}} ([x_{m}]_{-i}) - [x_m]_i$ and all $x$ in a neighborhood of $x^*$, $x \notin \Graph(g)$. As long as $\|\eta\| \leq c$, the desired result holds for $x$ in a neighborhood of $x^*$, $x \notin \Graph(g)$. But $x \in \Graph(g) \Rightarrow \mathcal{P}(x) = \mathcal{P}(x^*) \Rightarrow \nabla_x \tilde{U}(\mathcal{P}(x)) = 0$, in which case the inequality is trivially satisfied. \hfill \Box

Finally, the following lemma shows that the differential inequality (13) holds.

**Lemma 14.** Let $\Gamma$ be a non-degenerate potential game with mixed equilibrium $x^*$, and let $(x(t))_{t \geq 0}$ be a FP process. Then the inequality (13) holds for $x(t)$ in a neighborhood of $x^*$.

**Proof.** Let

$$\mathbf{P}(x) := \begin{pmatrix} \frac{\partial \tilde{p}_i}{\partial x_j} \\
\end{pmatrix}_{i=1, \ldots, \tilde{N}, j=1, \ldots, N}$$

where the partial derivatives are evaluated at $x$. The Jacobian of $\tilde{P}$ evaluated at $x$ is given by

$$\begin{pmatrix} \frac{\partial \tilde{p}_i}{\partial x_j} \end{pmatrix}_{i,j=1, \ldots, \tilde{N}, \ k, \ell = 1, \ldots, K_i - 1} = \begin{pmatrix} I & \mathbf{P}(x) \\
0 & 0 \end{pmatrix}.$$ 

Using the chain rule we may express the time derivative of the potential along the path $\tilde{P}(x(t))$ as

$$\frac{d}{dt} \tilde{U}(\tilde{P}(x(t))) = \nabla U(\tilde{P}(x(t))) \begin{pmatrix} I & \mathbf{P}(x) \\
0 & 0 \end{pmatrix} \dot{x} = \nabla_{x_m} U(\tilde{P}(x(t)))(\mathbf{I} \cdot \mathbf{P}(x)) \dot{x}.$$

For $i = 1, \ldots, \tilde{N}$, $k = 1, \ldots, \gamma_i - 1$ let $\eta_i^k(t) := \sum_{j=1}^N \sum_{\ell=\gamma_i}^{K_i-1} \frac{\partial \tilde{p}_i}{\partial x_j} x_{j,\ell}^\ast$, let $\eta_i(t) := (\eta_i^k(t))_{k=1}^{\gamma_i-1}$, and let $\eta(t) = (\eta_i(t))_{i=1}^\tilde{N}$. Multiplying out the right two terms above we get

$$\frac{d}{dt} \tilde{U}(\tilde{P}(x(t))) = \nabla_{x_m} U(\tilde{P}(x(t)))(\dot{x}_m + \eta(t)).$$

By Lemma 10 and Remark 12, if we restrict $x(t)$ to a sufficiently small neighborhood of $x^*$ then for any $z_i = (z_i^\ast, z_i^0) \in BR_{x_{-i}}(x_{-i}(t))$, $z_i^\ast = (z_i^k)_{k=1}^{\gamma_i-1}$, $z_i^0 = (z_i^k)_{k=\gamma_i}^{K_i-1}$, we have $z_i^\ast \in \overline{BR}_{x_{-i}}([x_m]_{-i})$ and $z_i^0 = 0$. We note two important consequences of this:

(i) If we restrict $x(t)$ to a sufficiently small neighborhood of $x^*$ and note that $\tilde{U}(\mathcal{P}(x(t))) = \tilde{U}(\mathcal{P}(x(t)))$, then by (24) we have

$$\frac{d}{dt} \tilde{U}(\mathcal{P}(x(t))) = \nabla \tilde{U}(\mathcal{P}(x(t))) \cdot \begin{pmatrix} z_1(t) + \eta_1(t) \\
\vdots \\
z_{\tilde{N}}(t) + \eta_{\tilde{N}}(t) \end{pmatrix}$$

$$= \sum_{i=1}^{\tilde{N}} \nabla_{x_i} \tilde{U}(\mathcal{P}(x(t))) \cdot (z_i(t) + \eta_i(t)),$$

where $z_i(t) \in \overline{BR}_{x_{-i}}([x_m(t)]_{-i}) - [x_m(t)]_i$.

(ii) We may force $\max_{i=1, \ldots, \tilde{N}} \|\eta_i\|$ to be arbitrarily small by restricting $x(t)$ to a neighborhood of $x^*$.

Consequence (i) follows readily by using the definition of FP (1). To show consequence (ii), note that by (1) we have $\dot{x}_i^\ast = z_i^\ast - x_i^k$ for all $i = 1, \ldots, \tilde{N}$, $k = 1, \ldots, K_i$, for all $z_i \in \overline{BR}_{x_{-i}} ([x_m]_{-i}) - [x_m]_i$ and all $x$ in a neighborhood of $x^*$, $x \notin \Graph(g)$.
for some $z_i \in \text{BR}_i(x_{-i})$. But, for $x$ in a neighborhood of $x^*$ and $k \geq \gamma_i$, we have shown above that $z_i^k = 0$, and hence $x_i^k = -x_i^k$. Due the ordering we assumed for $Y$, we have $[x^*]_i^k = 0$ for any $(i, k)$ such that $k \geq \gamma_i$. Hence, $x_i^k \to 0$ as $x \to x^*$, for any $(i, k)$ such that $k \geq \gamma_i$.

Furthermore, there exists a $c > 0$ such that $\left| \frac{\partial P_i(x)}{\partial x_j} \right| < c$, $1 = \ldots, \tilde{N}$, $k = 1, \ldots, \gamma_i - 1$, $j = 1, \ldots, N$, $\ell \geq \gamma_j$ uniformly for $x$ in a neighborhood of $x^*$ (see Lemma 27 in appendix). By the definition of $\eta_k$, this implies that $\max_{i=1, \ldots, N} \| \eta_i \|$ may be made arbitrarily small by restricting $x(t)$ to a sufficiently small neighborhood of $x^*$.

Now, let $x(t)$ be restricted to a sufficiently small neighborhood of $x^*$ so that $\| \eta_i(t) \|$ is small enough to apply Lemma 13 for each $i$. Applying Lemma 13 to (25) we get $\frac{d}{dt} \mathcal{U}(\mathcal{P}(x(t))) \geq \sum_{i=1}^N c_i \| \nabla x_i \mathcal{U}(\mathcal{P}(x(t))) \|$ for $x(t)$ in a neighborhood of $x^*$. By the equivalence of finite-dimensional norms, there exists a constant $c_1$ such that $\frac{d}{dt} \mathcal{U}(\mathcal{P}(x(t))) \geq c_1 \| \nabla \mathcal{U}(\mathcal{P}(x(t))) \|$ for $x(t)$ in a neighborhood of $x^*$.

Since $\Gamma$ is assumed to be (second-order) non-degenerate, $\mathcal{P}(x^*)$ is a non-degenerate critical point of $\mathcal{U}$. By Lemma 28 (see appendix) there exists a constant $c_2$ such that $c_1 \| \nabla \mathcal{U}(\bar{x}) \| \geq c_2 d(\bar{x}, \mathcal{P}(x^*))$ for all $\bar{x} \in \bar{X}$ in a neighborhood of $\mathcal{P}(x^*)$. Since $\mathcal{P}$ is continuous we have $\frac{d}{dt} \mathcal{U}(\mathcal{P}(x(t))) \geq c_2 d(\mathcal{P}(x(t)), \mathcal{P}(x^*))$ for $x(t)$ in a neighborhood of $x^*$.

5. Proof of Main Result. We will assume throughout this section that $\Gamma$ is a regular potential game. By Theorem 4, the ensuing results hold for almost all potential games.

For each mixed equilibrium $x^*$, let the set $\Lambda(x^*) \subset X$ be defined as

$$\Lambda(x^*) := \begin{cases} \{x^*\} & \text{if } x^* \text{ is completely mixed,} \\ \text{Graph}(g) & \text{otherwise,} \end{cases}$$

where $g$ is defined with respect to $x^*$ as in Section 4.1.

In this section we will prove Theorem 1 in two steps. First, we will show that for each mixed equilibrium $x^*$, the set $\Lambda(x^*)$ can only be reached in finite time from an $\mathcal{L}^\infty$-null set of initial conditions (see Proposition 15), where $\kappa$, defined in (4), is the dimension of $X$. Second, we will show that if a FP process converges to the set $\Lambda(x^*)$, then it must do so in finite time (see Proposition 22). Since $x^* \in \Lambda(x^*)$, Propositions 15 and 22 together show that for any mixed equilibrium $x^*$, the set of initial conditions from which FP converges to $x^*$ has $\mathcal{L}^\infty$-measure zero.

By Theorem 2 of [28] we see that in regular potential games, the set of NE is finite. Hence, Propositions 15 and 22 imply that FP can only converge to set of mixed strategy equilibria from a $\mathcal{L}^\infty$-null set of initial conditions. Since a FP process must converge to the set of NE in a potential game ([4], Theorem 5.5), this implies that Theorem 1 holds.

5.1. Finite-Time Convergence. The goal of this subsection is to prove the following proposition.

PROPOSITION 15. Let $\Gamma$ be a non-degenerate game and let $x^*$ be a mixed-strategy NE of $\Gamma$. The set $\Lambda(x^*)$ can only be reached by a FP process in finite time from a set
of initial conditions with $\mathcal{L}^\kappa$-measure zero. That is,

$$\mathcal{L}^\kappa(\{x_0 \in X : x(0) = x_0, \ x(t) \text{ is a FP process}, x(t) \in \Lambda(x^*) \text{ for some } t \in [0, \infty)\}) = 0.$$ 

Before proving the proposition we present some definitions and preliminary results. Let

$$I_{i,k,\ell} := \{(x_i, x_{-i}) \in X : U(y^k_i, x_{-i}) = U(y^\ell_1, x_{-i})\},$$

for $i = 1, \ldots, N, k, \ell = 1, \ldots, K_i, \ell \neq k$, be the set in which player $i$ is indifferent between his $k$-th and $\ell$-th actions.

If the game $\Gamma$ is non-degenerate, then each $I_{i,k,\ell}$ is the union of smooth surfaces with Hausdorff dimension at most $(k - 1)$ (see Lemma 32 in appendix). In particular, for each $x \in I_{i,k,\ell}$ there exists a vector $\nu \in \mathbb{R}^\kappa$ that is normal to $I_{i,k,\ell}$ at $x$. We refer to the set $I_{i,k,\ell}$ as an indifference surface of player $i$.

We define the set $\hat{Q} \subseteq X$ as follows. Let $\hat{Q}$ contain the set of points where two or more indifference surfaces intersect and their normal vectors do not coincide. Furthermore, if an indifference surface $I$ has a component $\tilde{I} \subseteq I$ with Hausdorff dimension less than $\kappa - 1$, then we put any points where $\tilde{I}$ intersects with another decision surface into $\hat{Q}$. Since each indifference surface is smooth with dimension at most $\kappa - 1$, $\hat{Q}$ has Hausdorff dimension at most $\kappa - 2$. Let

$$Q := \hat{Q} \cup \Lambda(x^*).$$

As shown in Section 4.1, if $x^*$ is non-degenerate, then the set $\text{Graph}(g)$ (and hence $\Lambda(x^*)$) has Hausdorff dimension at most $\kappa - 2$. Thus $Q$ has Hausdorff dimension at most $\kappa - 2$.  

The FP vector field (see (1)) is given by $FP : X \ni X$, where

$$FP(x) := BR(x) - x.$$ 

Let

$$Z := \{x \in X \setminus Q : x \in I_{i,k,\ell} \text{ for some } i, k, \ell \text{ with normal } \nu \text{ at } x, \text{and } \nu \cdot z = 0 \text{ for some } z \in FP(x)\}.$$ 

Since each $I_{i,k,\ell}$ has Hausdorff dimension $\kappa - 1$, $Z$ has Hausdorff dimension at most $\kappa - 1$. We define the relative boundary of $Z$, denoted here as $\partial Z$ as follows. If $Z$ has Hausdorff dimension $\kappa - 2$ or less, then let $\partial Z := Z$. If $Z$ has Hausdorff dimension $\kappa - 1$ then it may be expressed as the union of a finite number of smooth $(\kappa - 1)$-dimensional surfaces, denoted here as $(Z_s)_{s=1}^{N_z}$, $1 \leq N_z < \infty$, and a component with Hausdorff dimension at most $\kappa - 2$, denoted here as $Z'$. That is, $Z = (\bigcup_{s=1}^{N_z} Z_s) \cup Z'$. Each $Z_s, s = 1, \ldots, N_z$ is contained in some indifference surface, which we denote here as $I_s$. Define the relative interior of $Z_s$ (with respect to $I_s$) as $\text{ri} Z_s := \{x \in Z_s : \exists \epsilon > 0 \text{ s.t. } B(x, \epsilon) \cap I_s \subset Z_s\}$, and define the relative boundary of $Z_s$ as $\partial Z_s := \text{cl} Z_s \setminus \text{ri} Z_s$. We then define the relative boundary of $Z$ as

$$\partial Z := \left( \bigcup_{s=1}^{N_z} \partial Z_s \right) \cup Z'.$$

\footnote{Proposition 15 can easily be generalized to say that any set $A \subset X$ such that $\text{cl} A$ has Hausdorff dimension at most $\kappa - 2$, can only be reached in finite time from a set of $\mathcal{L}^\kappa$-measure zero by substituting $A$ for $\Lambda(x^*)$ throughout the section.}
Note that \( \partial \mathcal{Z} \) is a set with Hausdorff dimension at most \( \kappa - 2 \). By Lemma 35 in the appendix, the FP vector field is oriented tangentially along \( \mathcal{Z} \), in the sense that for any \( x \in \mathcal{Z} \) there holds \( \nu \cdot y = 0 \) for any vector \( \nu \) normal to \( \mathcal{Z} \) at \( x \), and any \( y \in \text{FP}(x) \). This implies that FP paths can only enter or exit \( \mathcal{Z} \) through \( \partial \mathcal{Z} \).

Let

\[
X^* := X \setminus (Q \cup \mathcal{Z})
\]

The following technical lemma will be used to show that FP is well posed within \( X^* \) (see Lemma 17). It is a consequence of the fact that the FP vector field can only have jumps that are tangential to indifference surfaces.

**Lemma 16.** Suppose \( x \in X^* \) is in some indifference surface \( \mathcal{I}_{i,k,\ell} \). Then there exists a constant \( c > 0 \) and a vector \( \nu \) that is normal to \( \mathcal{I}_{i,k,\ell} \) at \( x \), such that

\[
\nu \cdot z \geq c, \quad \forall z \in \text{FP}(\tilde{x})
\]

for all \( \tilde{x} \in X^* \) in a neighborhood of \( x \).

**Proof.** By the definition of \( \mathcal{I}_{i,k,\ell} \), if \( x \in \mathcal{I}_{i,k,\ell} \) then for all \( \tilde{x} \in X \) such that \( \tilde{x} = x \) we have \( \tilde{x} \in \mathcal{I}_{i,k,\ell} \). This implies that for any vector \( \nu \) that is normal to \( \mathcal{I}_{i,k,\ell} \), the \((i,m)\)-th component of \( \nu \) must be zero for all \( m = 1, \ldots, K_i - 1 \).

Suppose that \( x \in X^* \cap \mathcal{I}_{i,k,\ell} \). Since \( x \notin Q \), there is a neighborhood of \( x \) in which no indifference surface intersects with \( \mathcal{I}_{i,k,\ell} \). This implies that for \( \tilde{x} \) within a neighborhood of \( x \), there exists a neighborhood of \( \tilde{x} \), such that \( \tilde{x} \) is a vertex of \( X_{-i} \).

Together, these two facts imply that for all \( \tilde{x} \) in a neighborhood of \( x \), we have \( \nu \cdot z' = \nu \cdot z'' \) for all \( z' \in \text{BR}(x), z'' \in \text{BR}(\tilde{x}) \), for any vector \( \nu \) that is normal to \( \mathcal{I}_{i,k,\ell} \) at \( x \). Since \( x \notin \mathcal{Z} \), recalling the form of FP (27), this means we can choose a vector \( \nu \) that is normal to \( \mathcal{I}_{i,k,\ell} \) at \( x \) and a constant \( c > 0 \) such that \( \nu \cdot z > c \) for \( z \in \text{FP}(\tilde{x}) \) for all \( \tilde{x} \) in a neighborhood of \( x \).

The following lemma gives a well-posedness result for FP inside \( X^* \).

**Lemma 17.** For any \( x_0 \in X^* \), there exists a \( T \in (0, \infty) \) and a unique absolutely-continuous function \( x : [0, T) \to X^* \), with \( x(0) = x_0 \), solving the differential inclusion

\[
\dot{x}(t) \in \text{FP}(x(t))
\]

for almost all \( t \).

**Proof.** If \( x \in X^* \) is not on any indifference surface, then FP is single valued in a neighborhood of \( x \), and (1) is (locally) a Lipschitz differential equation with unique local solution.

Suppose that \( x_0 \in X^* \) is on an indifference surface \( \mathcal{I} \). By Lemma 16 there exists a constant \( c > 0 \) such that for all \( \tilde{x} \in \mathcal{I} \) of \( x \) we have \( \text{FP}(\tilde{x}) \cdot \nu > c \), where \( \nu \) is a normal vector to \( \mathcal{I} \) at \( x \). This implies that for \( \delta > 0 \) sufficiently small we have

\[
\{ t \in [-\delta, \delta] : x(t) \in \mathcal{I}, \text{ for any } i, k, \ell \} = \{ 0 \}.
\]

Now, let \( x_0 \in X^* \) and let \( (x(t))_{t \geq 0} \) and \( (z(t))_{t \geq 0} \) be two solutions to (1) with \( x(0) = z(0) = x_0 \). If \( (x(t))_{t \geq 0} \) never crosses an indifference surface, then the flow is always classical and the two solutions always coincide; i.e., \( x(t) = z(t), t \geq 0 \).

Suppose that \( (x(t))_{t \geq 0} \) does cross an indifference surface and let \( t^* \geq 0 \) be first time when such a crossing occurs. For \( t < t^* \), the flow is classical and we have \( x(t) = z(t) \) for \( t \leq t^* \).

By (29) we see that for \( \delta > 0 \) sufficiently small, \( x(t) \) is not in any indifference surface for \( t \in [t^* - \delta, t^* + \delta] \setminus \{ t^* \} \). Suppose that at time \( t = t^* + \delta \) we have \( x(t) =
\( \dot{x} \neq \dot{z} = z(t) \). Let \((\dot{x}(\tau))_{\tau \geq 0}\) and \((\dot{z}(\tau))_{\tau \geq 0}\) be solutions to the time-reversed FP flow with \( x(0) = \hat{x} \) and \( \hat{z}(0) = \hat{z} \).

Since \( \dot{x} \neq \dot{z} \), and since the time-reversed flow is classical for \( 0 \leq \tau < \delta \) (in particular, of the form \( \dot{x} = a + x \) for some constant \( a \)), we get \( \hat{x}(\delta) \neq \hat{z}(\delta) \). But this is impossible because the paths \((x(t))_{t \geq 0}\) and \((z(t))_{t \geq 0}\) are absolutely continuous and we already established that \( \hat{x}(\delta) = x(t^*) = z(t^*) = \hat{z}(\delta) \).

As a matter of notation, we say that \( \lambda \) is a signed measure on \( \mathbb{R}^n \) if there exists a Radon measure \( \mu \) on \( \mathbb{R}^n \) and a \( \mu \)-measurable function \( \sigma : \mathbb{R}^n \to \{-1, 1\} \) such that

\[
\lambda(K) = \int_K \sigma d\mu
\]

for all compact sets \( K \subset \mathbb{R}^n \). When convenient, we write \( \sigma \mu \) to denote the signed measure \( \lambda \) in (30).

Letting elements \( x \in X \) be written componentwise as \((x_s)_{s=1}^\kappa \), we recall [9] that a function \( u \in L^1(\Omega) \) (with \( \Omega \subseteq \mathbb{R}^n \), \( \Omega \) open) is a function of bounded variation (i.e., a BV function) if there exist finite signed Radon measures \( D_s u \) such that the integration by parts formula

\[
\int_{\Omega} u \frac{\partial \phi}{\partial x_s} \, dx = -\int_{\Omega} \phi \, dD_s u
\]

holds for all \( \phi \in C_c^\infty(\Omega) \). The measure \( D_s u \) is called the weak, or distributional, partial derivative of \( u \) with respect to \( x_s \). We let \( D.u := (D_s u)_{s=1,\ldots,\kappa} \).

The measure \( D.u \) can be uniquely decomposed into three parts [1]

\[
D.u = \nabla u \mathcal{L}^\kappa + C.u + J.u.
\]

Here \( J.u \) is supported on a set \( J_u \) with Hausdorff dimension \( \kappa - 1 \), and \( C.u \) is singular with respect to \( \mathcal{L}^\kappa \) and satisfies \( C.u(E) = 0 \) for all sets \( E \) with finite \( \mathcal{H}^{\kappa - 1} \) measure.

The \( L^1 \) function \( \nabla u \) is analogous to a classical derivative, and in particular if \( u \) is differentiable on an open set \( V \) then \( D.u = \nabla u \mathcal{L}^\kappa \) on that set, with \( \nabla u \) matching the classical derivative. Furthermore, if \( u \) jumps across a smooth \( (\kappa - 1) \)-dimensional hypersurface, then for \( x \) on the hypersurface we have

\[
D.u = J.u = (u^+-u^-)\nu d\mathcal{H}^{\kappa-1},
\]

where \( u^+ \) is the value of \( u \) on one side of the surface, \( u^- \) is the value on the other, and \( \nu \) is the normal vector pointing from \( u^- \) to \( u^+ \) [1].

A vector-valued function \( f \in L^1(\Omega : \mathbb{R}^\kappa) \) is a function of bounded variation if each of its components is also of bounded variation. Letting \( f \) be written componentwise as \( f = (f^s)_{s=1}^\kappa \), we write \( Df := (D_j f^s)_{j,s=i,\ldots,\kappa} \).

Next we define the divergence of a function \( f \in L^1(\Omega : \mathbb{R}^\kappa) \), denoted by \( D \cdot f \), as the measure

\[
D \cdot f := \sum_{s=1}^\kappa D_s f^s.
\]

Given a constant \( c \in \mathbb{R} \), we say that \( D \cdot f = c \) if \( D \cdot f = \frac{d}{dc} f^s \mathcal{L}^\kappa \), and \( \frac{d}{dc} f^s = c \), where \( \frac{d}{dc} f^s \) denotes the Radon-Nikodym derivative. The following lemma characterizes the divergence of the FP vector field. As a matter of notation, if a function \( f : X \to X \) satisfies \( f(x) \in \text{FP}(x) \) for all \( x \in X \) then we say \( f \in \text{FP} \).
Lemma 18. For every \( f \in \text{FP} \), the vector field \( f \) satisfies \( D \cdot f = -1 \).

The proof of this lemma follows from the fact that FP is piecewise linear, and any jumps in FP are tangential to indifference surfaces.

**Proof.** Suppose \( f \in \text{FP} \), and let \( f \) be written componentwise as \( f = (f_k^i)_{i=1, \ldots, N} \). Let \( i \in \{1, \ldots, N\} \) and \( k \in \{1, \ldots, K_i - 1\} \). Let \( D_{j, \ell} f_i^k \) denote the weak partial derivative of \( f_i^k \) with respect to \( x_j^\ell \), \( j = 1, \ldots, N \), \( \ell = 1, \ldots, K_j - 1 \), and let \( D f_i^k = (D_{j, \ell} f_i^k)_{j=1, \ldots, N, \ell=1,\ldots,K_j-1} \). Let \( J f_i^k = (J_{j, \ell} f_i^k)_{j=1, \ldots, N, \ell=1,\ldots,K_j-1} \) denote the jump component associated with \( D f_i^k \) (see (32)).

The vector field \( f \) is piecewise linear. Breaking up \( f \) over regions in which it is linear we see that \( \frac{dD f_i^k}{d\xi^\ell} = -1 \). It remains to show that \( D \cdot f \) has no singular component; i.e., under the decomposition (32), the measure \( D \cdot f \) has zero Cantor component and zero jump component.

Since \( f_i^k \) is piecewise linear and only jumps on the set \( \bigcup_{\ell=1, \ell \neq k} I_{i, k, \ell} \) which has finite \( \kappa - 1 \) measure, \( f_i^k \) has no Cantor part; that is, \( C f_i^k = (C_{j, \ell} f_i^k)_{j=1, \ldots, N, \ell=1, \ldots, \gamma_i} = 0 \) (see (32)). Hence, the singular component of \( D \cdot f \), which we denote here as \( S \), has no Cantor part and is given by \( S := \sum_{i=1}^N \sum_{K_i = 1}^{K_i - 1} J_{i, k, \ell} f_i^k \).

Suppose that \( x \in I_{i, k, \ell} \) for some \( \ell \) (recall \( \ell \neq k \)). Suppose \( \nu \) is a vector that is normal to \( I_{i, k, \ell} \) at \( x \). By the definition of \( I_{i, k, \ell} \), if \( x \in I_{i, k, \ell} \) then for all \( \hat{x} \in X \) such that \( \hat{x} - x = x - i \) we have \( \hat{x} \in I_{i, k, \ell} \). This implies that the \( (i, k) \)-th component of \( \nu \) must be zero. Since \( J f_i^k = (f_i^k) - (f_i^k)_{\nu} \nu^{\kappa - 1} \) for \( x \) on \( \bigcup_{\ell=1, \ell \neq k} I_{i, k, \ell} \) (see (33)), taking the \( (i, k) \)-th component we get \( J_{i, k, \ell} f_i^k \nu^{\kappa - 1} = 0 \).

Since this is true for every pair \( (i, k) \) we see that \( S = 0 \), and hence \( D \cdot f = -1 \) in the interior of \( X \). An identical argument holds on the boundary of \( X \), and hence, \( S = 0 \) and \( D \cdot f = -1 \).

The following lemma shows that for sets \( E \subseteq X^* \) with relatively smooth boundary, the surface integral of FP over the boundary of \( E \) is well defined.

**Lemma 19.** Let \( E \) be a subset of \( X^* \) with piecewise smooth boundary. For any functions \( f, g \in \text{FP} \) we have

\[
\int_{\partial E} f \cdot \nu_E d\mathcal{H}^{\kappa - 1} = \int_{\partial E} g \cdot \nu_E d\mathcal{H}^{\kappa - 1} = \int_{\partial E} \text{FP} \cdot \nu_E d\mathcal{H}^{\kappa - 1},
\]

where \( \nu_E \) denotes the outer normal vector of \( E \).

**Proof.** Suppose \( x \in X^* \) is not on any indifference surface \( I_{i, k, \ell} \). Then \( \text{FP}(x) \) maps to a singleton and \( f(x) = g(x) \).

Suppose \( x \in X^* \) is on an indifference surface \( I_{i, k, \ell} \). Let \( \nu_I \) denote a normal vector to \( I_{i, k, \ell} \). Since \( x \in X^* \), the vector field FP can only jump tangentially to \( \nu_I \). Using similar reasoning to the proof of Lemma 16, this implies that for any \( a, b \in \text{FP}(x) \) we have \( a \cdot \nu_I(x) = b \cdot \nu_I(x) \). Hence \( \text{FP}(x) \cdot \nu := a \cdot \nu, a \in \text{FP}(x) \) is well defined for such \( x \).

In particular, note that if \( x \in X^* \) is on some indifference surface \( I \) and \( \nu_I = \nu_E \) at \( x \), then \( f(x) \cdot \nu_E = \text{FP}(x) \cdot \nu_I \) for any function \( f \in \text{FP} \).

Let \( \bar{I} \) be the union of all indifference surfaces. Since \( \partial E \) is piecewise continuous and the indifference surfaces are smooth, the set \( S := \{ x \in X^* : x \in \bar{I} \cap \partial E, \nu_I(x) \neq \nu_{\partial E}(x) \} \) has \( \mathcal{H}^{\kappa - 1} \)-measure zero, where \( \nu_I(x) \) and \( \nu_{\partial E}(x) \) denote the normal vectors to \( \bar{I} \) and \( \partial E \) at \( x \).
We have shown that $f|_{(\partial E)\setminus S} = g|_{(\partial E)\setminus S}$ for all $f, g \in \text{FP}$, and $\mathcal{H}^{s-1}(S) = 0$, and hence,
\[
\int_{\partial E} f \cdot \nu_E d\mathcal{H}^{s-1} = \int_{\partial E} g \cdot \nu_E d\mathcal{H}^{s-1}, \quad \text{for all } f, g \in \text{FP}.
\]

There follows that, within $X^*$, the FP vector field compresses mass at a rate of $-1$. In particular, this implies that, within $X^*$, FP cannot map a set of positive measure to a set of zero measure in finite time.\footnote{We note that this result can also be derived as a consequence of Lemma 3.1 in [7]. For the sake of completeness and to simplify the presentation, we give a proof of the result here using the notation and tools introduced in the paper.}

**Lemma 20.** Let $E$ be a compact subset of $X^*$ with piecewise smooth boundary and finite perimeter. Then
\begin{equation}
\int_{\partial E} \text{FP} \cdot \nu_E d\mathcal{H}^{s-1} = -\mathcal{L}^s(E),
\end{equation}
where $\nu_E$ denotes the outer normal vector of $E$.

**Proof.** We first note that by Lemma 18 for every $f \in \text{FP}$ we have $\int_E dD \cdot f = -\mathcal{L}^s(E)$.

Let $(f_n)_{n \geq 1}$, $f_n : X^* \rightarrow X^*$ be a sequence of uniformly bounded $C^1$ functions such that $f_n \rightarrow f$ a.e. for some function $f : X^* \rightarrow X^*$ satisfying $f(x) \in \text{FP}(x)$ for all $x \in X^*$. (Such a sequence can be explicitly constructed by smoothing the FP vector field, e.g., [10].)

Let $\mathcal{F}$ and each $f_n$ be written pointwise, the dominated convergence theorem gives $\kappa$.

Let $f$ and each $f_n$ be written componentwise as $f = (f^n_s)_s=1$ and $f_n = (f^n_s)_s=1$. Let $D \cdot f_n = \sum_{s=1}^{\infty} D_s f^n_s$ be the divergence measure associated with $f_n$ and $D \cdot f = \sum_{s=1}^{\infty} D_s f^s$ the divergence measure associated with $f$. Since $f$ and $f_n$ are BV functions, by (31) we have
\[
- \int_{X^*} f^n_s \frac{\partial \phi}{\partial x_s} \, dx = \int_{X^*} \phi \, D_s f^n_s, \quad \text{and} \quad - \int_{X^*} f^s \frac{\partial \phi}{\partial x_s} \, dx = \int_{X^*} \phi \, D_s f^s
\]
for $n \in \mathbb{N}$, $s = 1, \ldots, \kappa$, for any $\phi \in C^1_c(X^*)$.

For a function $\phi \in C^1_c(X^*)$, there exists a constant $c > 0$ such that $|\frac{\partial \phi(x)}{\partial x_s}| < c$ for all $x \in X^*$. Since $(f_n)_{n \geq 1}$ is uniformly bounded, $|f_n(x) \frac{\partial \phi(x)}{\partial x_s}|$ is bounded by some constant $c > 0$ for all $x \in X^*$, and since $X^*$ is a bounded set, the constant function $c\chi_{X^*}$ (which dominates $|f_n \frac{\partial \phi}{\partial x_s}|$ on $X^*$) is integrable. Noting that $f_n \frac{\partial \phi}{\partial x_s} \rightarrow f \frac{\partial \phi}{\partial x_s}$ pointwise, the dominated convergence theorem gives
\begin{equation}
\lim_{n \rightarrow \infty} \int_{X^*} \phi \, D_s f^n_s = - \lim_{n \rightarrow \infty} \int_{X^*} f^n_s \frac{\partial \phi}{\partial x_s} \, dx = - \int_{X^*} f^s \frac{\partial \phi}{\partial x_s} \, dx = \int_{X^*} \phi \, D_s f^s.
\end{equation}

for $n \in \mathbb{N}$, $s = 1, \ldots, \kappa$. This implies that the sequence of measures $(D \cdot f_n)_{n \geq 1}$ converges weakly to $D \cdot f$ in the sense that for any $\phi \in C^1_c(X^*)$ there holds $\lim_{n \rightarrow \infty} \int_{X^*} \phi \, dD \cdot f_n = \int_{X^*} \phi \, dD \cdot f$. Letting $\phi$ approximate the characteristic function $\chi_E$, and noting that by Lemma 18 we have $(D \cdot f)(\partial E) = 0$, we see that $\lim_{n \rightarrow \infty} \int_{E^c} dD \cdot f_n = \int_{E^c} dD \cdot f$.\footnote{We note that this result can also be derived as a consequence of Lemma 3.1 in [7]. For the sake of completeness and to simplify the presentation, we give a proof of the result here using the notation and tools introduced in the paper.}
Hence,

\[-L^\kappa(E) = \int_E dD \cdot f\]

\[= \lim_{n \to \infty} \int_E dD \cdot f_n\]

\[= \lim_{n \to \infty} \int_{\partial E} f_n \cdot \nu_E d\mathcal{H}^{\kappa-1}\]

\[= \int_{\partial E} f \cdot \nu_E d\mathcal{H}^{\kappa-1}\]

\[= \int_{\partial E} \text{FP} \cdot \nu_E d\mathcal{H}^{\kappa-1},\]

where the third line follows from the Gauss-Green theorem [9], the fourth line follows from the dominated convergence theorem (by assumption, \(E\) has finite perimeter and a piecewise smooth boundary, and \(f\) is bounded), and the fifth line follows from Lemma 19.

We now prove Proposition 15.

**Proof.** We begin by noting that, by Lemma 34 in the appendix, \(\text{cl} Q\), has Hausdorff dimension at most \(\kappa - 2\).

Let \(\epsilon > 0\). By the definition of the Hausdorff measure ([9], Chapter 2), there exists a countable collection of balls \((B^j_\epsilon)_{j \geq 1}\), each with diameter less than \(\epsilon\), such that \(\text{cl} Q \cup \partial Z \subset \bigcup_{j \geq 1} B^j_\epsilon\) and \(\sum_{j=1}^{\infty} c \left(\frac{\text{diam} B^j_\epsilon}{2}\right)^{\kappa - 2} < 2\mathcal{H}^{\kappa-2}(\text{cl} Q \cup \partial Z)\), where \(c := \frac{\pi^{\kappa-2}}{\Gamma(\frac{\kappa-2}{2})+1}\), and where \(\Gamma\) in this context denotes the standard \(\Gamma\) function.

Since \(\partial Z\) is closed, \(\text{cl} Q \cup \partial Z\) is closed, and hence there exists a finite subcover \((B^j_\epsilon)_{j=1}^{N_\epsilon}\) such that \(\text{cl} Q \cup \partial Z \subset \bigcup_{j=1}^{N_\epsilon} B^j_\epsilon\). Let \(B_\epsilon := \bigcup_{j=1}^{N_\epsilon} B^j_\epsilon\), and let

\[X^*_\epsilon := X \setminus (B_\epsilon \cup Z)\]

Note that, by Lemma 33 in the appendix we have

\[(36) \lim_{\epsilon \to 0} \mathcal{H}^{\kappa-1}(\partial B_\epsilon) = 0.\]

Fix some time \(T > 0\), and for \(0 < t \leq T\), let

\[E_\epsilon(T - t) := \{x_0 \in X^*_\epsilon : x(0) = x_0, x(t) \text{ is a FP process,}\]

\[x(s) \in B_\epsilon, \text{ for some } 0 < s \leq t\}\]

and note that the boundary \(\partial E_\epsilon(T - t)\) is piecewise smooth. The set \(E_\epsilon(T - t)\) may be thought of as the set obtained by tracing paths backwards out of \(B_\epsilon\) from time \(T\) back to time \(T - t\). Let

\[V_\epsilon(t) := L^\kappa(E_\epsilon(T - t)).\]

Letting \(R_\epsilon\) denote the flux through \(\partial B_\epsilon\) into \(E_\epsilon(T - t)\) and again letting \(\nu\) denote
the outer normal to \( \partial E_\epsilon(T - t) \), for \( t > 0 \) we have
\[
\frac{d}{dt} V_\epsilon(t) = \int_{\partial E_\epsilon(T - t) \setminus \partial B_e} -\text{FP} \cdot \nu \, dx
\geq R_\epsilon + \int_{\partial E_\epsilon(T - t)} -\text{FP} \cdot \nu \, dx
\leq R_\epsilon + \mathcal{L}^\epsilon(E_\epsilon(T - t))
= R_\epsilon + V_\epsilon(t),
\]
where the third line follows by Lemma 20.

Noting that \( \|\text{FP}\|_\infty < \infty \), the flux through \( \partial B_e \) is bounded by
\[
R_\epsilon \leq \mathcal{H}^{\epsilon - 1}(\partial B_e)\|\text{FP}\|_\infty =: \bar{R}_\epsilon.
\]
By (36) we have \( \mathcal{H}^{\epsilon - 1}(\partial B_e) \to 0 \) as \( \epsilon \to 0 \), and hence \( \bar{R}_\epsilon \to 0 \) as \( \epsilon \to 0 \).

Using the integral form of Gronwall’s inequality, (37) and (38) give \( V_\epsilon(t) \leq t\bar{R}_\epsilon \epsilon^t \), \( 0 < t \leq T \). In particular, this means that
\[
\mathcal{L}^\epsilon(E_\epsilon(0)) \leq \bar{R}_\epsilon e^T,
\]
where the right hand side goes to zero as \( \epsilon \to 0 \). Sending \( \epsilon \to 0 \), we see that the set \( W(T) := \{ x_0 \in X^* : \ x(0) = x_0, \ x(t) \text{ is a FP process}, \ x(s) \in Q \cup \partial Z \text{ for some } 0 < s \leq T \} \) has \( \mathcal{L}^\epsilon \)-measure zero. 

Since paths may only enter \( Z \) through the boundary \( \partial Z \), this means that the set of points in \( X \) from which \( Z \) can be reached within time \( T \) is contained in \( W(T) \cup Z \). Furthermore, the set of points from which \( Q \cup Z \) can be reached within time \( T \) is contained in \( W(T) \cup Z \cup Q \), which is a \( \mathcal{L}^\epsilon \)-measure zero set. Since this is true for every \( T > 0 \), we get the desired result.

**Remark 21.** We note that the proof given above implies that \( Q \cup Z \) can only be reached in finite time from a \( \mathcal{L}^\epsilon \)-measure zero subset of \( X^* \). Since \( X^* := X \setminus (Q \cup Z) \), this, combined with Lemma 17, implies that for almost every initial condition \( x_0 \in X^* \), there exists a unique FP process \( x \) with \( x(0) = x_0 \). Since \( \mathcal{L}^\epsilon(X \setminus X^*) = 0 \), this implies that for almost every \( x_0 \in X \), there exists a unique FP process \( x \) with \( x(0) = x_0 \).

### 5.2. Infinite-Time Convergence

The following proposition shows that it is not possible to converge to \( \Lambda(x^*) \) in infinite time.

**Proposition 22.** Let \( \Gamma \) be a non-degenerate potential game and let \( x^* \) be a mixed-strategy equilibrium. Suppose \( (x(t))_{t \geq 0} \) is a FP process and \( x(t) \to x^* \). Then \( x(t) \) converges to \( \Lambda(x^*) \) in finite time.

**Proof.** From the definitions of \( \Lambda(x^*) \) and \( P \) we see that
\[
x(t) \to \Lambda(x^*) \iff P(x(t)) \to P(x^*).
\]
If we integrate (13), use the fact \( P(x(t)) \to P(x^*) \), and set \( e(t) := d(P(x(t)), P(x^*)) \), then we find that
\[
\dot{U}(P(x^*)) - \dot{U}(P(x(t))) \geq c_2 \int_t^\infty e(s) \, ds.
\]
Using (12) above we get
\[
\quad c_2 \geq \int_t^\infty e(s) \, ds,
\]
with $c = c_1/c_2$. Now, using Markov’s inequality we have that

$$L^1 \left( \{ s : e(s) > e(t)/2, s > t \} \right) \leq \frac{2}{e(t)} \int_t^\infty e(s) \, ds \leq 2ce(t).$$

Let $t$ be fixed. Recursively applying the above inequality we find that

$$L^1 \left( \{ s : e(s) > 0, s > t \} \right) \leq 4ce(t).$$

Thus if $\mathcal{P}(x(t))$ converges to $\mathcal{P}(x^*)$, it must reach it for the first time in finite time.

By construction $\mathcal{P}(x) = \mathcal{P}(x^*)$ if and only if $x \in \Lambda(x^*)$. Hence, if $x(t)$ converges to $\Lambda(x^*)$ it must reach it for the first time in finite time.

By (13) we have $\frac{d}{dt} \tilde{U}(\mathcal{P}(x(t))) \geq 0$ in a neighborhood of $\mathcal{P}(x^*)$. Since $\Gamma$ is non-degenerate, the Hessian of $\tilde{U}$ is invertible at $\mathcal{P}(x^*)$, and for all $\tilde{x} \in \tilde{X}$ in a punctured ball around $\mathcal{P}(x^*)$ we have $\tilde{U}(\tilde{x}) \neq \tilde{U}(\mathcal{P}(x^*))$. Thus, if $x(t) \to x^*$ and $\mathcal{P}(x(T)) = \mathcal{P}(x^*)$ (i.e., $x(T) \in \Lambda(x^*)$) for some $T \geq 0$, then we must have $\mathcal{P}(x(t)) = \mathcal{P}(x^*)$ (i.e., $x(t) \in \Lambda(x^*)$ for all $t \geq T$). Contrariwise, we would have $\tilde{U}(\mathcal{P}(x^*)) = \tilde{U}(\mathcal{P}(x(T))) < \lim_{s \to \infty} \tilde{U}(\mathcal{P}(x(s))) = \tilde{U}(\mathcal{P}(x^*))$, which is a contradiction.

**Appendix.**

**Lemma 23.** Suppose $\Gamma$ is a degenerate game. At any mixed equilibrium there are at least two players using mixed strategies.

**Proof.** Suppose that $x^*$ is an equilibrium in which only one player uses a mixed strategy—say, player 1. Let $C_i = \text{car}(x^*)$ and $\gamma_i = |C_i|$. Then the mixed strategy Hessian is given by $\nabla U(x) = \left(\frac{\partial^2 U(x)}{\partial x_i \partial x_k^*}\right)_{k,t = 1, \ldots, \gamma_i} = 0$, where the equality to zero follows since $U$ is linear in $x_1$.

**Lemma 24.** Let $x \in X$ and $i = 1, \ldots, N$. Assume $Y_i$ is ordered so that $y_i^1 \in BR_i(x_{-i})$. Then:

(i) For $k = 1, \ldots, K_i - 1$ we have $\frac{\partial U(x)}{\partial x_i^k} \leq 0$.

(ii) For $k = 1, \ldots, K_i - 1$, we have $y_i^{k+1} \in BR_i(x_{-i})$ if and only if $\frac{\partial U(x)}{\partial x_i^k} = 0$. In particular, combined with (i) this implies that $y_i^{k+1} \notin BR_i(x_{-i}) \iff \frac{\partial U(x)}{\partial x_i^k} < 0$.

**Proof.** (i) Differentiating (5) we find that

$$\frac{\partial U(x)}{\partial x_i^k} = U(y_i^{k+1}, x_{-i}) - U(y_i^1, x_{-i}).$$

(i) Since $y_i^1$ is a best response, we must have $U(y_i^1, x_{-i}) \geq U(y_i^{k+1}, x_{-i})$ for any $k = 1, \ldots, K_i - 1$. Hence $\frac{\partial U(x)}{\partial x_i^k} \leq 0$.

(ii) Follows readily from (5).

**Lemma 25.** Let $x \in X$. If $y_i^k \in BR_i(x_{-i})$ then $\frac{\partial U(x)}{\partial x_i^k} \geq 0$.

**Proof.** The result follows readily from (43).

**Lemma 26.** Suppose $x^*$ is an equilibrium and $y_i^k \in \text{car}(x^*)$, $k \geq 2$. Then $\frac{\partial U(x^*)}{\partial x_i^k} = 0$.

**Proof.** Since $U$ is multilinear, $y_i^k$ must be a pure-strategy best response to $x^*_{-i}$. The result then follows from Lemma 24.
Lemma 27. There exists a $c > 0$ such that $|\frac{\partial^k g(x)}{\partial x^j}| < c$, $i = 1, \ldots, \tilde{N}, k = 1, \ldots, \gamma_i - 1, j = 1, \ldots, N$, $\ell \geq \gamma_j$ for $x$ in a neighborhood of $x^\ast$.

Proof. Differentiating (10) we see that $\frac{\partial^k g(x)}{\partial x^j} = -\frac{\partial^k f(x)}{\partial x^j}$, $i = 1, \ldots, \tilde{N}, k = 1, \ldots, \gamma_i - 1, j = 1, \ldots, N$, $\ell \geq \gamma_j$.

By the definition of $g$ we have $F(x_p, g(x_p), u) = 0$ for all $x_p$ in a neighborhood of $x_p^\ast$. Hence,

$$0 = D_{x_p} F(x_p, g(x_p), u) = D_{x_p} F(x_p, x_{m}', u) |_{x_{m}' = g(x_p)} + D_{x_m} F(x_p, x_{m}, u) D_{x_p} g(x_p),$$

By (7) and (8) we see that $D_{x_m} F(x_p, x_{m}, u) = H(x)$. Since the equilibrium $x^\ast$ is assumed to be non-degenerate, $H(x^\ast)$ is invertible and the above implies that

$$D_{x_p} g(x_p^\ast) = H(x^\ast)^{-1} D_{x_p} F(x_p^\ast, x_{m}^\ast).$$

Using (8) and the multilinearity of $U$, one may readily verify that $D_{x_p} F(x_p^\ast, x_{m}^\ast, u)$ is entrywise finite. Since $g$ is continuously differentiable, it follows that each entry of

$$\left( \frac{\partial^k h(x)}{\partial x_j} \right)_{i=1, \ldots, \tilde{N}, k=1, \ldots, \gamma_i - 1 \atop j=1, \ldots, N, \ell \geq \gamma_j} = -D_{x_p} g(x_p)$$

is uniformly bounded for $x = (x_p, x_m)$ in a neighborhood of $x^\ast$. □

Lemma 28. Suppose $V : \mathbb{R}^n \to \mathbb{R}$ is twice differentiable. Suppose $x^\ast$ is a critical point of $V$ and the Hessian of $V$ at $x^\ast$, denoted by $H(x^\ast)$, is invertible. Then there exists a constant $c$ such that $\|\nabla V(x)\| \geq c d(x^\ast, x)$ for all $x$ in a neighborhood of $x^\ast$.

Proof. Suppose the claim is false. Then for any $\epsilon > 0$ there exists a sequence $(x^\ast_k)_k \subset B(x^\ast, \epsilon)$ such that $\|\nabla V(x_k^\ast)\| < \frac{1}{k} d(x_k^\ast, x^\ast)$. Let $(x_k^\ast)_k \geq 1$ be such a sequence that furthermore satisfies $\lim_{k \to \infty} d(x_k^\ast, x^\ast) = 0$. Let $y_k \in \mathbb{R}^n$, $t_k \in \mathbb{R}$ be such that $x_k^\ast = x^\ast + t_k y_k$, $\|y_k\| = 1$. Since $(y_k^\ast)_k \geq 1$ is a sequence on the unit sphere in $\mathbb{R}^n$ it has a convergent subsequence; say, $y_k \to y$ as $j \to \infty$. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(t) := V(x^\ast + ty)$. Using the continuity of $\nabla V$ we see that for any $c > 0$ we have $|f'(t)| < ct$ for all $t$ sufficiently small. Since $x^\ast$ is a critical point of $V$ we have $f'(0) = 0$. Hence

$$f''(0) = \lim_{t \to 0} \left| \frac{f'(t) - f'(0)}{t} \right| = \lim_{t \to 0} \left| \frac{f'(t)}{t} \right| < c.$$

Letting $c \to 0$ we see that $f''(0) = 0$. But this means $0 = f''(0) = y^T H(x^\ast) y$, implying the Hessian is singular, which is a contradiction. □

The following lemma characterizes the level sets of polynomial functions. Before presenting the lemma we require the following definition.

Definition 29. Given a polynomial $p : \mathbb{R}^n \to \mathbb{R}$, $n \geq 1$, let

$$Z(p) := \{ x \in \mathbb{R}^n : p(x) = 0 \}$$

be the zero-level set of $p$.

Lemma 30. Let $p(x) : \mathbb{R}^n \to \mathbb{R}$, $n \geq 1$ be a polynomial that is not identically zero. Then $\mathcal{L}^n(Z(p)) = 0$.
Proof. We will prove the result using an inductive argument.

Suppose first that \( n = 1 \) so that \( p : \mathbb{R} \to \mathbb{R} \). Let \( k \) denote the degree of \( p \). Since \( p \) is not identically zero, the fundamental theorem of algebra implies that \( p \) has at most \( k \) zeros. Hence \( \mathcal{L}^1(Z(p)) = 0 \).

Now, suppose that \( n \geq 2 \) and for any polynomial \( \tilde{p} : \mathbb{R}^{n-1} \to \mathbb{R} \) there holds \( \mathcal{L}^{n-1}(Z(\tilde{p})) = 0 \). We may write

\[
p(x, x_n) = \sum_{j=0}^{k} p_j(x)x_n^j,
\]

where \( k \) is the degree of \( p \) in the variable \( x_n \), \( x = (x_1, \ldots, x_{n-1}) \), the functions \( p_j \), \( j = 0, \ldots, k \) are polynomials in \( n-1 \) variables, and where at least one \( p_j \) is not identically zero.

If \( (x, x_n) \) is such that \( p(x, x_n) = 0 \) then there are two possibilities: Either (i) \( p_0(x) = \cdots = p_k(x) = 0 \), or (ii) \( x_n \) is the root of the one-variable polynomial \( p_x(t) := \sum_{j=1}^{k} p_j(x)t^j \).

Let \( A \) and \( B \) be the subsets of \( \mathbb{R}^n \) where (i) and (ii) hold respectively, so that \( Z(p) = \overline{A \cup B} \). For any \( x_n \in \mathbb{R} \) we have \( (x, x_n) \in \overline{A} \iff x \in Z(p_j), \forall j = 1, \ldots, k \). By the induction hypothesis, we have \( Z(p_j) = 0 \) for at least one \( j \), and hence \( \int_{\mathbb{R}^{n-1}} \chi_A(x, x_n) \, dx = 0 \) for any \( x_n \in \mathbb{R} \), where we include the argument in the characteristic function \( \chi_A \), in order to emphasize the dependence on both \( x \) and \( x_n \).

This implies that \( x_n \mapsto \int_{\mathbb{R}^{n-1}} \chi_{(x,x_n) \in A} \, dx \) is a measurable function (it’s identically zero) and

\[
\mathcal{L}^n(A) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^{n-1}} \chi_A(x, x_n) \, dx \, dx_n = 0.
\]

By the fundamental theorem of algebra, for any \( x \in \mathbb{R}^{n-1} \) there are at most \( k \) values \( t \in \mathbb{R} \) such that \( (x, t) \in B \), and hence \( \int_{\mathbb{R}} \chi_B(x, x_n) \, dx_n = 0 \). As before, this implies that \( x \mapsto \int_{\mathbb{R}} \chi_B(x, x_n) \, dx_n \) is a measurable function and

\[
\mathcal{L}^n(B) = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \chi_B(x, x_n) \, dx_n \, dx = 0.
\]

Since \( Z(p) = \overline{B \cup A} \), this proves the desired result. \( \square \)

Remark 31. Note that if \( p \equiv 0 \), then \( Z(p) = \mathbb{R}^n \). Thus, in general, if \( p : \mathbb{R}^n \to \mathbb{R} \) is a polynomial, then Lemma 30 implies that either \( Z(p) = \mathbb{R}^n \) or \( \mathcal{L}^n(Z(p)) = 0 \).

Lemma 32. Suppose \( \Gamma \) is a non-degenerate potential game. Then each indifference surface \( \mathcal{I}_{i,k,\ell} \), as defined in (26), is a union of smooth surfaces with Hausdorff dimension at most \( \kappa - 1 \).

Proof. Throughout the proof, when we refer to the dimension of a set we mean the Hausdorff dimension. Let \( i \in \{1, \ldots, N\} \), \( k, \ell \in \{1, \ldots, K_i\} \), \( k \neq \ell \) and let \( \mathcal{I} := \mathcal{I}_{i,k,\ell} \), where \( \mathcal{I}_{i,k,\ell} \) is as defined in (26). Note that \( \mathcal{I} \) is the zero-level set of the polynomial \( p(x) := U(y^*_i, x_{-i}) - U(y^*_i, x_{-i}) \). By Lemma 30 and Remark 31 we see that either \( \mathcal{L}^n(\mathcal{I}) = 0 \), or \( \mathcal{I} = X \). Being the level set of a polynomial, if \( \mathcal{L}^n(\mathcal{I}) = 0 \), then \( \mathcal{I} \) is the union of smooth surfaces with dimension at most \( \kappa - 1 \).

Suppose that \( \mathcal{I} \) has dimension greater than \( \kappa - 1 \). Then by the above, we see that \( \mathcal{I} = X \). Since \( \Gamma \) is a finite normal-form game, there exists at least one equilibrium \( x^* \in X \). Letting \( x^* \) be written componentwise as \( x^* = ([x^*_i]_{j=1}^N)_{i=1}^{m=1} \cdots K_i \), we see that if \( [x^*_i]_i > 0 \), then \( x^* \in \mathcal{I} = X \) implies that \( x^* \) is a second-order degenerate
equilibrium. Otherwise, if $[x^*_i]^k = 0$, then $x^*_i \in I = X$ implies that $x^*$ is a first-order degenerate equilibrium. In either case we see that $x^*$ is a degenerate equilibrium, and hence $\Gamma$ is a degenerate game, which is a contradiction.

Since $I$ was an arbitrary indifference surface, we see that if $\Gamma$ is a non-degenerate game, then every indifference surface has dimension at most $\kappa - 1$.

**Lemma 33.** Let $B_\epsilon$ be as defined in the proof of Proposition 15. Then, $\mathcal{H}^{\kappa-1}(\partial B_\epsilon) \to 0$ as $\epsilon \to 0$.

**Proof.** Following standard notation (see [9], Chapter 2), for $0 \leq s < \infty$, $0 < \delta \leq \infty$, and $A \subset \mathbb{R}^n$, let

$$
\mathcal{H}^s(A) := \inf \left\{ \sum_{j=1}^{\infty} \alpha(s) \left( \frac{\text{diam} C_j}{2} \right)^s : A \subset \bigcup_{j=1}^{\infty} C_j, \ \text{diam} C_j \leq \delta \right\},
$$

where $\alpha(s) := \frac{\pi^s}{\Gamma(s+1)}$, and where $\Gamma$ in this context denotes the $\Gamma$ function.

By our construction of $(B^j_\epsilon)_{j \geq 1}$, for every $\epsilon > 0$ we have

$$
\sum_{j=1}^{\infty} \alpha(s) \left( \frac{\text{diam} B^j_\epsilon}{2} \right)^{\kappa-2} < 2 \mathcal{H}^{\kappa-2}(Q \cup \partial Z) < \infty.
$$

Since $\text{diam} B^j_\epsilon \leq \epsilon$ for every $\epsilon > 0$, $j \in \mathbb{N}$, this gives

$$
\lim_{\epsilon \to 0} \mathcal{H}^{\kappa-1}(\partial B_\epsilon) \leq \lim_{\epsilon \to 0} \mathcal{H}^\epsilon(B_\epsilon)
$$

$$
\leq \lim_{\epsilon \to 0} \sum_{j=1}^{\infty} \alpha(s) \left( \frac{\text{diam} B^j_\epsilon}{2} \right)^{\kappa-2}
$$

$$
\leq \lim_{\epsilon \to 0} \epsilon \sum_{j=1}^{\infty} \alpha(s) \left( \frac{\text{diam} B^j_\epsilon}{2} \right)^{\kappa-2}
$$

$$
= \lim_{\epsilon \to 0} \epsilon \mathcal{H}^{\kappa-2}(Q \cup \partial Z) = 0.
$$

By the definition of the Hausdorff measure we have $\mathcal{H}^{\kappa-1}(\partial B_\epsilon) := \sup_{\delta > 0} \mathcal{H}^{\kappa-1}(\partial B_\epsilon)$. Hence, the above implies $\lim_{\epsilon \to 0} \mathcal{H}^{\kappa-1}(\partial B_\epsilon) = 0$.

**Lemma 34.** Let $Q$ be defined as in Section 5.1. Then $\text{cl} Q$ has Hausdorff dimension at most $\kappa - 2$.

**Proof.** Let $A$ be the subset of $X$ where two or more decision surfaces intersect. Let $N \subset A$ be the subset of $X$ where two or more decision surfaces intersect and their normal vectors coincide. Define the *relative interior* of $N$ with respect to $A$ as

$$
\text{ri} N := \{ x \in N : \exists \epsilon > 0 \text{ s.t. } B(x, \epsilon) \cap A \subset N \},
$$

and define the *relative boundary* of $N$ with respect to $A$ as

$$
\partial N := \text{cl} N \setminus \text{ri} N.
$$

Since each indifference surface has Hausdorff dimension $\kappa - 1$, $N$ has Hausdorff dimension at most $\kappa - 1$. In particular, $N$ is the union of a finite number of smooth
\[ k - 1 \text{ dimensional surfaces and a component with Hausdorff dimension at most } k - 2. \]
This implies that the relative boundary of \( N \) has Hausdorff dimension at most \( k - 2 \).

Let \( Q \) be as defined in Section 5.1. Note that the closure of \( \bar{Q} \) satisfies \( \cl \bar{Q} \subseteq V \cup N \). Since the sets \( \bar{Q} \) and \( \partial N \) have Hausdorff dimension at most \( k - 2 \), the set \( \cl \bar{Q} \) also has Hausdorff dimension at most \( k - 2 \).

Let \( \Lambda(x^*) \) be as defined in Section 5. If \( \Lambda(x^*) = \{ x^* \} \), then \( \Lambda(x^*) \) is closed and has Hausdorff dimension 0. Otherwise, \( \Lambda(x^*) \) is defined as the graph of \( g \). In Section 4.1 it was shown that \( \Graph(g) \) has Hausdorff dimension at most \( k - 2 \). Since \( g \) is a smooth function, the closure of \( \Graph(g) \) has Hausdorff dimension at most \( k - 2 \).

Recall that \( Q \) is defined as \( Q = \bar{Q} \cup \Lambda(x^*) \) and hence \( \cl Q = \cl \bar{Q} \cup \cl \Lambda(x^*) \). Since \( \cl \bar{Q} \) and \( \cl \Lambda(x^*) \) each have Hausdorff dimension at most \( k - 2 \), \( \cl Q \) also has Hausdorff dimension at most \( k - 2 \).

**Lemma 35.** Let \( Z \) be as defined (28). Then for any \( x \in Z \) there holds \( \nu \cdot y = 0 \) for any vector \( \nu \) normal to \( Z \) at \( x \), and any \( y \in \FP(x) \).

**Proof.** Suppose \( x \in X \) and \( x \) is in some indifference surface \( \mathcal{I}_{i,k,\ell} \). Suppose \( \nu \) is a vector that is normal to \( \mathcal{I}_{i,k,\ell} \) at \( x \). By the definition of \( \mathcal{I}_{i,k,\ell} \), if \( x \in \mathcal{I}_{i,k,\ell} \) then for all \( \hat{x} \in X \) such that \( \hat{x}_{i,-i} = x_{i,-i} \) we have \( \hat{x} \in \mathcal{I}_{i,k,\ell} \). This implies that the \((i, k)\)-th component of \( \nu \) must be zero for every \( k = 1, \ldots, K_{i} - 1 \).

For \( x \in X \), let \( \mathcal{N}(x) := \{(i, k) : i \in \{1, \ldots, N\}, k \in \{1, \ldots, K_{i} - 1\}, \quad x \in \mathcal{I}_{i,k,\ell} \text{ for some } \ell = 1, \ldots, K_{i} - 1, \quad \ell \neq k\} \). Letting \( \FP^{k}_{i} \) be the \((i, k)\)-th component map of \( \FP \), note that by the definition of an indifference surface, \( \FP^{k}_{i}(x) \) is single valued for every pair \((i, k) \notin \mathcal{N}(x) \).

Suppose \( x \in X \setminus Q \) is in at least one decision surface \( \mathcal{I} \) and let \( \nu \) be a vector that is normal to \( \mathcal{I} \) at \( x \). Note that \( x \notin Q \) implies that if \( x \) is contained in any other decision surface \( \mathcal{I} \neq \mathcal{I} \) at \( x \), then \( \nu \) is also normal to \( \mathcal{I} \) at \( x \). Letting \( \nu \) be written componentwise as \( \nu = (\nu_{i}^{k})_{i=1}^{N}, \quad k = 1, \ldots, K_{i} - 1 \), the above discussion implies that \( \nu_{i}^{k} = 0 \) for every pair \((i, k) \in \mathcal{N}(x) \).

Now suppose \( x \in Z \). By the definition of \( Z \) we have \( x \notin Q \) and \( x \) is in at least one decision surface \( \mathcal{I} \). Let \( \nu \) be a vector that is normal to \( \mathcal{I} \) at \( x \). By the definition of \( Z \), there exists some \( y \in \FP(x) \) such that \( y \cdot \nu = 0 \). Breaking this down in terms of components in \( \mathcal{N}(x) \) we have

\[
0 = y \cdot \nu = \sum_{(i,k) \in \mathcal{N}(x)} y_{i}^{k} \nu_{i}^{k} + \sum_{(i,k) \notin \mathcal{N}(x)} y_{i}^{k} \nu_{i}^{k}.
\]

The first sum is zero since \( \nu_{i}^{k} = 0 \) for all \((i, k) \in \mathcal{N}(x) \). Consequently, the second sum must also be zero. But we have shown above that \( \FP^{k}_{i}(x) \) is single valued for any \((i, k) \notin \mathcal{N}(x) \). Hence, for any \( \tilde{y} \in \FP(x) \) we have \( \tilde{y}_{i}^{k} = y_{i}^{k} \) for all \((i, k) \notin \mathcal{N}(x) \), and in particular, \( \sum_{(i,k) \notin \mathcal{N}(x)} \tilde{y}_{i}^{k} \nu_{i}^{k} = \sum_{(i,k) \notin \mathcal{N}(x)} y_{i}^{k} \nu_{i}^{k} = 0 \). Moreover, since \( \nu_{i}^{k} = 0 \) for all \((i, k) \in \mathcal{N}(x) \) we have \( \sum_{(i,k) \in \mathcal{N}(x)} \tilde{y}_{i}^{k} \nu_{i}^{k} = 0 \), which implies

\[
\tilde{y} \cdot \nu = \sum_{(i,k) \in \mathcal{N}(x)} \tilde{y}_{i}^{k} \nu_{i}^{k} + \sum_{(i,k) \notin \mathcal{N}(x)} \tilde{y}_{i}^{k} \nu_{i}^{k} = 0.
\]

Since \( \tilde{y} \in \FP(x) \) was arbitrary, this proves the desired result. \( \blacksquare \)

**References**


