We propose a novel discrete signal processing framework for the representation and analysis of datasets with complex structure. Such datasets arise in many social, economic, biological, and physical networks. Our framework extends traditional discrete signal processing theory to structured datasets by viewing them as signals represented by graphs, so that signal coefficients are indexed by graph nodes and relations between them are represented by weighted graph edges. We discuss the notions of signals and filters on graphs, and define the concepts of the spectrum and Fourier transform for graph signals. We demonstrate their relation to the generalized eigenvector basis of the graph adjacency matrix and study their properties. As a potential application of the graph Fourier transform, we consider the efficient representation of structured data that utilizes the sparseness of graph signals in the frequency domain.

Index Terms—Graph signal processing, graph signal, graph filter, graph spectrum, graph Fourier transform, generalized eigenvectors, sparse representation.

1. INTRODUCTION

Recently we have been observing a growth of interest in the efficient techniques for representation, analysis and processing of large datasets emerging in various fields and applications, such as sensor and transportation networks, internet and world wide web, image and video databases, and social and economic networks. These datasets share a common trait: their elements are related to each other in a structured manner, for example, through similarities or dependencies between data elements. This relational structure is often represented with graphs, in which data elements correspond to nodes, relation between elements are represented by edges, and the strength or significance of relations is reflected in edge weights.

The analysis and processing of structured data has been studied in multiple ways. Graph properties, such as degree distributions, node centrality and betweenness, and clustering, are often used to infer the community structure and interaction in social and economic networks [1, 2]. Inference and learning of structured datasets can be performed using graphical models [3, 4] by viewing data elements as random variables and expressing their probabilistic dependencies between each other with graph edges. Data learning, clustering, and classification has been approached using spectral graph theory [5]. A common feature of these approaches, however, is that they analyze the graphs that represents the relational structure of datasets, rather than the datasets themselves. Another technique for the representation and spectral analysis of data based on the Laplacian matrix of the graph and its eigenvectors has become popular recently [6, 7]. This technique is more similar to existing signal processing approaches, and to our work in particular; however, it is restricted to undirected graphs with real, non-negative edge weights.

We propose a framework, called discrete signal processing on graphs (DSPG), for the representation, processing, and analysis of structured datasets that can be represented by graphs. Our framework extends the traditional discrete signal processing (DSP) theory that studies signals with linear structure, such as time series and space signals, e.g. images, to signals with complex, non-linear structure. We discuss the notions of signals and filters on graphs, and then define the concepts of spectral decomposition, spectrum, and Fourier transform for graph signals. We identify their relation to the generalized eigenvectors of the adjacency matrices of representation graphs and study their properties. As a potential application of the graph Fourier transform, we consider efficient data representation and compression. In particular, we demonstrate that if a graph signal is sparsely represented in the spectral domain, i.e. its frequency content is dominated by few frequencies, then it can be efficiently approximated with only a few spectrum coefficients.

2. SIGNALS AND FILTERS ON GRAPHS

In this section we discuss the notions of graph signals and filters. These concepts are defined and studied in [8].

Graph signals. If we consider a quantitative dataset for which we are given information about the relationship between its elements, we can represent it as a numerical-valued signal indexed by a graph. For example, for a set of sensor measurements, the relation between measurements from different sensors can be expressed through the physical distance between sensors. For a collection of researchers and their publication records, the relation can be given by their collaborations and publication coauthoring. Assuming that the dataset is finite, we can treat it as a set of vectors

\[
S = \{s : s = (s_0, \ldots, s_{N-1})^T, s_n \in \mathbb{C}\}.
\]

Then, we can represent the relation between coefficients \(s_n\) of \(s\) with a graph \(G = (V, A)\), so that \(V = \{v_0, \ldots, v_{N-1}\}\) is a set of \(N\) nodes, and \(A\) is a \(N \times N\) weighted adjacency matrix. Each coefficient \(s_n\) corresponds to (is indexed by) node \(v_n\), and the weight \(A_{n,m}\) of the directed edge from \(v_m\) to \(v_n\) expresses the degree of relation of \(s_n\) to \(s_m\). Note that edge weights \(A_{n,m}\) can take arbitrary real or complex values (for example, if data elements are negatively correlated). We call a signal \(s\) indexed by a graph \(G\) a graph signal.

Graph signals, in general, can be complex-valued. Furthermore, they can be added together and scaled by constant coefficients. Hence, they form a vector space. If no additional assumptions are made on their values, the set \(S\) of graph signals corresponds to the \(N\)-dimensional complex vector space \(S = \mathbb{C}^N\).

We illustrate representation graphs with several examples. The graph in Fig. 1(a) represents a finite, periodic discrete time series [9, 10]. It is a directed, cyclic graph, with directed edges of the same weight, reflecting the causality of a time series. The periodicity of the time series is captured by the edge from \(v_{N-1}\) to \(v_0\). The two-dimensional rectangular lattice graph in Fig. 1(b) represents a digital image. Each pixel corresponds to a node that is connected to the nodes that index its four adjacent pixels. This relation is symmetric, hence all edges are undirected. If no additional information is available, all edge weights \(a_{ij}\) and \(b_{ij}\) are equal, with a possible exception of boundary nodes which may have directed edges and different

This work was supported in part by AFOSR grant FA95501210087.
Having defined the concepts of graph signals and filters, we now discuss the spectral decomposition, spectrum, and Fourier transform for graph signals. These concepts are related to the Jordan normal form of the adjacency matrix A; this topic is discussed in [12].

**Spectral decomposition.** The spectral decomposition of a signal space $\mathcal{S}$ corresponds to the identification of subspaces $\mathcal{S}_k$, $0 \leq k < K$, of $\mathcal{S}$ that are invariant to filtering. For a signal $s_k \in \mathcal{S}_k$ from a subspace $\mathcal{S}_k$, the output $\tilde{s}_k = h(A)s_k$ of any filter $h(A)$ is also a signal from the same subspace $\mathcal{S}_k$. The signal $s \in \mathcal{S}$ then can be represented as

$$s = s_0 + s_1 + \ldots + s_{K-1},$$

(4)

where $s_k \in \mathcal{S}_k$. Decomposition (4) is uniquely determined for every graph signal $s \in \mathcal{S}$ if and only if 1) invariant subspaces $\mathcal{S}_k$ have zero intersection, i.e., $\mathcal{S}_k \cap \mathcal{S}_m = \{0\}$ for any $k \neq m$; 2) $\dim \mathcal{S}_0 + \ldots + \dim \mathcal{S}_{K-1} = \dim \mathcal{S} = N$; and 3) each $\mathcal{S}_k$ is irreducible, i.e., it cannot be decomposed into smaller invariant subspaces.

Consider the Jordan decomposition of $A$:

$$A = V J V^{-1}.$$  

(5)

Here, $J$ is the Jordan normal form and $V$ is the matrix of generalized eigenvectors. Let $\lambda_m$ denote an arbitrary eigenvalue of $A$, and $v_{m,0}, \ldots, v_{m,r}$ denote a Jordan chain of generalized eigenvectors corresponding to this eigenvalue. Then $S_m = \text{span}\{v_{m,0}, \ldots, v_{m,r}\}$ is a vector subspace of $\mathcal{S}$ with this Jordan chain as its basis. Any signal $s_m \in S_m$ has a unique expansion in this basis:

$$s_m = \tilde{s}_m = \sum_{r=0}^N A_{n,m} \tilde{s}_m \iff \tilde{s} = As.$$  

(2)

Similarly to traditional linear, time-invariant DSP theory, we consider linear, shift-invariant filters for graph signals. As demonstrated in [8], any linear, shift-invariant graph filter is necessarily a matrix polynomial in the adjacency matrix $A$ of the form

$$h(A) = h_0 I + h_1 A + \ldots + h_L A^L,$$  

(3)

with possibly complex coefficients $h_\ell \in \mathbb{C}$. Furthermore, any graph filter (3) can be represented by at most $N$ coefficients; and if it is invertible, its inverse also is a matrix polynomial in $A$ of the form (3).

### 3. GRAPH FOURIER TRANSFORM

Having defined the concepts of graph signals and filters, we now discuss the spectral decomposition, spectrum, and Fourier transform for graph signals.
Graph Fourier transform. The spectral decomposition of $S$ expands each signal $s \in S$ in the basis given by the union of all generalized eigenvectors. This expansion can be written as

$$s = V \hat{s},$$

where the vector of expansion coefficients is given by

$$\hat{s} = V^{-1} s.$$  

We call the basis of generalized eigenvectors the graph Fourier basis, and the expansion (10) the graph Fourier transform. We denote the graph Fourier transform matrix as

$$F = V^{-1}. \quad (11)$$

Following the conventions of classical DSP, we call the coefficients $\hat{s}_n$ in (10) the spectrum of a signal $s$. The inverse graph Fourier transform is given by (9); it reconstructs the signal from its spectrum.

**Discussion.** The connection (10) between the graph Fourier transform and the Jordan decomposition (5) highlights some desirable properties of representation graphs. For instance, graphs with transformed and the Jordan decomposition (5) is interpreted as the representation of the signal in terms of stable, unchangeable opinions or preferences.

Graph signals can thus represent a social network. In this case filtering can be viewed as an opinion or a preference of individuals, and the indexing graph may contain a characteristic of a social network, such as changeable opinions or preferences.

Efficient signal representation is required in multiple areas of signal processing, such as storage, compression, and transmission. Some widely-used techniques are based on expanding signals into suitable bases with the expectation that most information about the signal is captured with few basis functions. For example, some image compression standards, e.g. JPEG and JPEG 2000, expand images into cosine or wavelet bases, which yield high-quality approximations for smooth images [15].

If a representation basis corresponds to a Fourier basis in some signal model, we say that signals are sparse in the frequency domain if their spectrum is dominated by only a few frequencies, i.e. they are accurately approximated by a few Fourier basis functions. As we demonstrate in the following examples, graphs signals can be sparse in their respective frequency domain, which makes their Fourier bases useful for efficient signal representation and compression$^1$. For simplicity of the discussion and calculations, we consider signals represented by undirected graphs. In this case, as discussed in Section 3, corresponding graph Fourier transforms are orthogonal matrices, and the Fourier bases are orthogonal. The advantage of an orthogonal basis is that selecting spectrum components with largest magnitudes minimizes the approximation error in the least-squares sense. The approach discussed here also extends to directed graphs with general Fourier bases.

**Compression algorithm.** Given an orthogonal graph Fourier basis, we compress a graph signal $s$ by keeping only $C$ of its spectrum coefficients ($\hat{s}_k$) that have largest magnitudes. Without loss of generality, assume that $|\hat{s}_0| \geq |\hat{s}_1| \geq \ldots \geq |\hat{s}_N|$. Then the signal reconstructed after compression is

$$\tilde{s} = F^{-1} (\hat{s}_0, \ldots, \hat{s}_{C-1}, 0, \ldots, 0)^T. \quad (12)$$

If for $0 \leq k < K$, signals $s_k$ are approximated as $\tilde{s}_k$, each with $C$ largest-magnitude coefficients of their spectrum, we calculate the average approximation error as

$$err(C) = \frac{\sum_{k=0}^{K-1} ||s_k - \tilde{s}_k||_2}{\sum_{k=0}^{K-1} ||s_k||_2}.$$  

**Image compression.** As the first example, we consider the graph representation of images using the graph in Fig. 1(b). As follows from the figure, we make a simplifying assumptions that edge weights depend only on their row or column, as shown. Then, given a specific image, we determine the edge weights $a_n$ and $b_m$ by minimizing the distortion caused by the graph shift:

$$\left\{ \begin{array}{c} a_0, \ldots, a_{N-2}, \\ b_0, \ldots, b_{M-2} \end{array} \right\} = \arg \min_{a_n, b_m \in \mathbb{C}} ||As - s||_2.$$  

Here, $s$ is a vectorized representation of the image. As demonstrated in [17], this is a least-squares minimization problem.

For the evaluation of this image representation technique, we consider $K = 4$ images shown in Fig. 2, all of size $256 \times 256$. Table 1 shows average errors (13) obtained for different fractions of spectrum coefficients used for approximation. For comparison, we also consider three standard orthogonal transforms: the discrete Fourier (DFT), cosine (DCT), and wavelet (DWT) transforms. As can be observed from the results, the graph Fourier transform leads to smallest errors regardless of the number of spectrum coefficients used for approximation.

$^1$Eigenvectors of the graph Laplacian matrix have also been considered for the compression of graph signals [16]. In contrast, our approach use the generalized eigenvectors of the graph adjacency matrix.
Compressions of sensor measurements. Another example we consider is the representation of measurements from a non-uniformly distributed sensor field. In particular, we consider a set of daily temperature measurements from weather stations located near 150 major US cities [18]. Data from each sensor is a separate time series; however, compressing each time series separately requires buffering measurements from multiple days before they can be compressed for storage or transmission. Instead, we consider graph signals constructed from daily snapshots of all 150 measurements. We construct the representation graph, shown in Fig. 1(c), using geographical distances between sensors. Each sensor corresponds to a node \( v_n \), \( 0 \leq n < 150 \), and is connected to 8 nearest sensors with undirected edges weighted by the normalized inverse exponents of the squared distances: if \( d_{nm} \) denotes the distance between the \( n \)th and \( m \)th sensors and \( v_{nm} \) is connected to \( v_n \), then

\[
A_{n,m} = \frac{e^{-d_{nm}}}{\sqrt{\sum_{k \in N_n} e^{-d_{nk}} \sum_{k \in N_m} e^{-d_{mk}}}}.
\]  

(14)

We consider a full year of 365 daily measurements from each sensor, and evaluate the representation efficiency by calculating the average approximation error (13) over \( K = 365 \) days. For comparison, we also consider compressing each separate time series of measurements from each station with DFT and DCT, and calculating average errors over \( K = 150 \) stations. The results are shown in Table 2. The graph Fourier transform yields smallest errors for all fractions of spectrum coefficients used for approximation.

<table>
<thead>
<tr>
<th>Transform</th>
<th>Fraction of coefficients used (C/N)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>2%</td>
</tr>
<tr>
<td>Graph FT</td>
<td>10%</td>
</tr>
<tr>
<td>DFT</td>
<td>14%</td>
</tr>
<tr>
<td>DCT</td>
<td>12%</td>
</tr>
</tbody>
</table>

Table 2. Average approximation errors for digital images.

5. CONCLUSIONS

We have proposed a framework for discrete signal processing of signals indexed by graphs. We discussed the notions of graph signals and filters, and defined the concepts of spectral decomposition, spectrum, and Fourier transform for graph signals. We identified their relation to the Jordan decomposition of the adjacency matrices of representation graphs. As a potential application of the graph Fourier transform, we demonstrated that graph signals can be sparsely represented in their frequency domain, and thus efficiently approximated using a few Fourier basis functions with little approximation error.

6. REFERENCES