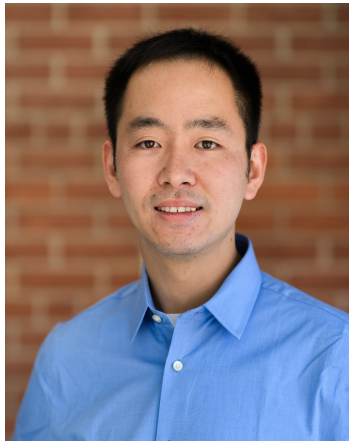
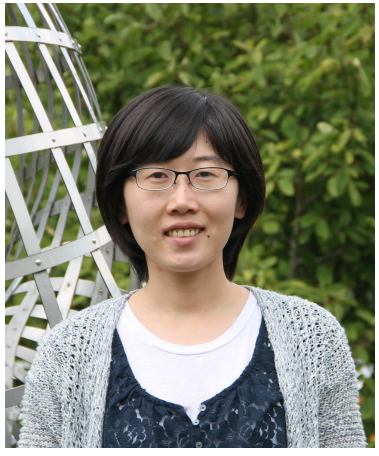
Taming Nonconvexity in Information Science



Yuxin Chen
Princeton



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ITW 2018 Tutorial

Guangzhou, China

Acknowledgement

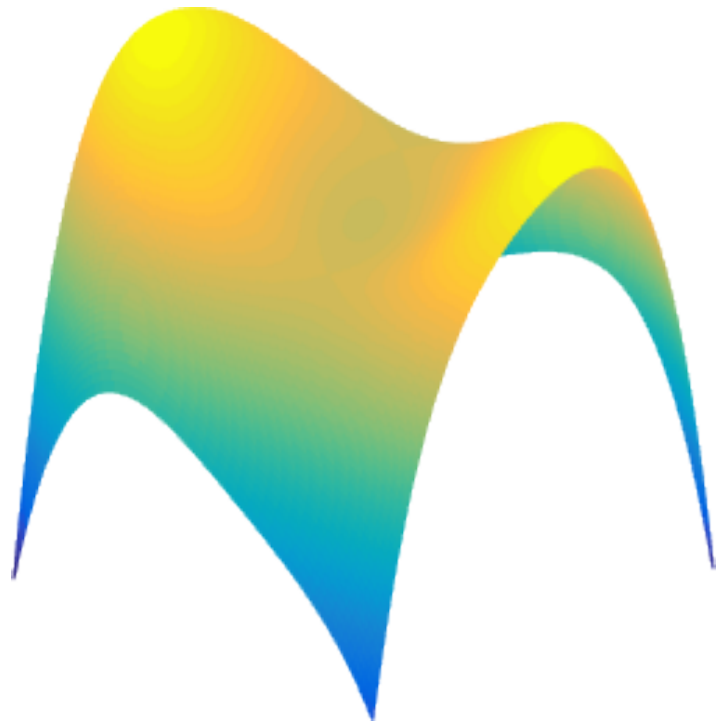
- Our collaborators: Emmanuel J. Candès, Jianqing Fan, Yuanxin Li, Yingbin Liang, Yue M. Lu, Cong Ma, Kaizheng Wang, Huishuai Zhang

- This work is supported in part by ARO W911NF-18-1-0303, AFOSR FA9550-19-1-0030 and FA9550-15-1-0205, ONR N00014-18-1-2142, and NSF ECCS-1818571 and CCF-1806154.

Nonconvex estimation problems are everywhere

Empirical risk minimization is usually nonconvex

$$\text{minimize}_{\boldsymbol{x}} \quad f(\boldsymbol{x}; \text{data}) \quad \rightarrow \quad \text{loss function may be nonconvex}$$



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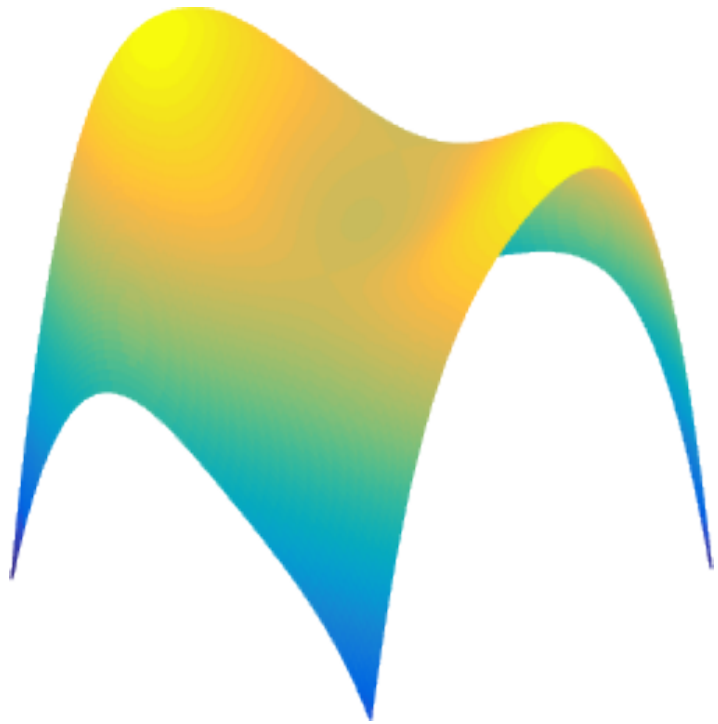
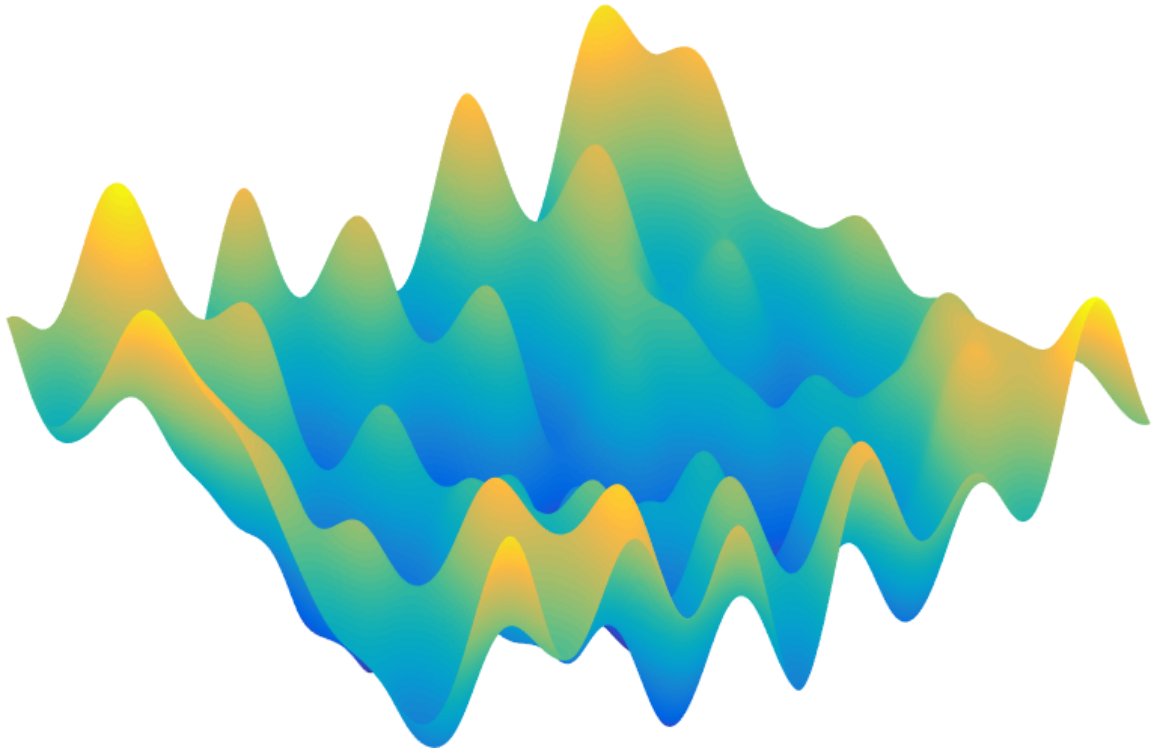
$$\text{minimize}_{\boldsymbol{x}} \quad f(\boldsymbol{x}; \text{data}) \quad \rightarrow \quad \text{loss function may be nonconvex}$$

- low-rank matrix completion
- blind deconvolution
- dictionary learning
- mixture models
- deep learning
- ...



Nonconvex optimization may be super scary



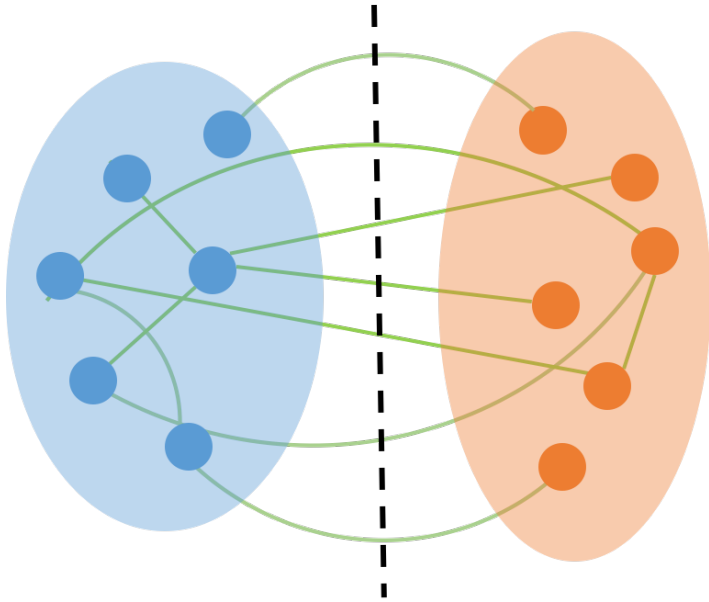
There may be bumps everywhere and exponentially many local optima

e.g. 1-layer neural net (Auer, Herbster, Warmuth '96; Vu '98)

Example: solving quadratic programs is hard

Finding maximum cut in a graph is about solving a quadratic program

$$\begin{aligned} \text{maximize}_{\boldsymbol{x}} \quad & \boldsymbol{x}^\top \boldsymbol{W} \boldsymbol{x} \\ \text{subj. to} \quad & x_i^2 = 1, \quad i = 1, \cdots, n \end{aligned}$$



Example: solving quadratic programs is hard

"I can't find an efficient algorithm, but neither can all these people."

figure credit: coding horror

$1,000,000 question

One strategy: convex relaxation

Can relax into convex problems by

- finding convex surrogates (e.g. matrix completion)
- lifting into higher dimensions (e.g. Max-Cut)

Example of convex surrogate: matrix completion

figure credit: Candès et al.

Netflix challenge

Predict unseen ratings

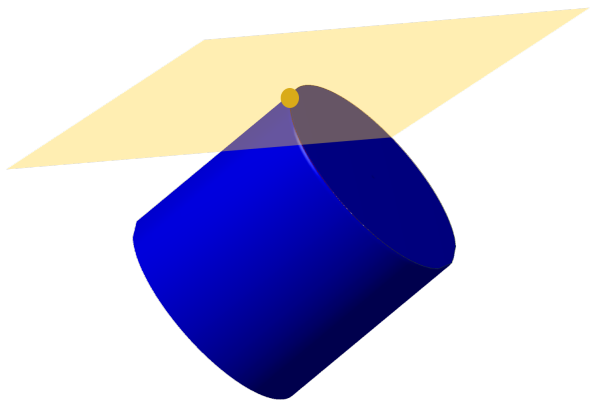
Example of convex surrogate: matrix completion

— Fazel '02, Recht, Parrilo, Fazel '10, Candès, Recht '09

minimize$_{\boldsymbol{M}}$ rank$(\boldsymbol{M})$ subj. to data constraints

⇩ cvx surrogate

minimize$_{\boldsymbol{M}}$ nuc-norm$(\boldsymbol{M})$ subj. to data constraints



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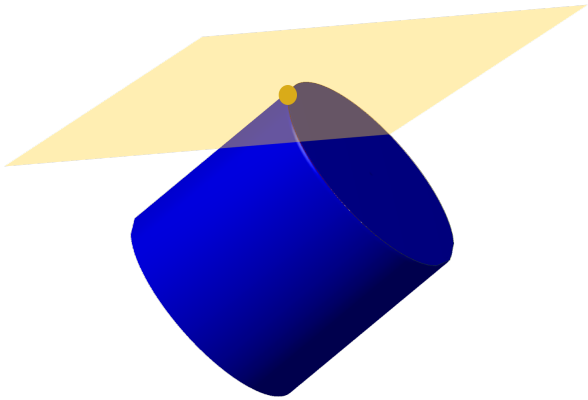

Robust variation used everyday by Netflix Candès, Li, Ma, Wright '10

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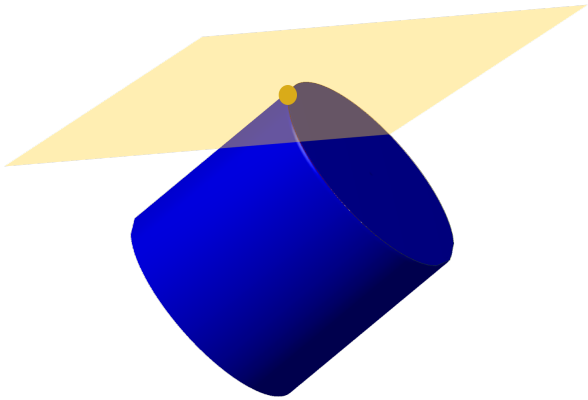
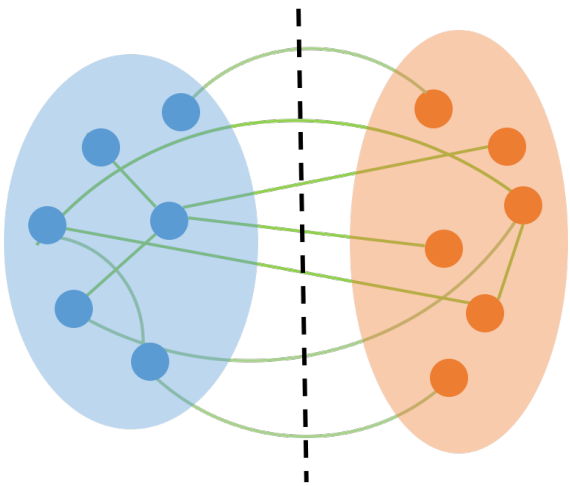


Robust variation used everyday by Netflix Candès, Li, Ma, Wright '10

Problem: operate in *full* matrix space even though $\boldsymbol{X}$ is low-rank

Example of lifting: Max-Cut

— Goemans, Williamson '95



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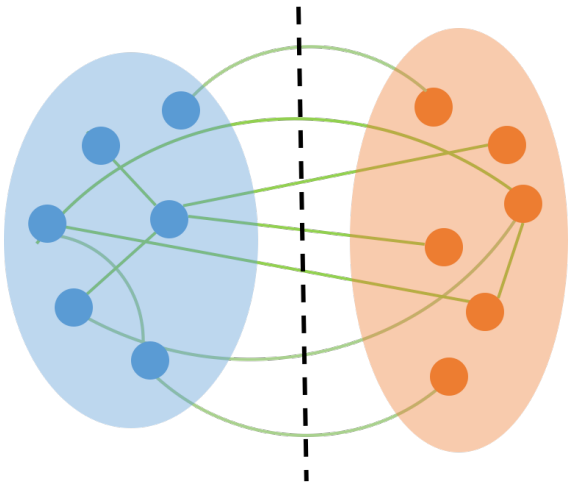

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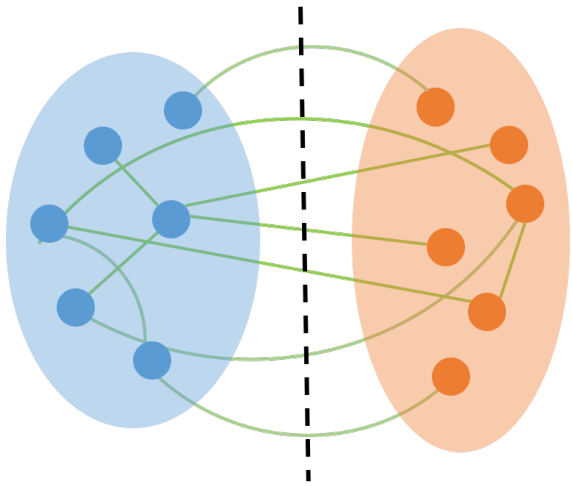

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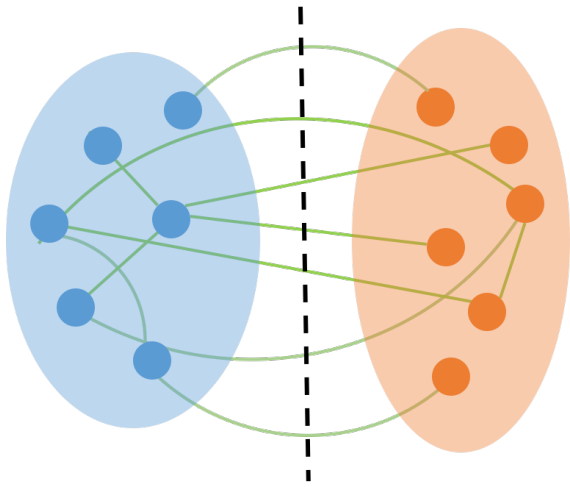

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Problem: explosion in dimensions ($\mathbb{R}^n \to \mathbb{R}^{n \times n}$)

How about optimizing nonconvex problems directly without lifting?

Nonconvex optimization

Complicated nonconvex problems are solved on a daily basis via simple algorithms such as stochastic gradient descent

Nonconvex optimization

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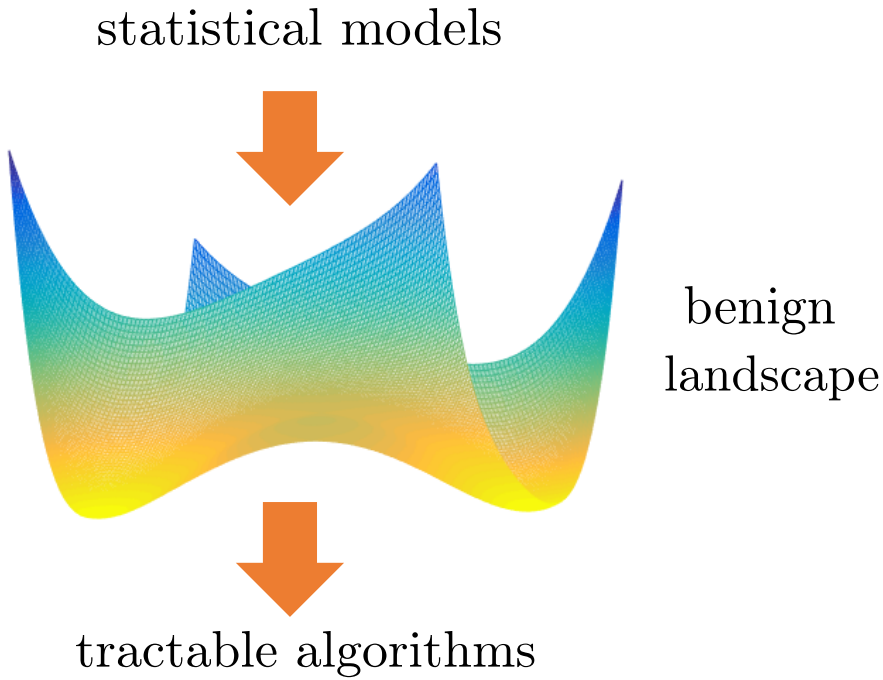
- How come simple nonconvex algorithms work so well in practice?

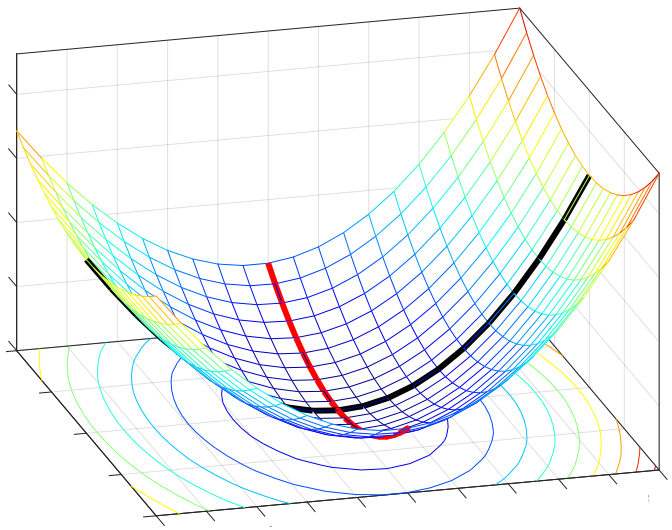
Statistical models come to rescue



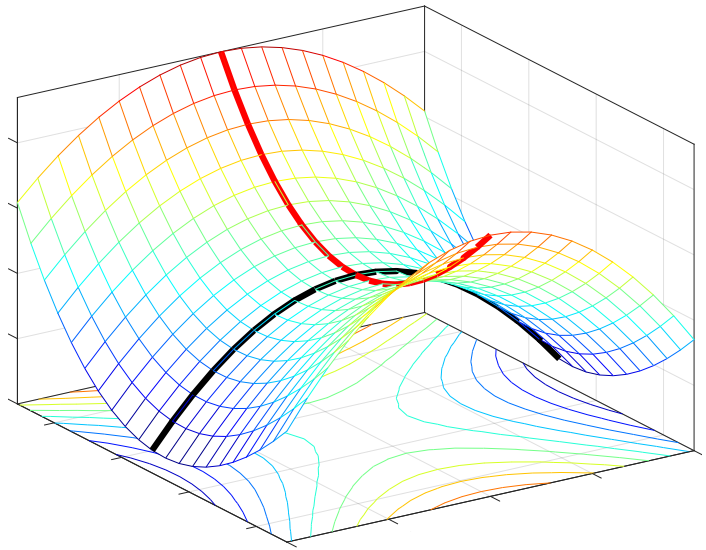
When data are generated by certain statistical models, problems are often much nicer than worst-case instances

Sometimes they are much nicer than we think

Under certain statistical models,
we see benign global geometry: no spurious local optima



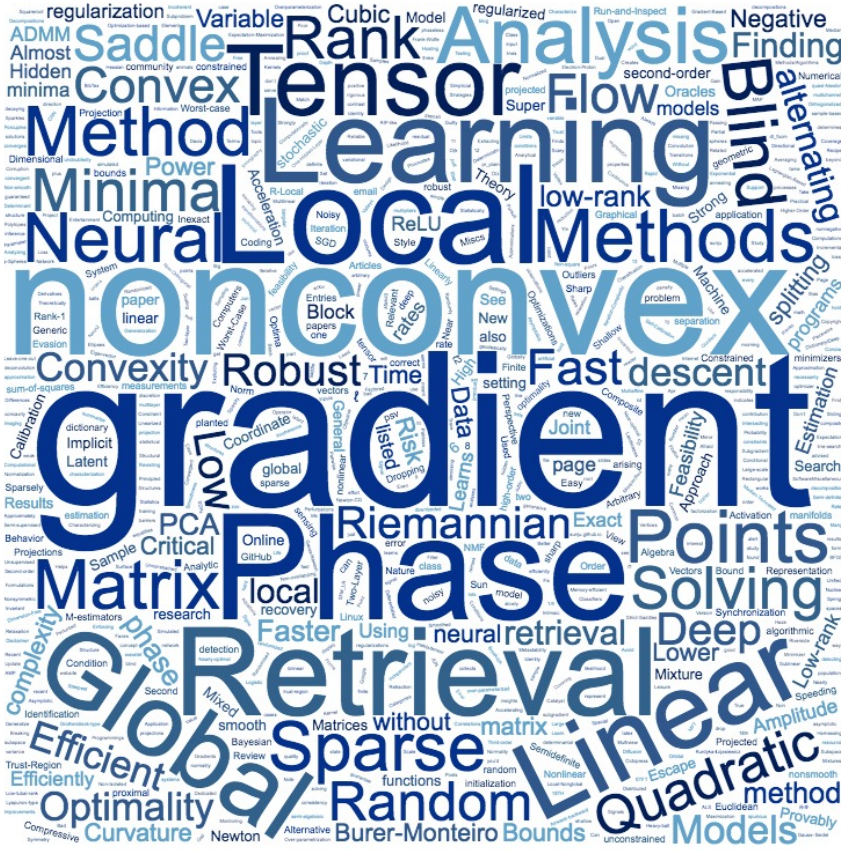
global minimum



saddle point

Even the simplest possible nonconvex methods
might be remarkably efficient under suitable statistical models

Nonconvex optimization with performance guarantees



Phase retrieval: Gerchberg-Saxton '72, Netrapalli et al. '13, Candès, Li, Soltanolkotabi '14, Chen, Candès '15, Cai, Li, Ma '15, Zhang et al. 16, Wang et al. '16, Sun et al. '16, Ma et al. '17, Chen et al. '18, ...

Matrix completion: Keshavan et al. '09, Jain et al. '09, Hardt '13, Sun, Luo '15, Chen, Wainwright '15, Zheng, Lafferty '16, Ge et al. '16, Jin et al. '16, Ma et al. '17, ...

Matrix sensing: Jain et al. '13, Tu et al. '15, Zheng, Lafferty '15, Bhojanapalli et al. 16, Li, Zhu, Tang '18, ...

Blind deconvolution / demixing: Li et al. '16, Lee et al. '16, Ling, Strohmer '16, Huang, Hand '16, Ma et al. '17, Zhang et al. '18, Li, Bresler '18, Dong, Shi '18, ...

Dictionary learning: Arora et al. '14, Sun et al. '15, Chatterji, Bartlett '17, ...

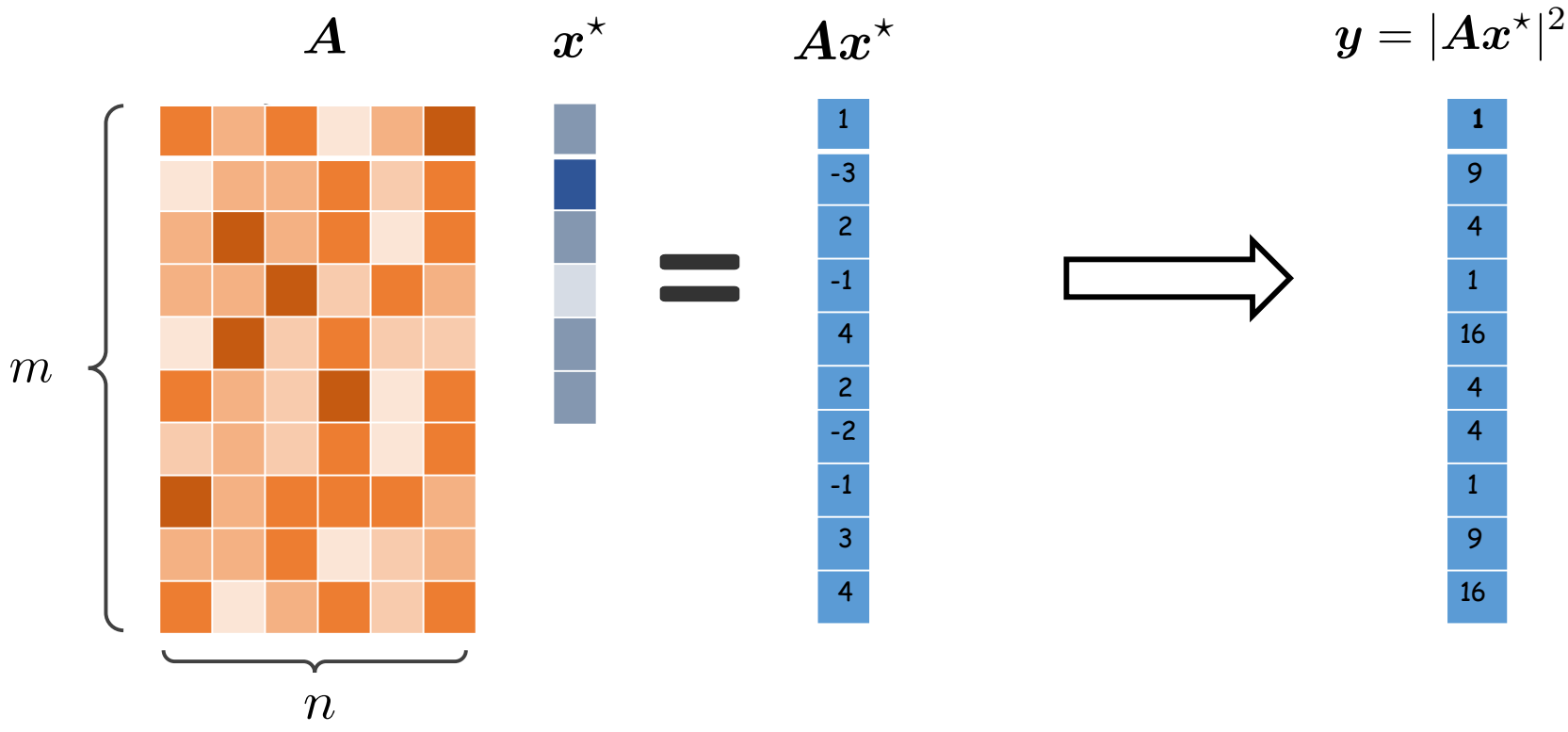
Robust principal component analysis: Netrapalli et al. '14, Yi et al. '16, Gu et al. '16, Ge et al. '17, Cherapanamjeri et al. '17, ...

- *http://sunju.org/research/nonconvex/*
- *"Nonconvex Optimization Meets Low-Rank Matrix Factorization: An Overview," Chi, Lu, Chen '18*

Outline

- Part I: Overview
- Part II: Phase retrieval: a case study
  - Spectral initialization
  - Local refinement: algorithm and analysis
- Part III: Low-rank matrix estimation
- Part IV: Closing remarks

Solving quadratic systems of equations



Recover $\boldsymbol{x}^\star \in \mathbb{R}^n$ from m random quadratic measurements

$$y_k \quad = \quad (\boldsymbol{a}_k^\top \boldsymbol{x}^\star)^2, \qquad k = 1, \ldots, m$$

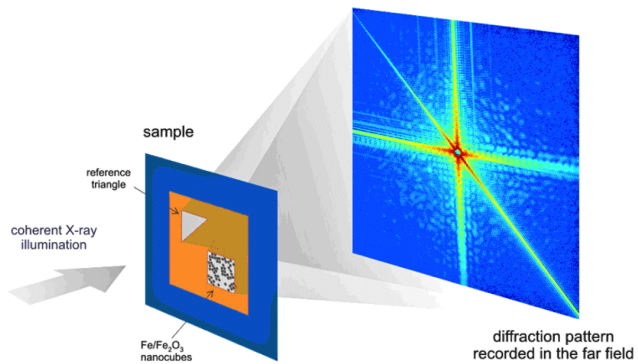
assume w.l.o.g. $\|\boldsymbol{x}^\star\|_2 = 1$

Motivation: phase retrieval

Detectors record intensities of diffracted rays

- electric field $x(t_1, t_2) \longrightarrow$ Fourier transform $\widehat{x}(f_1, f_2)$

figure credit: Stanford SLAC



intensity of electrical field: $\left|\widehat{x}(f_1, f_2)\right|^2 = \left|\int x(t_1, t_2) e^{-i2\pi(f_1 t_1 + f_2 t_2)} \mathrm{d}t_1 \mathrm{d}t_2\right|^2$

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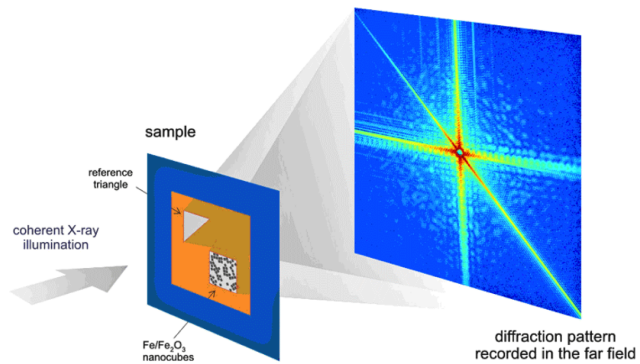

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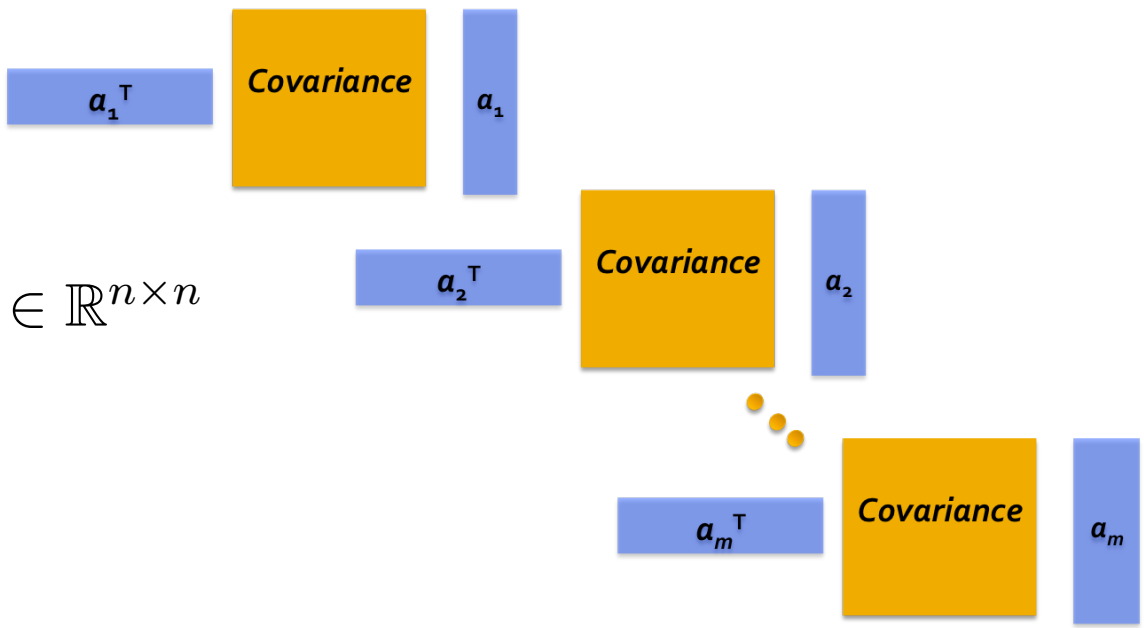
Phase retrieval: recover signal $x(t_1, t_2)$ from intensity $\left|\widehat{x}(f_1, f_2)\right|^2$

Motivation: covariance estimation from quadratic sketches

— *Chen, Chi, Goldsmith '13, Cai, Zhang '13*

- **Data**: m quadratic measurements about *low-rank* covariance matrix $\boldsymbol{\Sigma}$

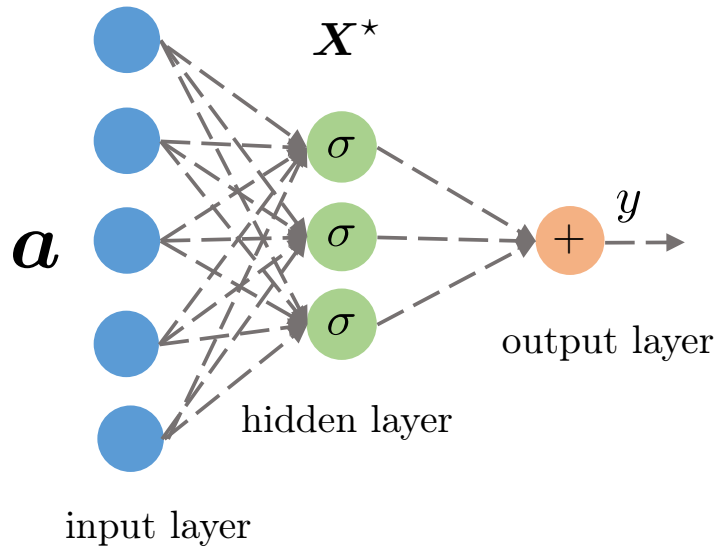
$$y_i = \boldsymbol{a}_i^\top \boldsymbol{\Sigma} \boldsymbol{a}_i + \text{noise}, \qquad i = 1, \cdots, m$$



- **Goal:** recover $\boldsymbol{\Sigma} \in \mathbb{R}^{n \times n}$

Motivation: learning neural nets with quadratic activation

— *Soltanolkotabi, Javanmard, Lee '17, Li, Ma, Zhang '17*



input features: $\boldsymbol{a}$; weights: $\boldsymbol{X}^\star = [\boldsymbol{x}_1^\star, \cdots, \boldsymbol{x}_r^\star]$

output: $$y = \sum_{i=1}^{r} \sigma(\boldsymbol{a}^\top \boldsymbol{x}_i^\star)$$

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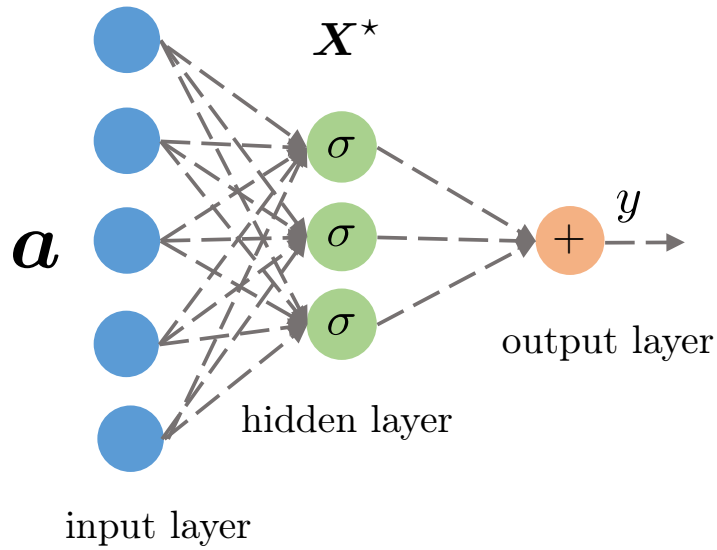

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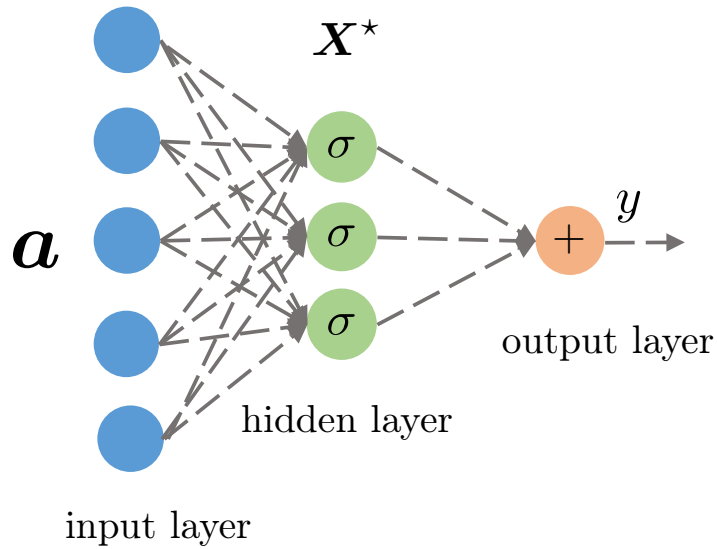

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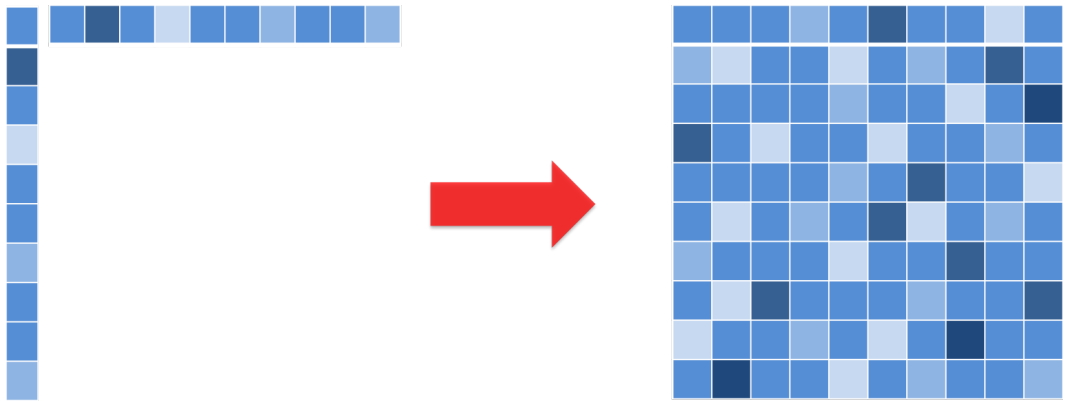
We consider simplest model when $r = 1$

An equivalent view: low-rank factorization

Introduce $\boldsymbol{X} = \boldsymbol{x}\boldsymbol{x}^\top$ to linearize constraints

$$y_k = (\boldsymbol{a}_k^\top \boldsymbol{x})^2 = \boldsymbol{a}_k^\top (\boldsymbol{x}\boldsymbol{x}^\top)\boldsymbol{a} \quad \Longrightarrow \quad y_k = \boldsymbol{a}_k^\top \boldsymbol{X} \boldsymbol{a}_k$$



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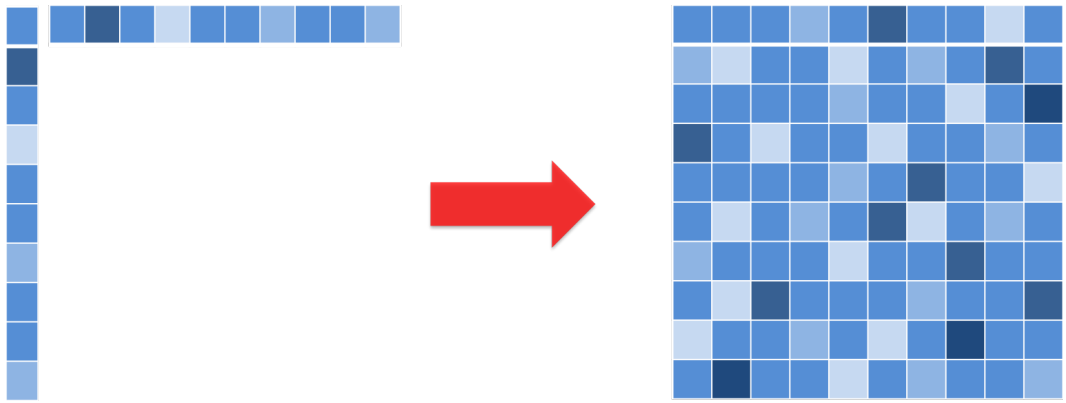

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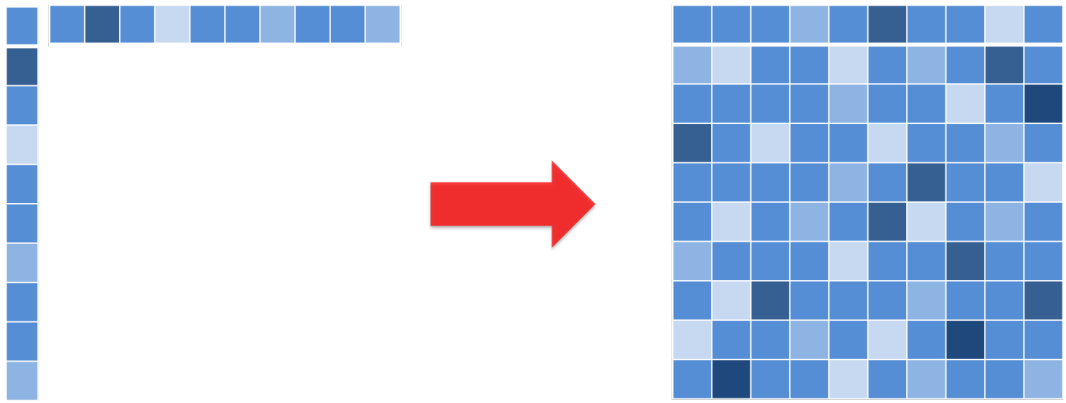

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Solving quadratic systems is essentially low-rank matrix completion

A natural least squares formulation

$$\text{given:} \qquad y_k \;=\; (\boldsymbol{a}_k^\top \boldsymbol{x}^\star)^2, \quad 1 \le k \le m$$

$$\Downarrow$$

$$\text{minimize}_{\boldsymbol{x}\in\mathbb{R}^n} \quad f(\boldsymbol{x}) = \frac{1}{4m}\sum_{k=1}^{m} \Big[\big(\boldsymbol{a}_k^\top \boldsymbol{x}\big)^2 - y_k \Big]^2$$

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- **pros:** often exact as long as sample size is sufficiently large
- **cons:** $f(\cdot)$ is highly nonconvex
  $\longrightarrow$ *computationally challenging!*

Wirtinger flow (Candès, Li, Soltanolkotabi '14)

$$\text{minimize}_{\boldsymbol{x}} \quad f(\boldsymbol{x}) = \frac{1}{4m} \sum_{k=1}^{m} \left[\left(\boldsymbol{a}_k^\top \boldsymbol{x} \right)^2 - y_k \right]^2$$

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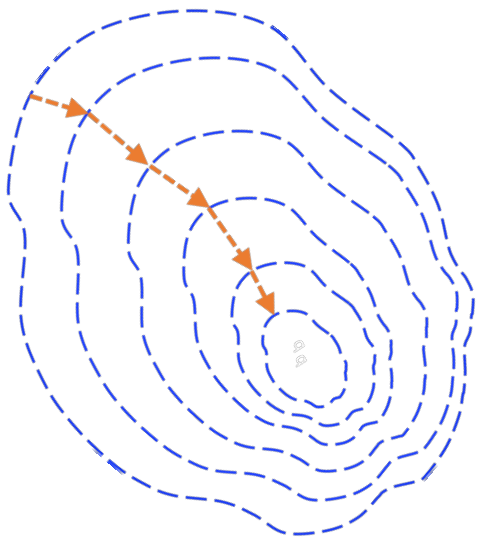
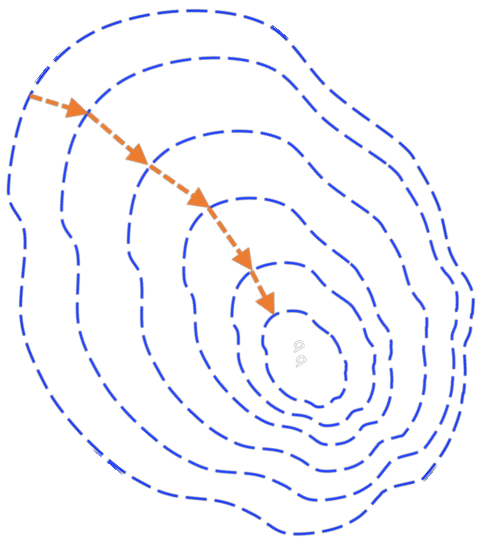


- **spectral initialization:** $\boldsymbol{x}^0 \leftarrow$ leading eigenvector of certain data matrix

Wirtinger flow (Candès, Li, Soltanolkotabi '14)

$$\text{minimize}_{\boldsymbol{x}} \quad f(\boldsymbol{x}) = \frac{1}{4m}\sum_{k=1}^{m}\left[\left(\boldsymbol{a}_k^\top \boldsymbol{x}\right)^2 - y_k\right]^2$$



- **spectral initialization:** $\boldsymbol{x}^0 \leftarrow$ leading eigenvector of certain data matrix
- **gradient descent:**

$$\boldsymbol{x}^{t+1} = \boldsymbol{x}^t - \eta\,\nabla f(\boldsymbol{x}^t), \qquad t = 0, 1, \cdots$$

Spectral initialization

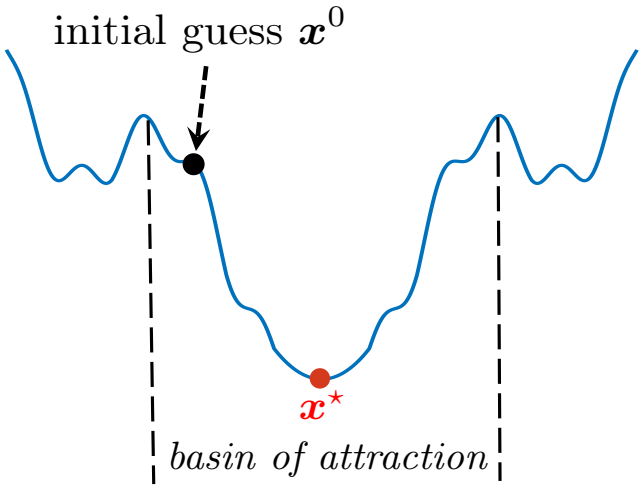
$\boldsymbol{x}^0 \longleftarrow$ leading eigenvector of

$$\boldsymbol{Y} := \frac{1}{m} \sum_{k=1}^{m} y_k \, \boldsymbol{a}_k \boldsymbol{a}_k^\top$$

Rationale: under random Gaussian design $\boldsymbol{a}_i \overset{\text{ind.}}{\sim} \mathcal{N}(\boldsymbol{0}, \boldsymbol{I})$,

$$\mathbb{E}[\boldsymbol{Y}] := \mathbb{E}\left[\frac{1}{m} \sum_{k=1}^{m} \boldsymbol{y}_k \boldsymbol{a}_k \boldsymbol{a}_k^\top\right] = \underbrace{\|\boldsymbol{x}^\star\|_2^2 \, \boldsymbol{I} + 2\boldsymbol{x}^\star \boldsymbol{x}^{\star\top}}_{\text{leading eigenvector: } \pm\boldsymbol{x}^\star}$$

Rationale of two-stage approach



1. initialize within $\underbrace{\text{local basin sufficiently close to } \boldsymbol{x}^\star}$

(restricted) strongly convex; no saddles / spurious local mins

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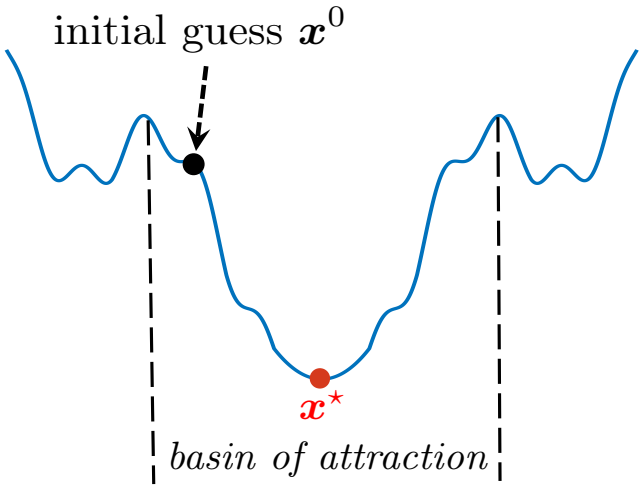

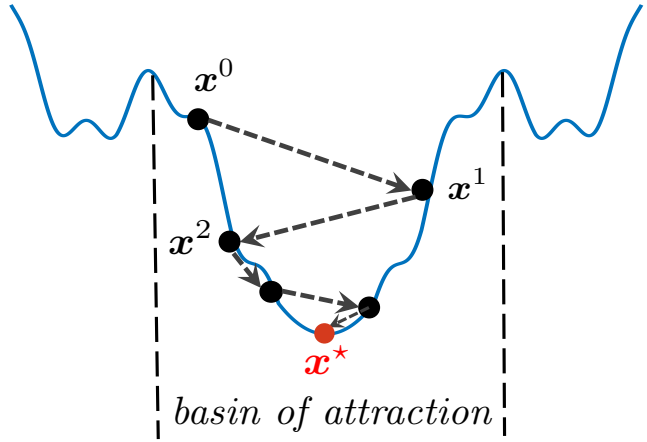

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2. iterative refinement

A highly incomplete list of two-stage methods

phase retrieval:

- Netrapalli, Jain, Sanghavi '13
- Candès, Li, Soltanolkotabi '14
- Chen, Candès '15
- Cai, Li, Ma '15
- Wang, Giannakis, Eldar '16
- Zhang, Zhou, Liang, Chi '16
- Kolte, Ozgur '16
- Zhang, Chi, Liang '16
- Soltanolkotabi '17
- Vaswani, Nayer, Eldar '16
- Chi, Lu '16
- Wang, Zhang, Giannakis, Akcakaya, Chen '16
- Tan, Vershynin '17
- Ma, Wang, Chi, Chen '17
- Duchi, Ruan '17
- Jeong, Gunturk '17
- Yang, Yang, Fang, Zhao, Wang, Neykov '17
- Qu, Zhang, Wright '17
- Goldstein, Studer '16
- Bahmani, Romberg '16
- Hand, Voroninski '16
- Wang, Giannakis, Saad, Chen '17
- Barmherzig, Sun '17
- ...

other problems:

- Keshavan, Montanari, Oh '09
- Sun, Luo '14
- Chen, Wainwright '15
- Tu, Boczar, Simchowitz, Soltanolkotabi, Recht '15
- Zheng, Lafferty '15
- Balakrishnan, Wainwright, Yu '14
- Chen, Suh '15
- Chen, Candès '16
- Li, Ling, Strohmer, Wei '16
- Yi, Park, Chen, Caramanis '16
- Jin, Kakade, Netrapalli '16
- Huang, Kakade, Kong, Valiant '16
- Ling, Strohmer '17
- Li, Ma, Chen, Chi '18
- Aghasi, Ahmed, Hand '17
- Lee, Tian, Romberg '17
- Li, Chi, Zhang, Liang '17
- Cai, Wang, Wei '17
- Abbe, Bandeira, Hall '14
- Chen, Kamath, Suh, Tse '16
- Zhang, Zhou '17
- Boumal '16
- Zhong, Boumal '17
- ...

Computational cost

$$\boldsymbol{A}\boldsymbol{x} := \left[\boldsymbol{a}_k^\top \boldsymbol{x}\right]_{1\le k\le m}$$

- **Spectral initialization:** leading eigenvector $\to$ a few applications of $\boldsymbol{A}$ and $\boldsymbol{A}^\top$

$$\frac{1}{m}\sum_{k=1}^{m} y_k\, \boldsymbol{a}_k \boldsymbol{a}_k^\top = \frac{1}{m}\boldsymbol{A}^\top \operatorname{diag}\{y_k\}\, \boldsymbol{A}$$

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- **Iterations:** one application of $\boldsymbol{A}$ and $\boldsymbol{A}^\top$ per iteration

$$\boldsymbol{x}^{t+1} = \boldsymbol{x}^t - \eta \nabla f(\boldsymbol{x}^t)$$

Asymptotic notation

- $f(n) \lesssim g(n)$ or $f(n) = O(g(n))$ means

$$\lim_{n\to\infty} \frac{|f(n)|}{|g(n)|} \leq \text{const}$$

- $f(n) \gtrsim g(n)$ means

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- $f(n) \asymp g(n)$ means

$$\text{const}_1 \leq \lim_{n\to\infty} \frac{|f(n)|}{|g(n)|} \leq \text{const}_2$$

First theory of WF

$$\mathsf{dist}(\boldsymbol{x}^t, \boldsymbol{x}^\star) := \min\{\|\boldsymbol{x}^t \pm \boldsymbol{x}^\star\|_2\}$$

Theorem 1 (Candès, Li, Soltanolkotabi '14)

Under i.i.d. Gaussian design, WF with spectral initialization achieves

$$\mathsf{dist}(\boldsymbol{x}^t, \boldsymbol{x}^\star) \lesssim \left(1 - \frac{\eta}{4}\right)^{t/2} \|\boldsymbol{x}^\star\|_2,$$

with high prob., provided that step size $\eta \lesssim 1/n$ *and sample size:* $m \gtrsim n \log n$

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with high prob., provided that step size and sample size:

- Iteration complexity: $O(n \log \frac{1}{\epsilon})$
- Sample complexity: $O(n \log n)$
- Derived based on (worst-case) local geometry

Improved theory of WF

$$\mathrm{dist}(\boldsymbol{x}^t, \boldsymbol{x}^\star) := \min\{\|\boldsymbol{x}^t \pm \boldsymbol{x}^\star\|_2\}$$

Theorem 2 (Ma, Wang, Chi, Chen '17)

Under i.i.d. Gaussian design, WF with spectral initialization achieves

$$\mathrm{dist}(\boldsymbol{x}^t, \boldsymbol{x}^\star) \lesssim \left(1 - \frac{\eta}{2}\right)^t \|\boldsymbol{x}^\star\|_2$$

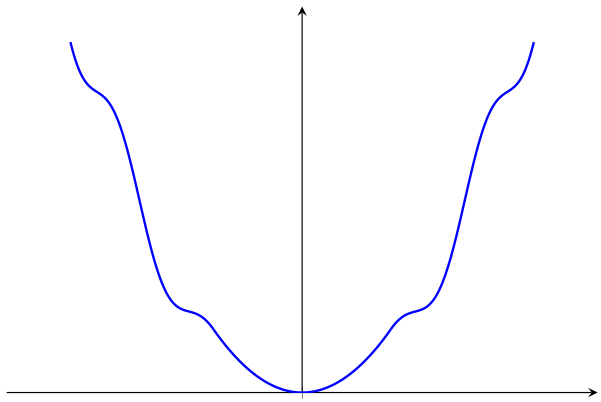
with high prob., provided that step size $\eta \asymp 1/\log n$ *and sample size* $m \gtrsim n \log n$.

- Iteration complexity: $O(n \log \frac{1}{\epsilon}) \searrow O(\log n \log \frac{1}{\epsilon})$
- Sample complexity: $O(n \log n)$
- Derived based on finer analysis of GD trajectory

Gradient descent theory revisited

Consider unconstrained optimization problem

$$\text{minimize}_{\boldsymbol{x}} \qquad f(\boldsymbol{x})$$



Two standard conditions that enable geometric convergence of GD

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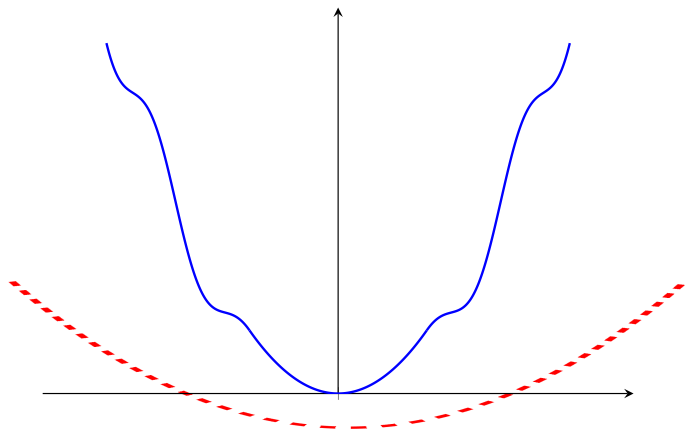

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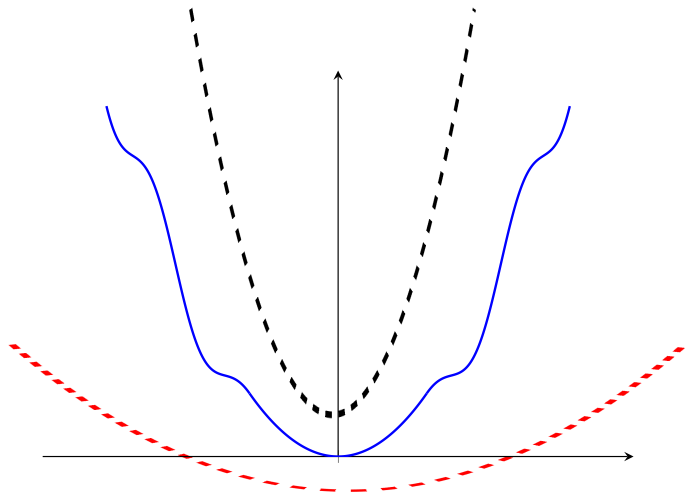

Two standard conditions that enable geometric convergence of GD

- (local) restricted strong convexity (or regularity condition)
- (local) smoothness

$$\nabla^2 f(\boldsymbol{x}) \succ \mathbf{0} \quad \text{and} \quad \text{is well-conditioned}$$

Gradient descent theory revisited

f is said to be α-strongly convex and β-smooth if

$$\mathbf{0} \preceq \alpha \boldsymbol{I} \preceq \nabla^2 f(\boldsymbol{x}) \preceq \beta \boldsymbol{I}, \qquad \forall \boldsymbol{x}$$

ℓ_2 **error contraction:** GD with $\eta = 1/\beta$ obeys

$$\|\boldsymbol{x}^{t+1} - \boldsymbol{x}^\star\|_2 \leq \left(1 - \frac{\alpha}{\beta}\right) \|\boldsymbol{x}^t - \boldsymbol{x}^\star\|_2$$

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- Condition number β/α determines rate of convergence
- Attains ε-accuracy within $O(\frac{\beta}{\alpha} \log \frac{1}{\varepsilon})$ iterations

What does this optimization theory say about WF?

Gaussian designs: $\boldsymbol{a}_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_n), \quad 1 \le k \le m$

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Consequence (Candès et al '14): WF attains ε-accuracy within $O\big(n \log \frac{1}{\varepsilon}\big)$ iterations if $m \asymp n \log n$

Generic optimization theory gives pessimistic bounds

WF converges in $O(n)$ iterations

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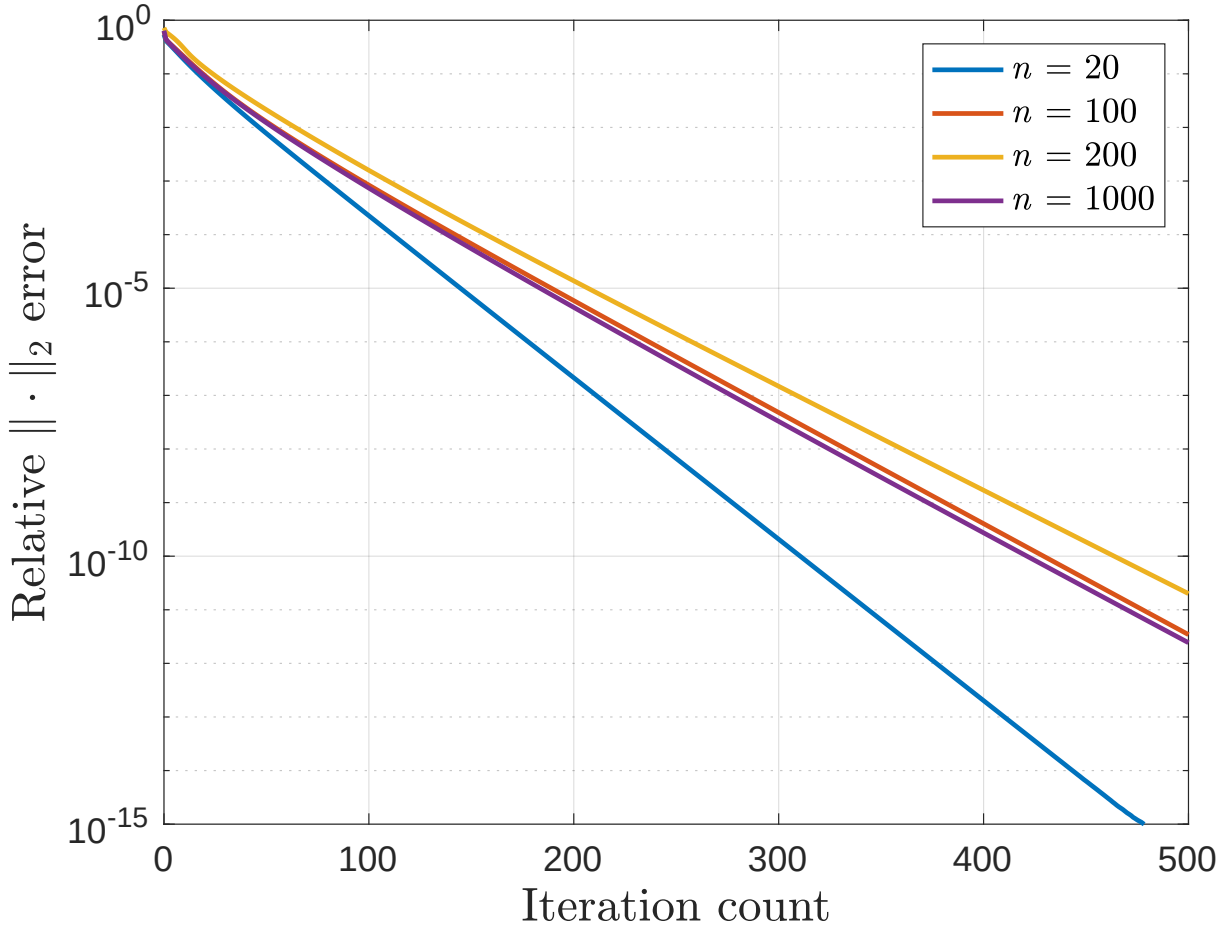
Does it capture what really happens?

Numerical efficiency with $\eta_t = 0.1$



Vanilla GD (WF) converges fast for a constant step size!

A second look at gradient descent theory

Which local region enjoys both strong convexity and smoothness?

$$\nabla^2 f(\boldsymbol{x}) = \frac{1}{m}\sum_{k=1}^{m}\left[3\left(\boldsymbol{a}_k^\top \boldsymbol{x}\right)^2 - \left(\boldsymbol{a}_k^\top \boldsymbol{x}^\star\right)^2\right]\boldsymbol{a}_k\boldsymbol{a}_k^\top$$

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- Not sufficiently smooth if $\boldsymbol{x}$ and $\boldsymbol{a}_k$ are too close (coherent)

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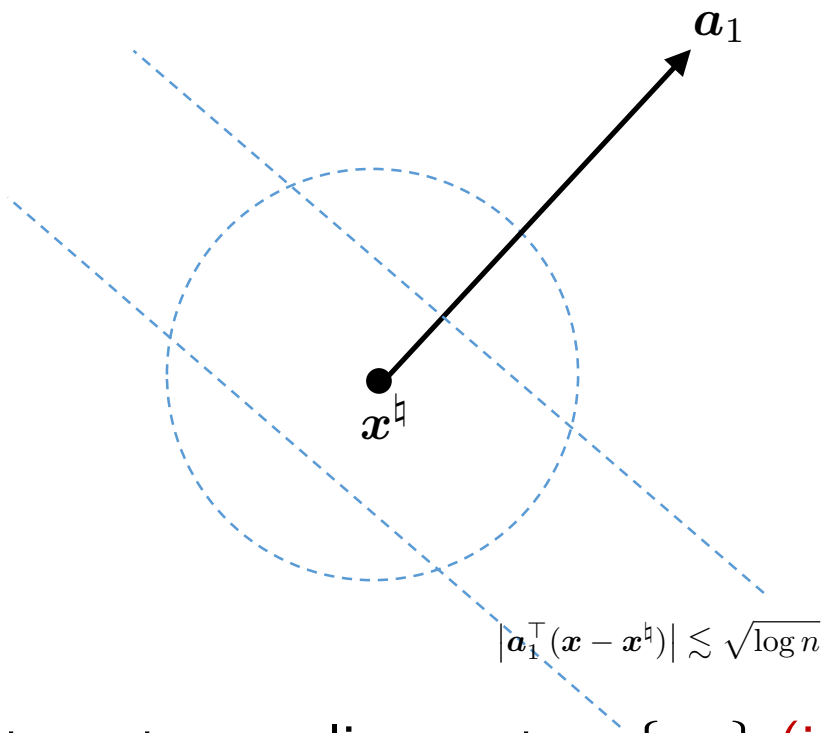

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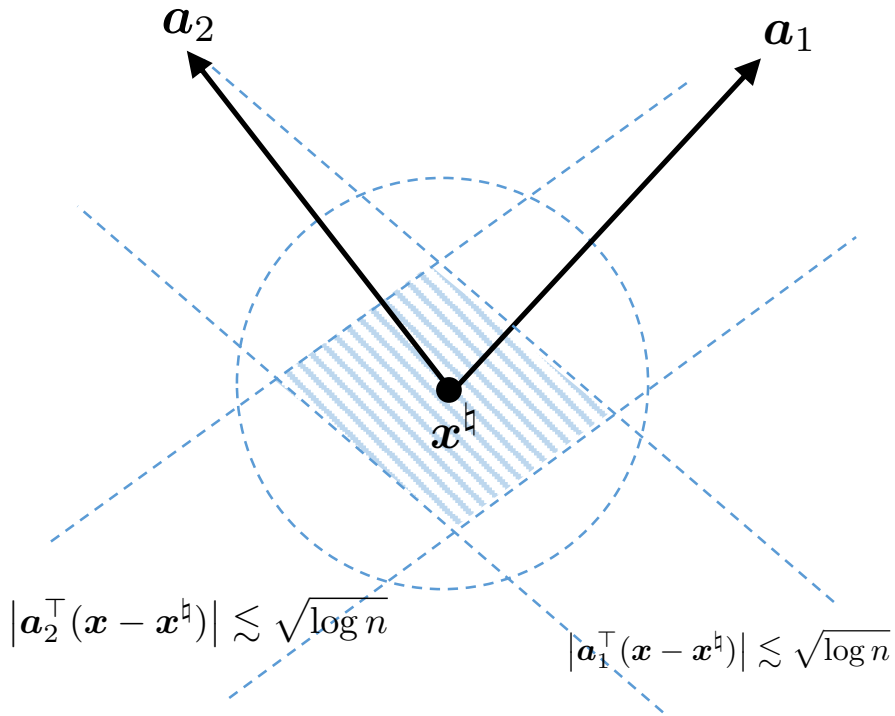

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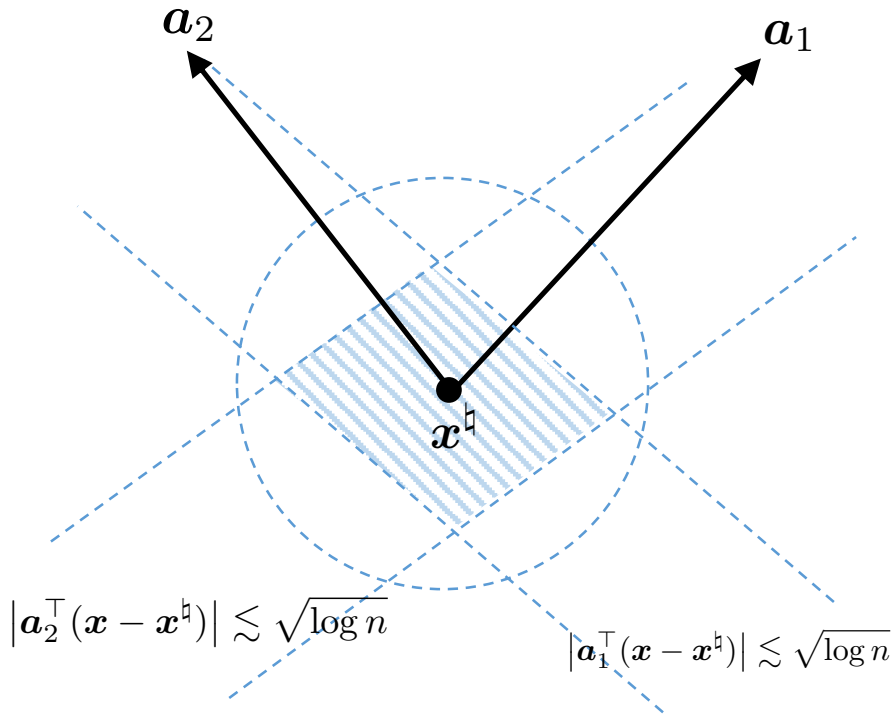

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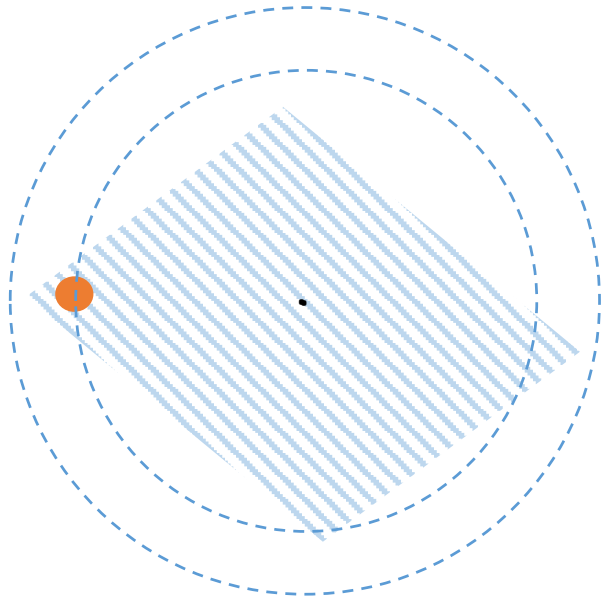
Prior works suggest enforcing regularization (e.g. truncation, projection, regularized loss) to promote incoherence

Encouraging message: GD is implicitly regularized

region of local strong convexity + smoothness



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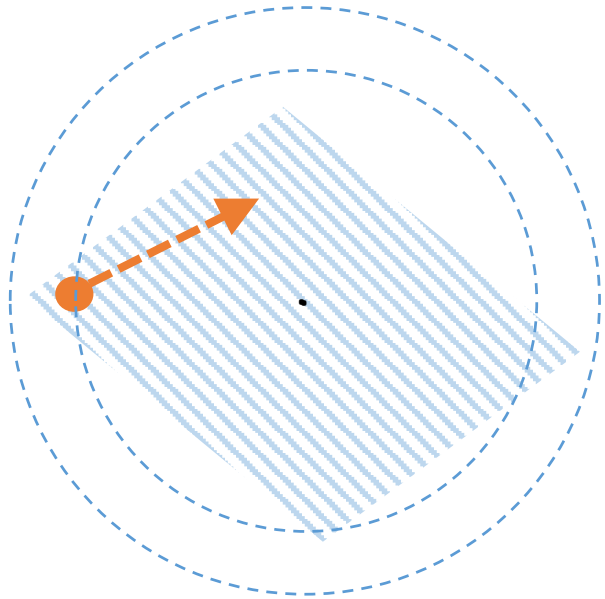

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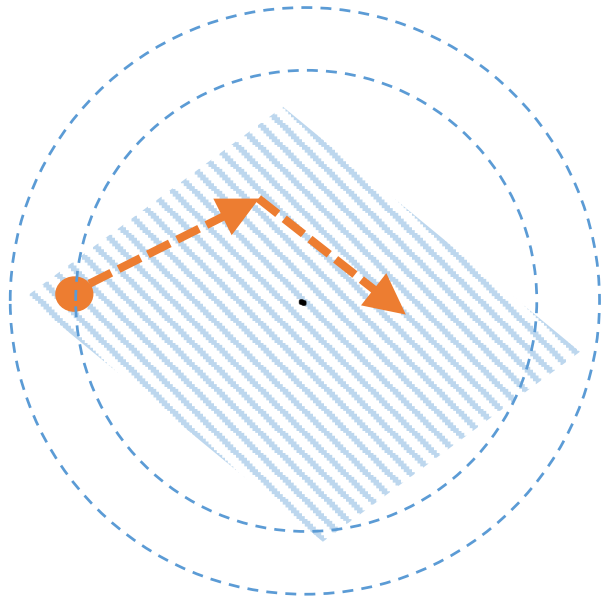

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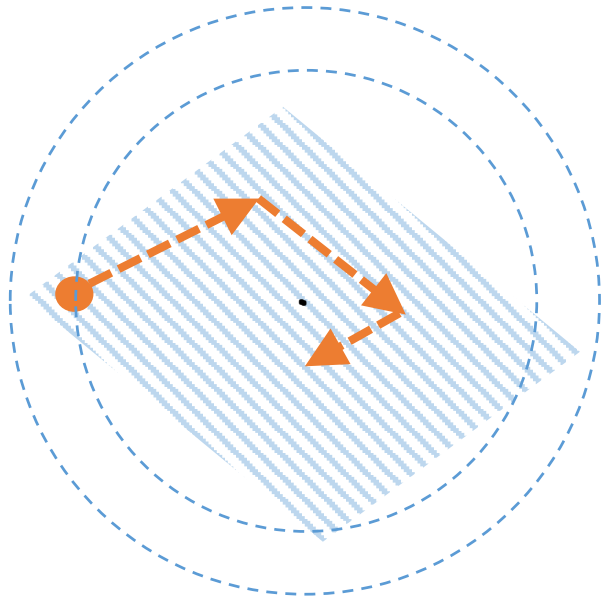

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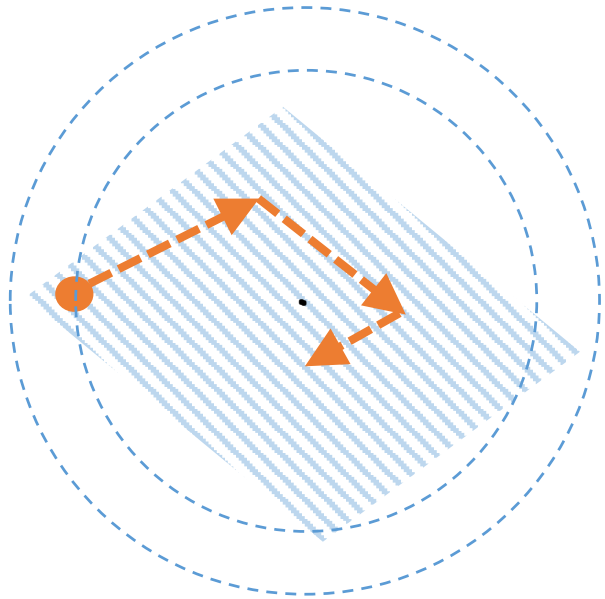

GD implicitly forces iterates to remain incoherent with $\{\boldsymbol{a}_k\}$

$$\max_k \left|\boldsymbol{a}_k^\top(\boldsymbol{x}^t - \boldsymbol{x}^\star)\right| \lesssim \sqrt{\log n}\,\|\boldsymbol{x}^\star\|_2, \quad \forall t$$

— cannot be derived from generic optimization theory; relies on finer statistical analysis for entire trajectory of GD

Theoretical guarantees for local refinement stage

Theorem 3 (Ma, Wang, Chi, Chen '17)

Under i.i.d. Gaussian design, WF with spectral initialization achieves

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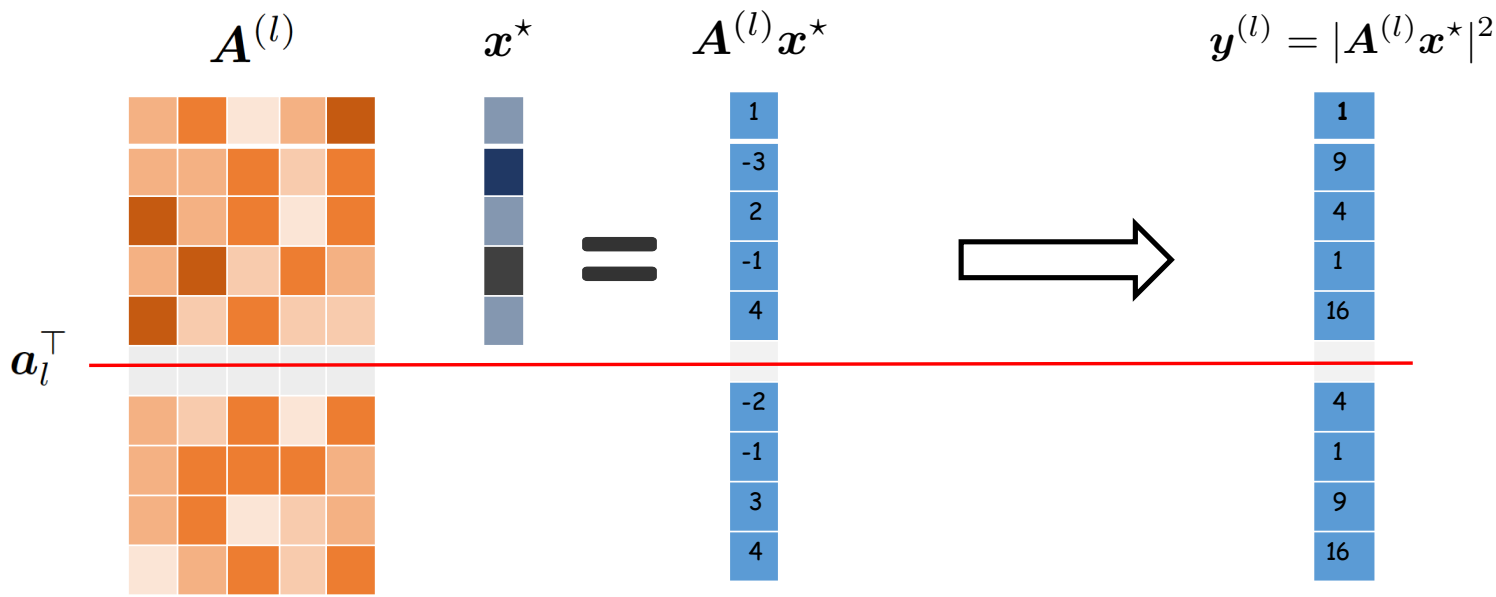
- $\max_k \left|\boldsymbol{a}_k^\top \boldsymbol{x}^t\right| \lesssim \sqrt{\log n}\, \|\boldsymbol{x}^\star\|_2$ *(incoherence)*
- $\mathrm{dist}(\boldsymbol{x}^t, \boldsymbol{x}^\star) \lesssim \left(1 - \frac{\eta}{2}\right)^t \|\boldsymbol{x}^\star\|_2$ *(linear convergence)*

provided that step size $\eta \asymp 1/\log n$ *and sample size* $m \gtrsim n \log n$.

- Attains ε accuracy within $O\big(\log n \, \log \frac{1}{\varepsilon}\big)$ iterations

Key proof idea: leave-one-out analysis

For each $1 \le l \le m$, introduce leave-one-out iterates $\boldsymbol{x}^{t,(l)}$ by dropping lth measurement



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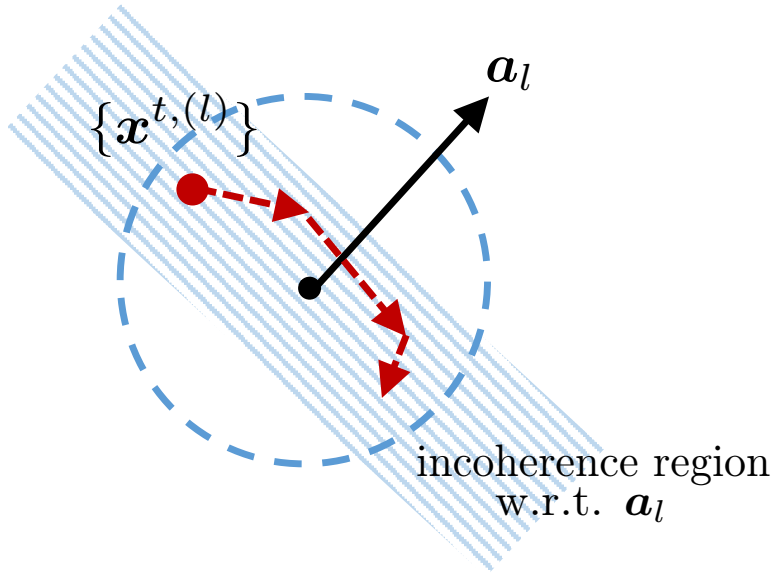

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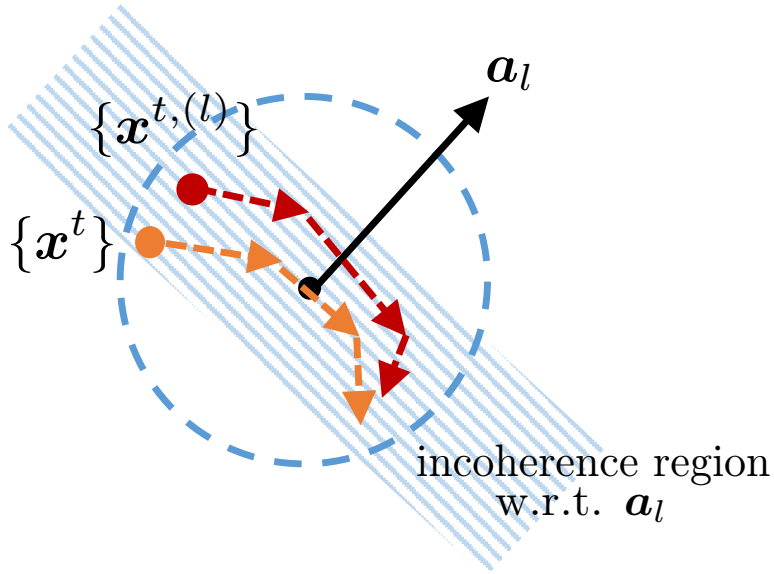

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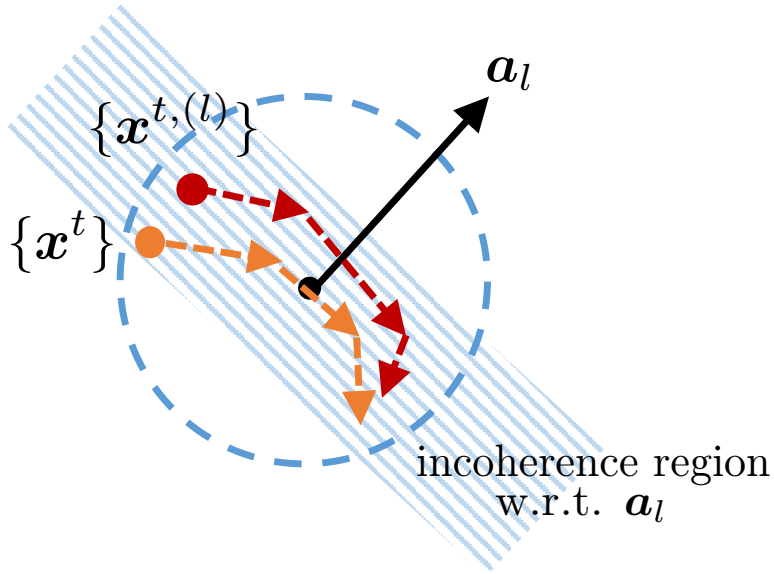

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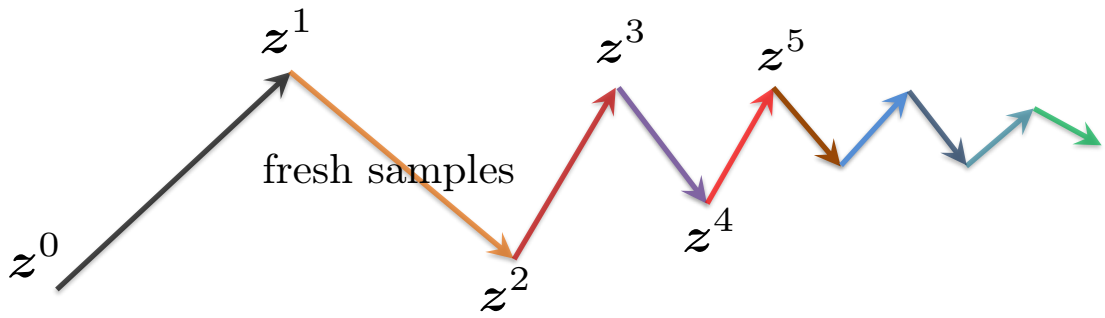
$$\Longrightarrow \quad \boldsymbol{x}^t \text{ is } \underbrace{\text{nearly independent of}}_{\text{nearly orthogonal to}} \boldsymbol{a}_l$$

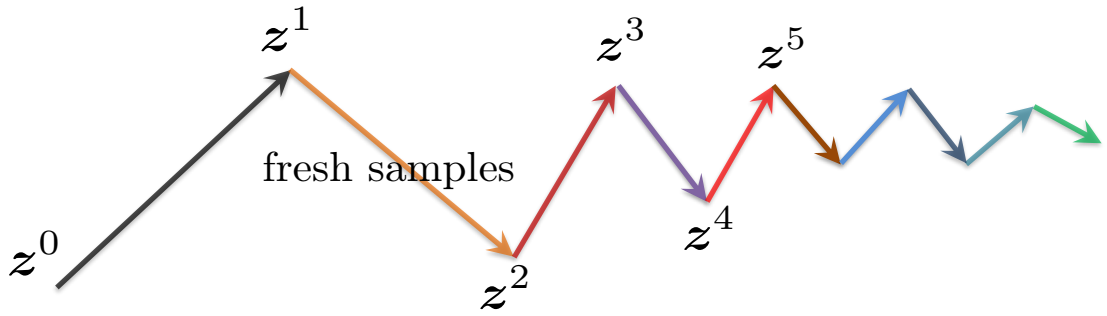
No need of sample splitting

- Several prior works use sample-splitting: require fresh samples at each iteration; not practical but helps analysis

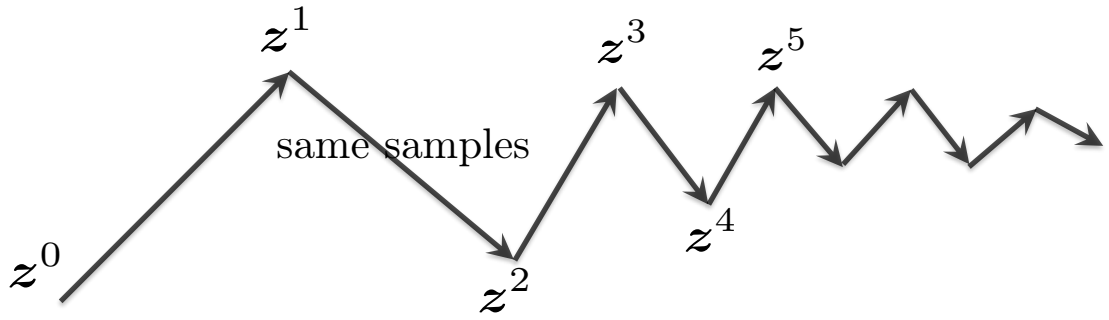


No need of sample splitting

- Several prior works use sample-splitting: require fresh samples at each iteration; not practical but helps analysis



- **This tutorial:** reuses all samples in all iterations



Questions

So far we have presented theory for

$$\text{spectral initialization} \;+\; \text{vanilla gradient descent (WF)}$$

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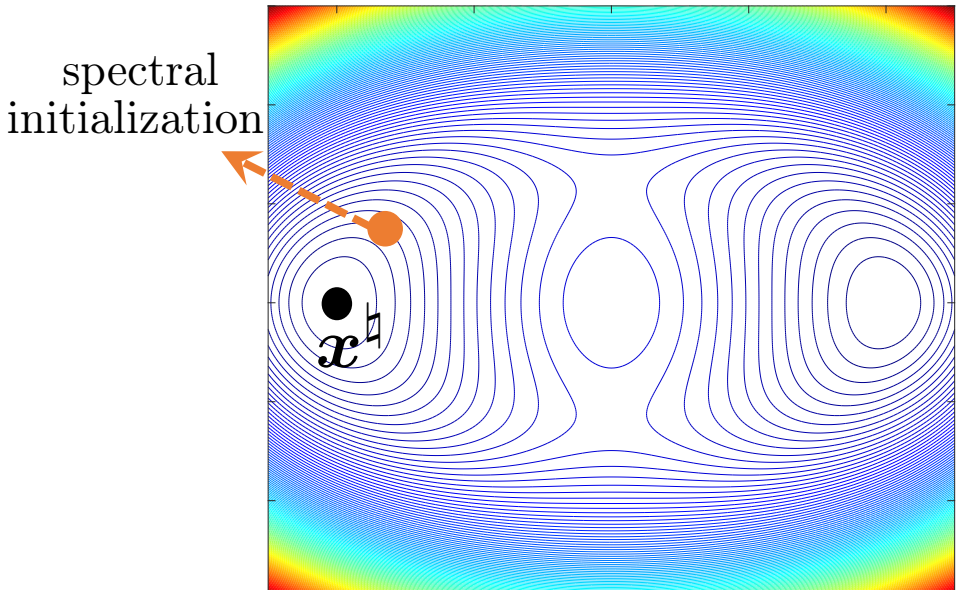
$$\text{spectral initialization} \ + \ \text{vanilla gradient descent (WF)}$$

Questions:

- Is carefully-designed initialization necessary for fast convergence?
- Can we further improve sample complexity?
- Robustness vis a vis noise and outliers?

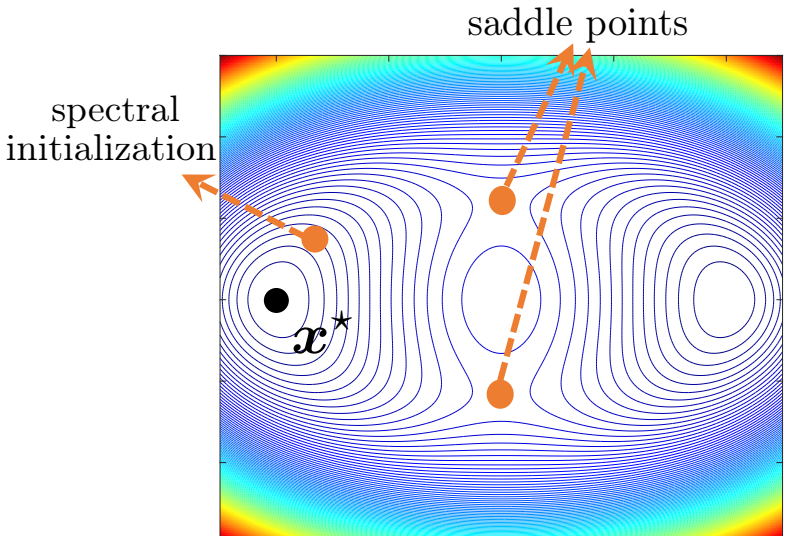
Is carefully-designed initialization necessary for fast convergence?

Initialization



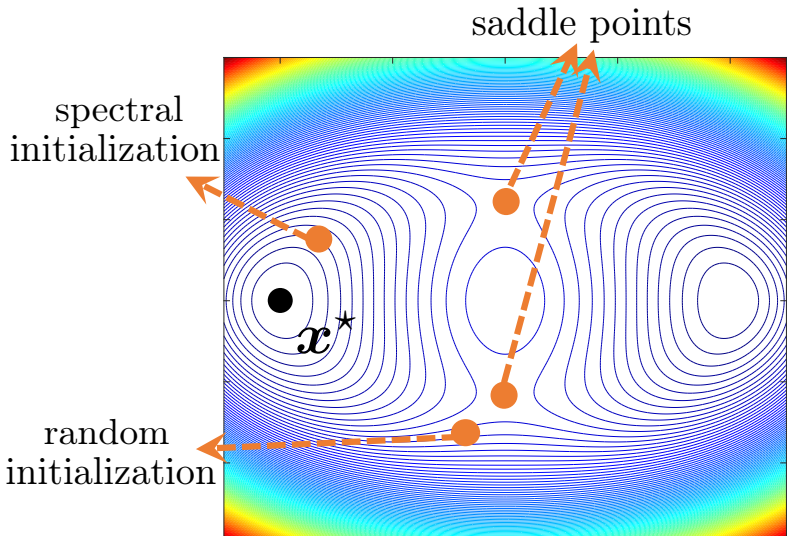
- Spectral initialization gets us reasonably close to truth

Initialization



- Spectral initialization gets us reasonably close to truth
- Cannot initialize GD from anywhere, e.g. it might get stucked at local stationary points (e.g. saddle points)

Initialization

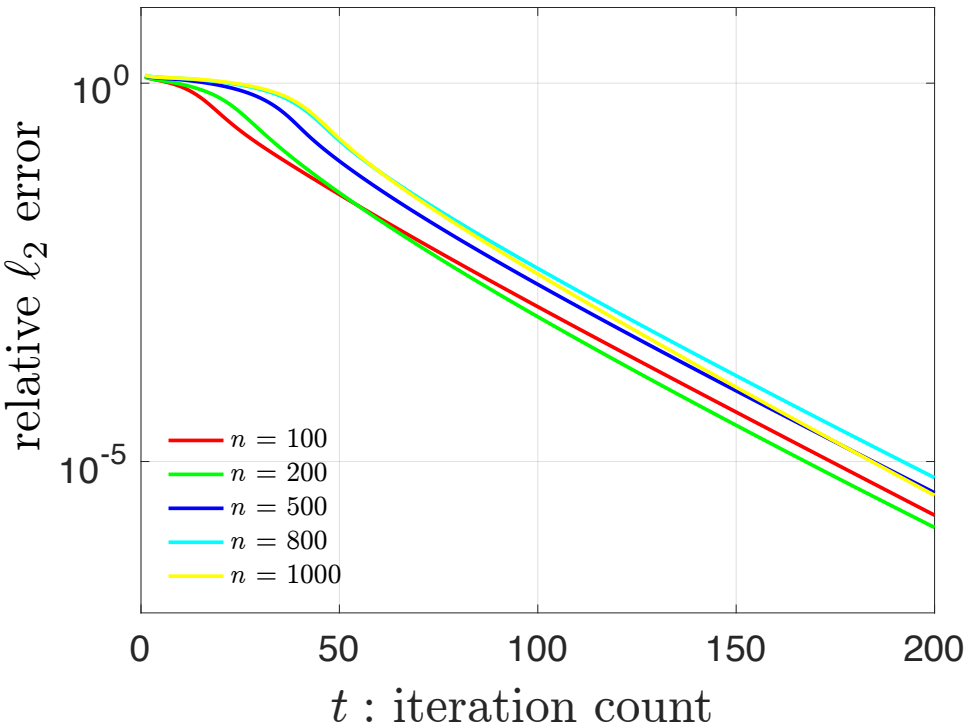


- Spectral initialization gets us reasonably close to truth
- Cannot initialize GD from anywhere, e.g. it might get stucked at local stationary points (e.g. saddle points)

Can we initialize GD randomly, which is simpler and model-agnostic?

Numerical efficiency of randomly initialized GD

$$\eta_t = 0.1,\ \boldsymbol{a}_i \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_n),\ m = 10n,\ \boldsymbol{x}^0 \sim \mathcal{N}(\boldsymbol{0}, n^{-1}\boldsymbol{I}_n)$$



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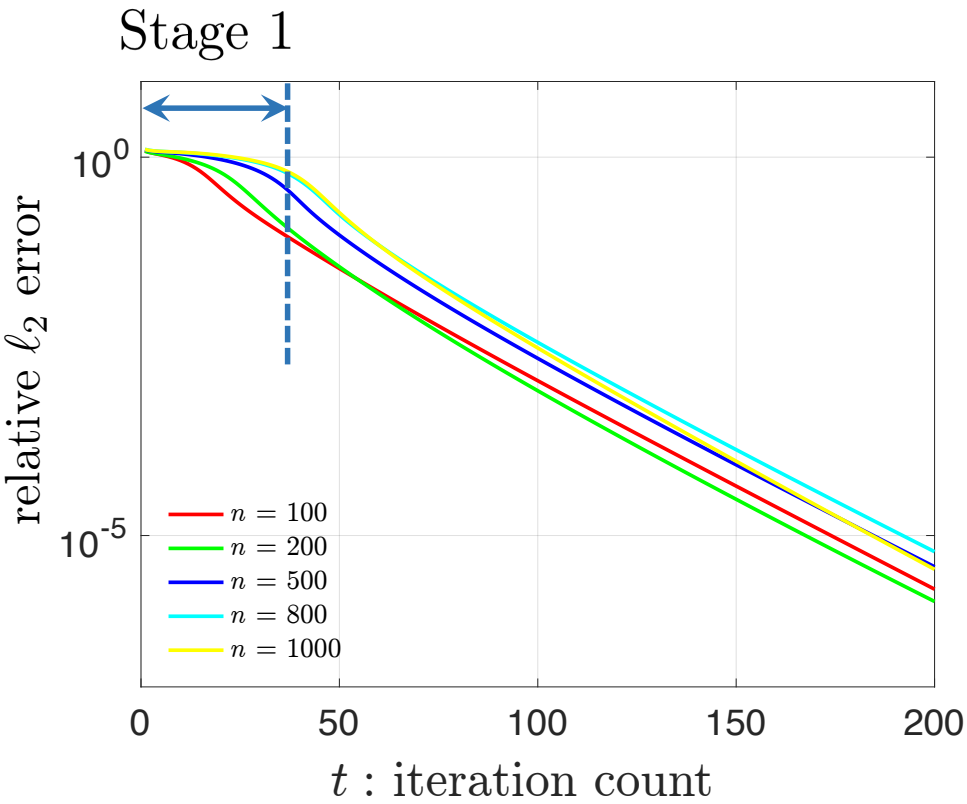

Randomly initialized GD enters local basin within **a few iterations**

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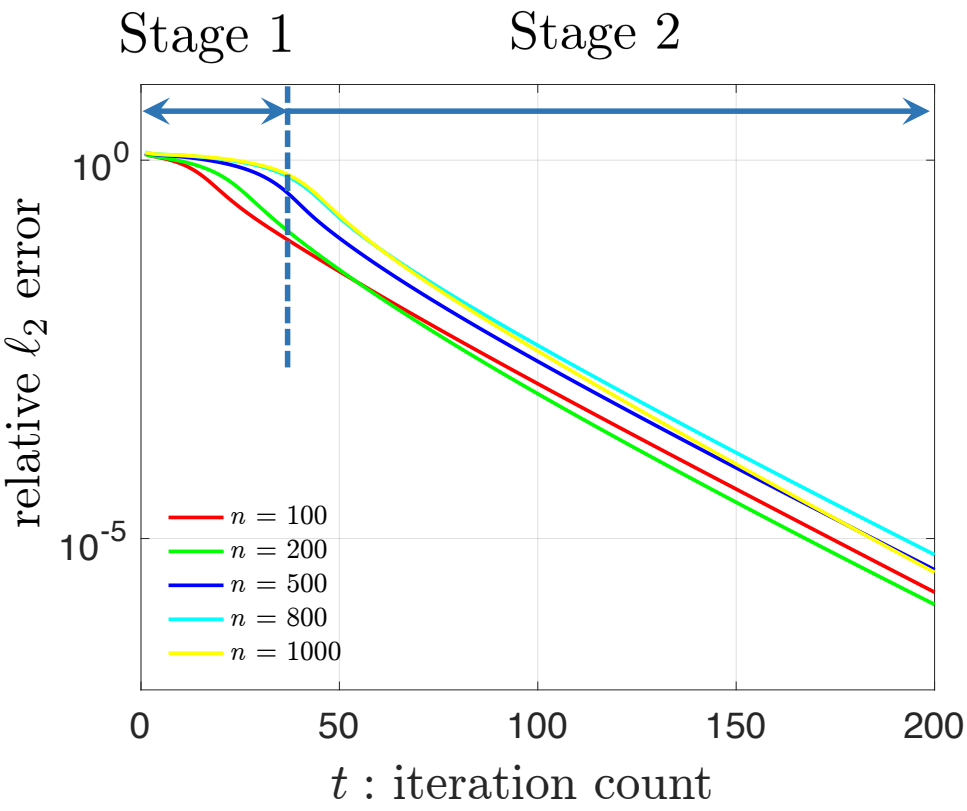
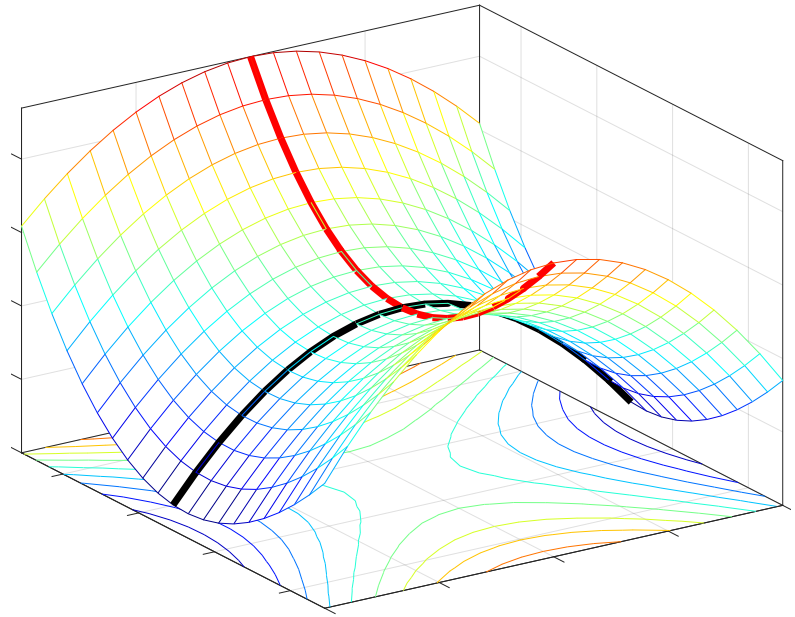


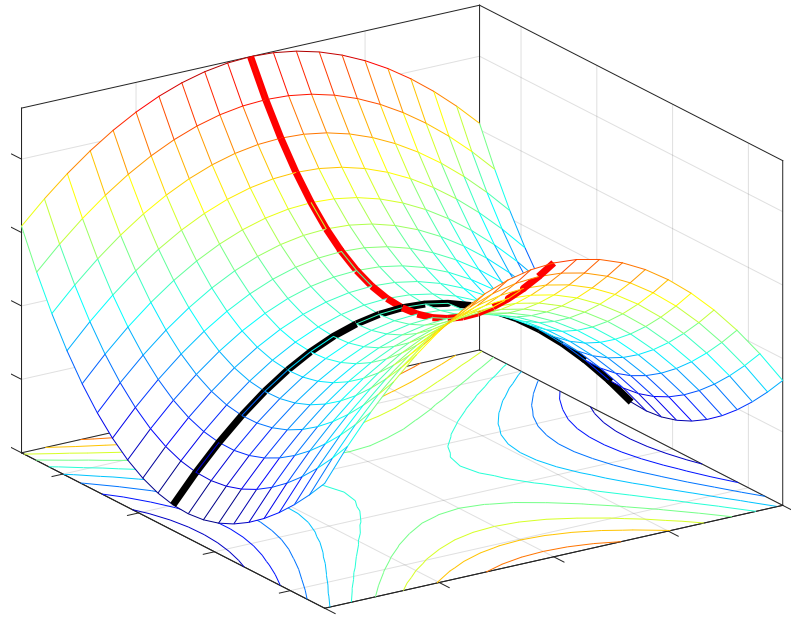
Randomly initialized GD enters local basin within **a few iterations**

A geometric analysis



- if $m \gtrsim n \log^3 n$, then (Sun et al. '16)
  - there is no spurious local mins
  - all saddle points are strict (i.e. associated Hessian matrices have at least one sufficiently negative eigenvalue)

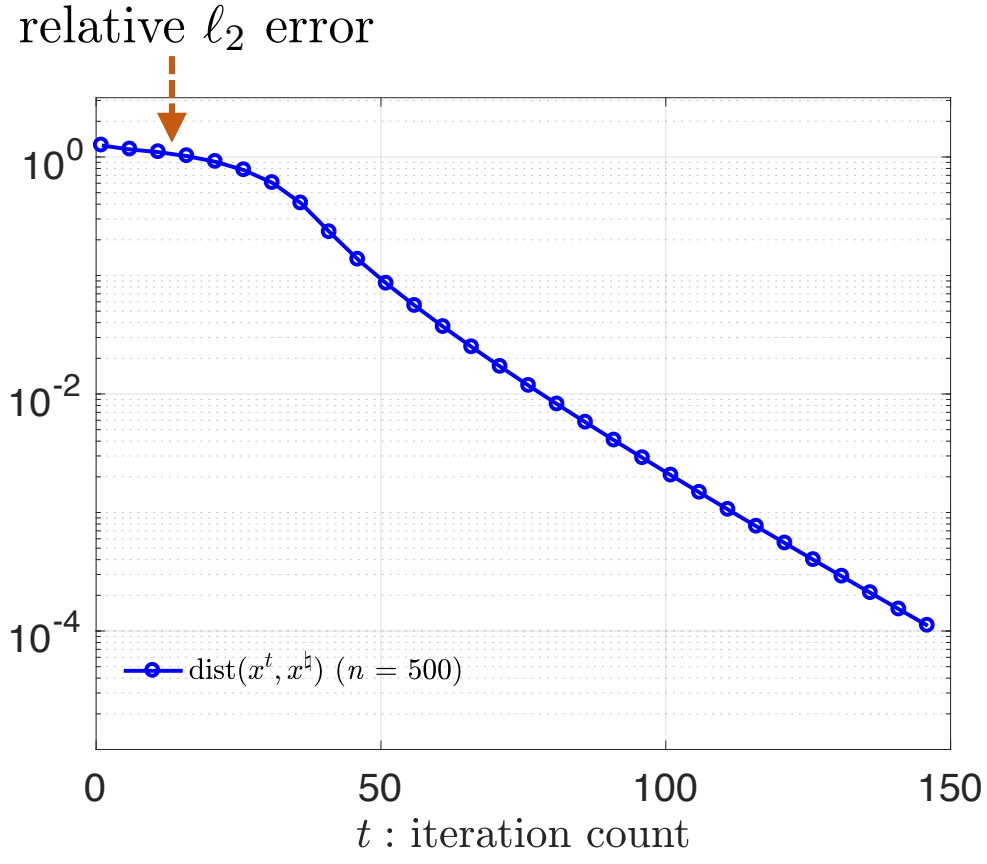
A geometric analysis



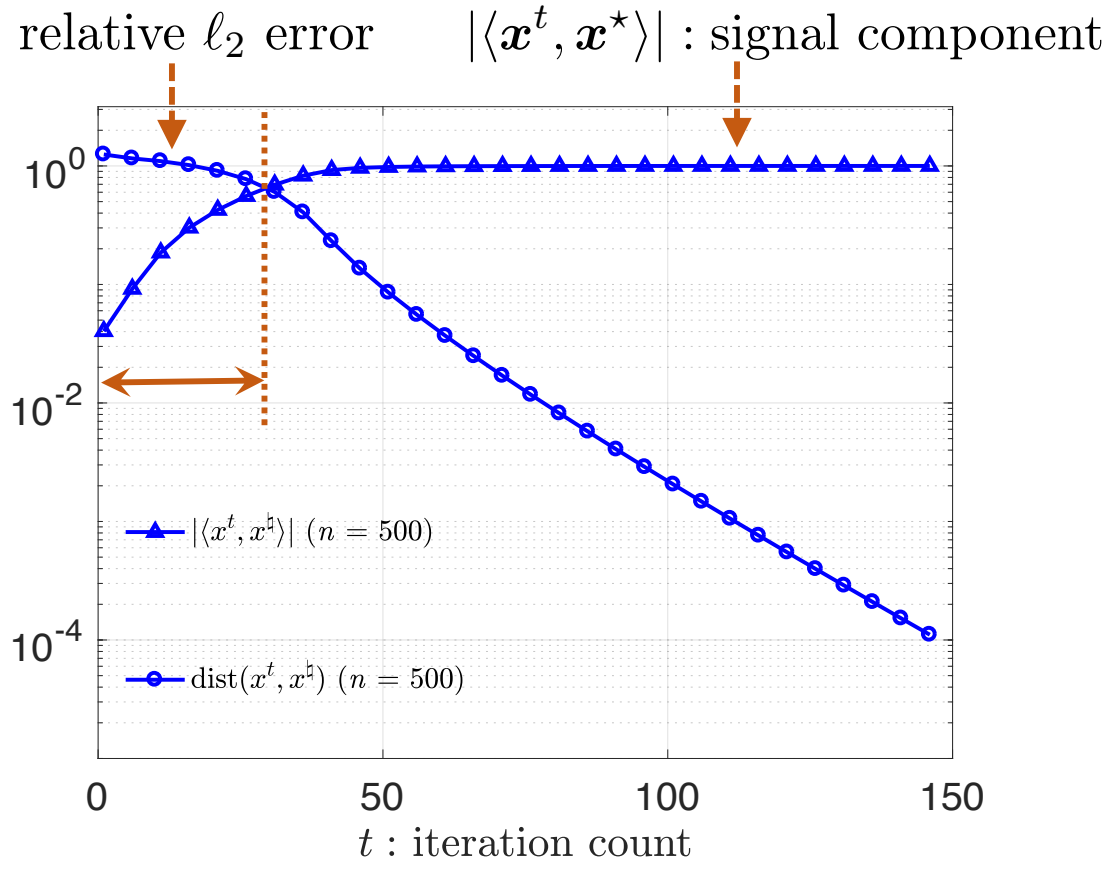
- With such benign landscape, GD with random initialization converges to global min almost surely (Lee et al. '16)

No convergence rate guarantees for vanilla GD!

Exponential growth of signal strength in Stage 1

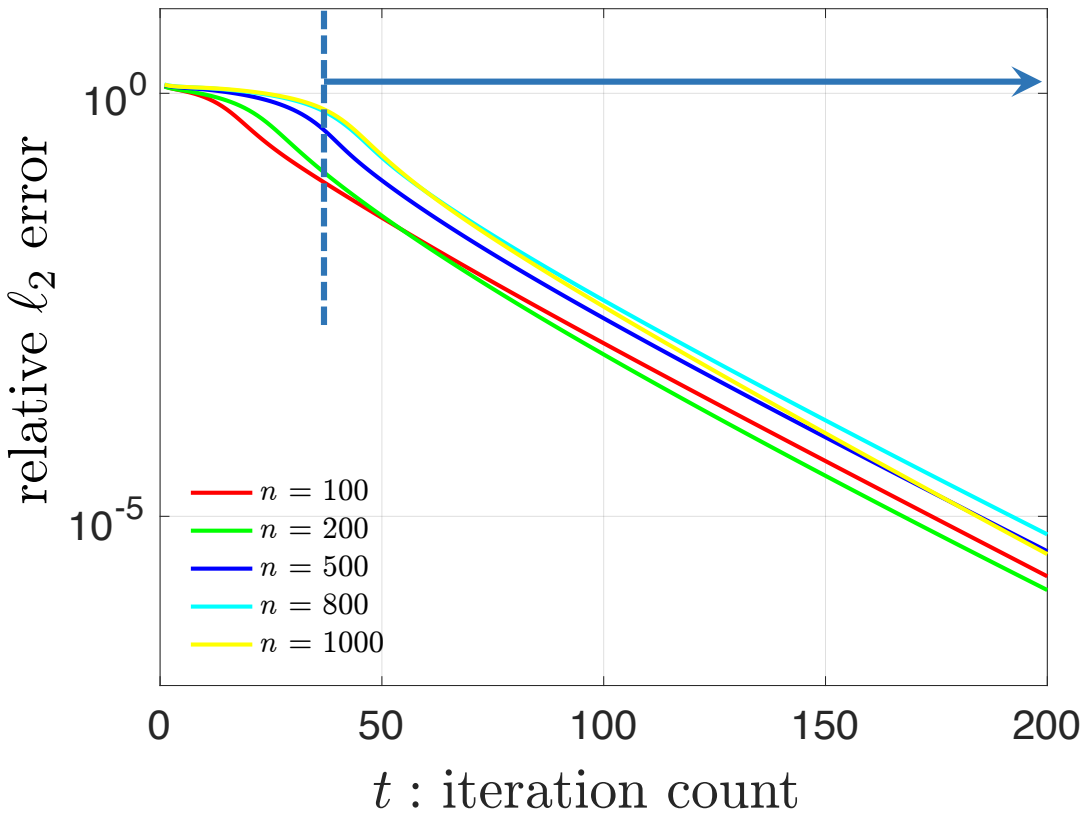


Exponential growth of signal strength in Stage 1

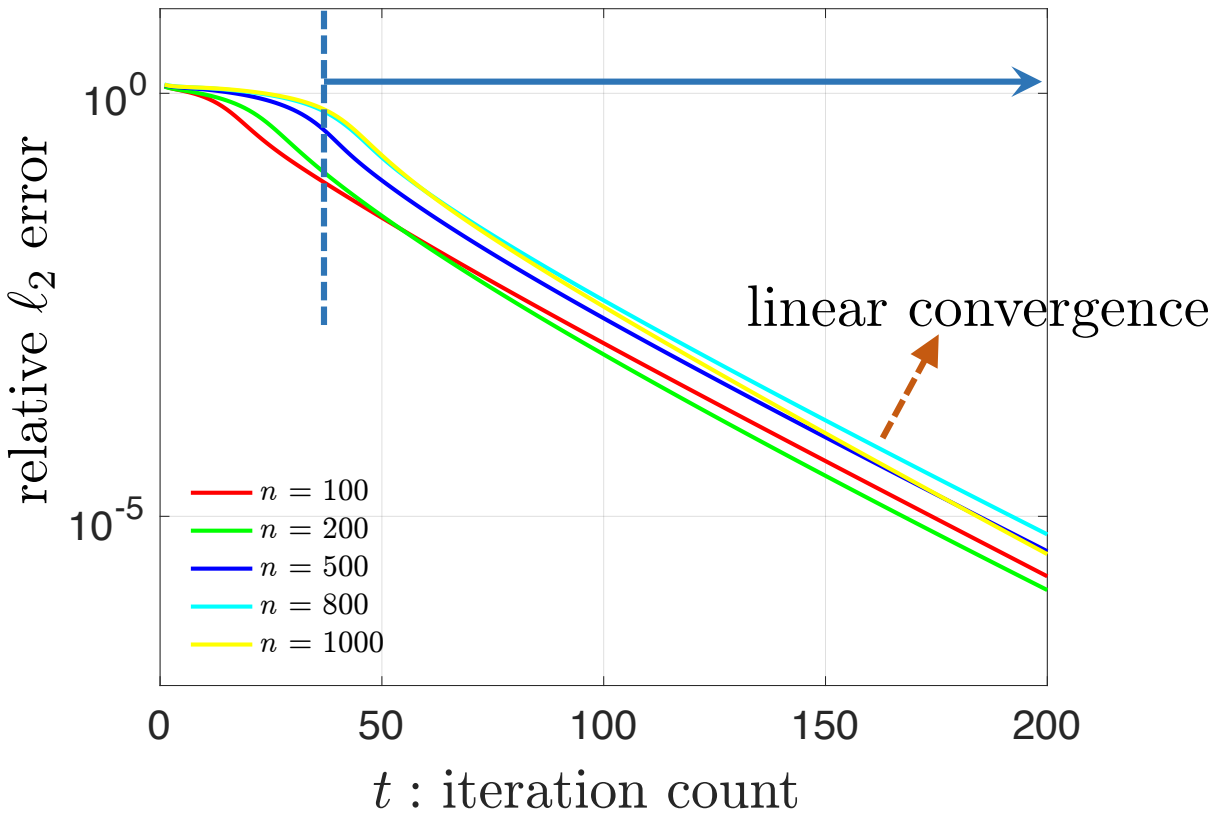


Numerically, $O(\log n)$ iterations are enough to enter local region

Linear / geometric convergence in Stage 2



Linear / geometric convergence in Stage 2



Numerically, GD converges linearly within local region

Theoretical guarantees for randomly initialized GD

These numerical findings can be formalized when $\boldsymbol{a}_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_n)$:

Theorem 4 (Chen, Chi, Fan, Ma '18)

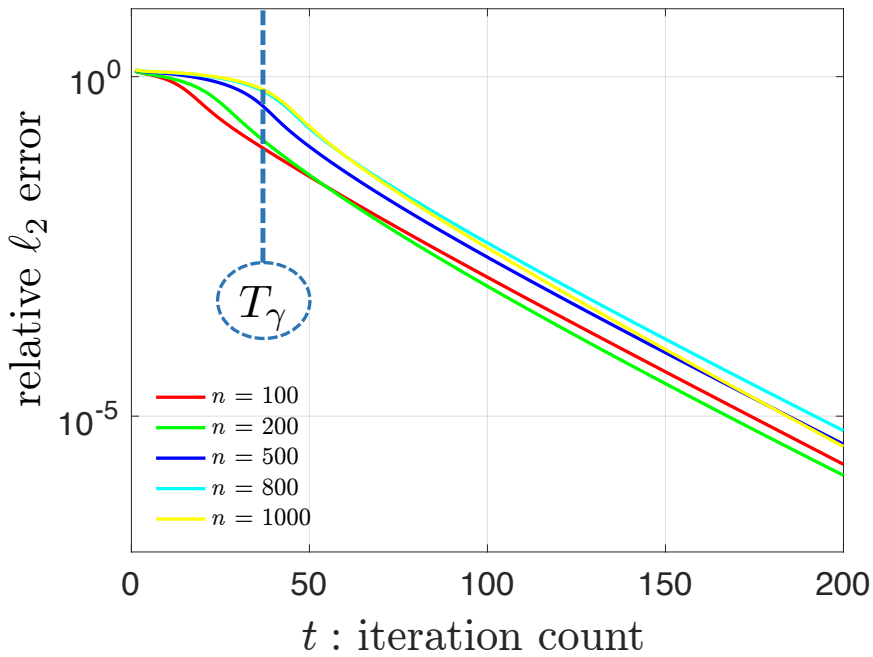
Under i.i.d. Gaussian design, GD with $\boldsymbol{x}^0 \sim \mathcal{N}(\boldsymbol{0}, n^{-1}\boldsymbol{I}_n)$ *achieves*

$$\mathrm{dist}(\boldsymbol{x}^t, \boldsymbol{x}^\star) \le \gamma(1-\rho)^{t-T_\gamma}\|\boldsymbol{x}^\star\|_2, \qquad t \ge T_\gamma$$

for $T_\gamma \lesssim \log n$ *and some constants* $\gamma, \rho > 0$*, provided that step size* $\eta \asymp 1$ *and sample size* $m \gtrsim n\,\mathrm{poly}\log m$

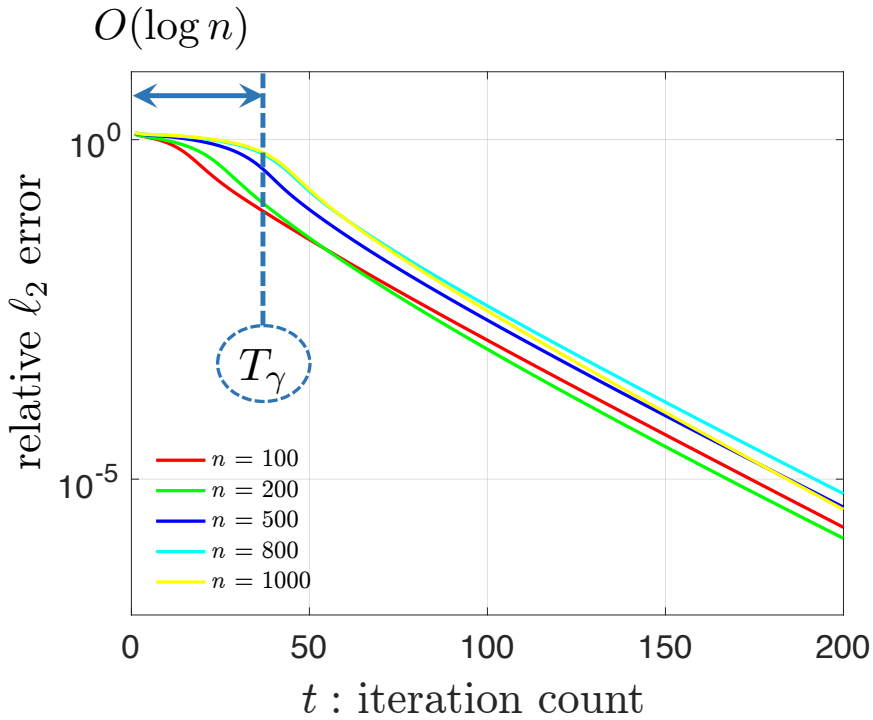
Theoretical guarantees for randomly initialized GD

$$\mathrm{dist}(\boldsymbol{x}^t, \boldsymbol{x}^\star) \leq \gamma(1-\rho)^{t-T_\gamma}\|\boldsymbol{x}^\star\|_2, \quad t \geq T_\gamma \asymp \log n$$



Theoretical guarantees for randomly initialized GD

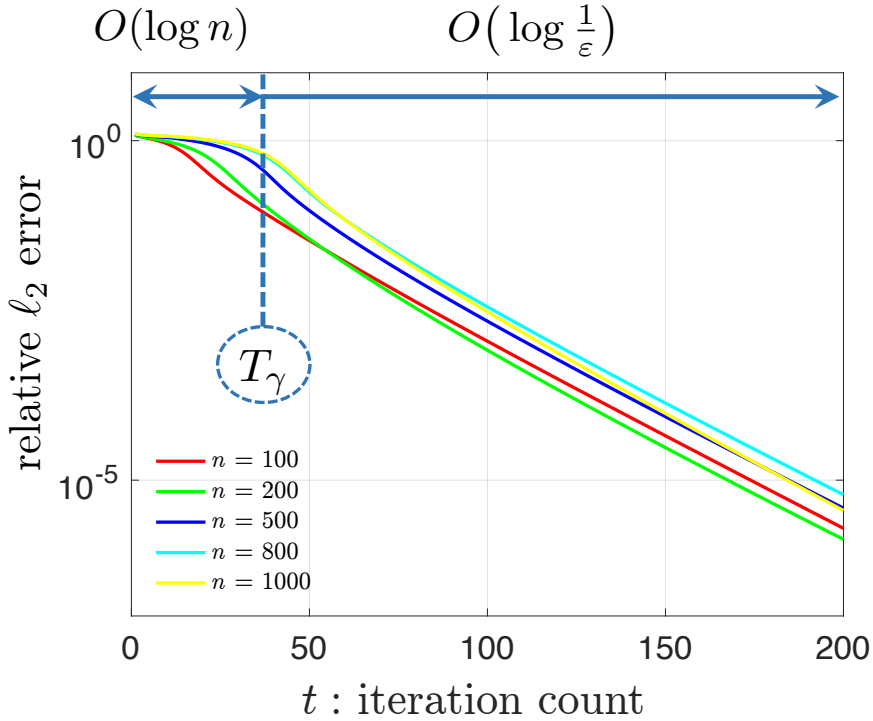
$$\mathrm{dist}(\boldsymbol{x}^t, \boldsymbol{x}^\star) \leq \gamma(1-\rho)^{t-T_\gamma}\|\boldsymbol{x}^\star\|_2, \quad t \geq T_\gamma \asymp \log n$$



- *Stage 1:* takes $O(\log n)$ iterations to reach $\mathrm{dist}(\boldsymbol{x}^t, \boldsymbol{x}^\star) \leq \gamma$

Theoretical guarantees for randomly initialized GD

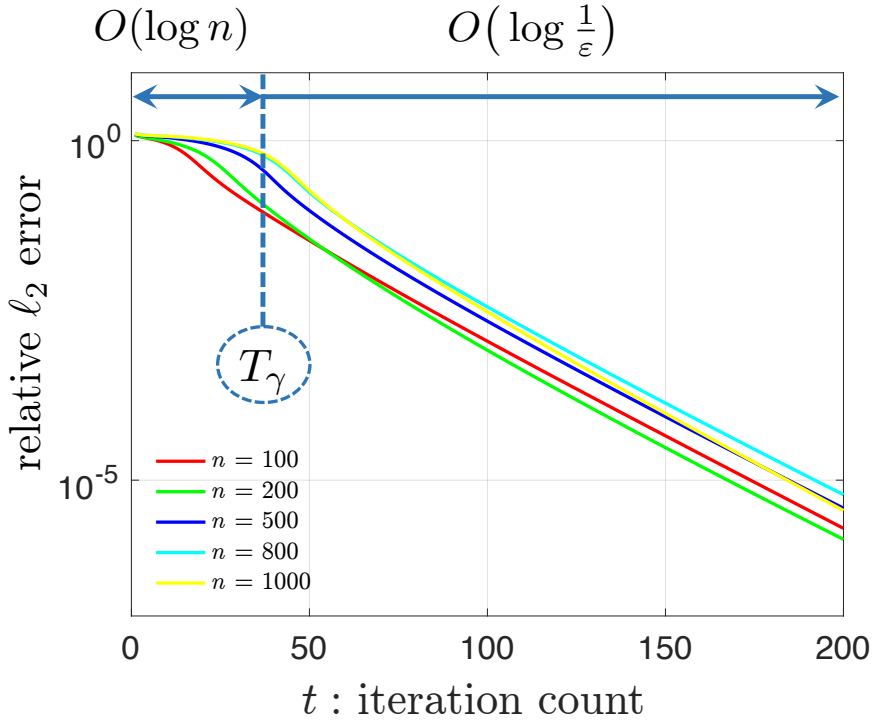
$$\mathrm{dist}(\boldsymbol{x}^t, \boldsymbol{x}^\star) \leq \gamma(1-\rho)^{t-T_\gamma}\|\boldsymbol{x}^\star\|_2, \quad t \geq T_\gamma \asymp \log n$$



- *Stage 1:* takes $O(\log n)$ iterations to reach $\mathrm{dist}(\boldsymbol{x}^t, \boldsymbol{x}^\star) \leq \gamma$
- *Stage 2:* linear convergence

Theoretical guarantees for randomly initialized GD

$$\mathrm{dist}(\boldsymbol{x}^t, \boldsymbol{x}^\star) \le \gamma(1-\rho)^{t-T_\gamma}\|\boldsymbol{x}^\star\|_2, \quad t \ge T_\gamma \asymp \log n$$



- *near-optimal compututational cost*:

  — $O\big(\log n + \log \frac{1}{\varepsilon}\big)$ iterations to yield ε accuracy

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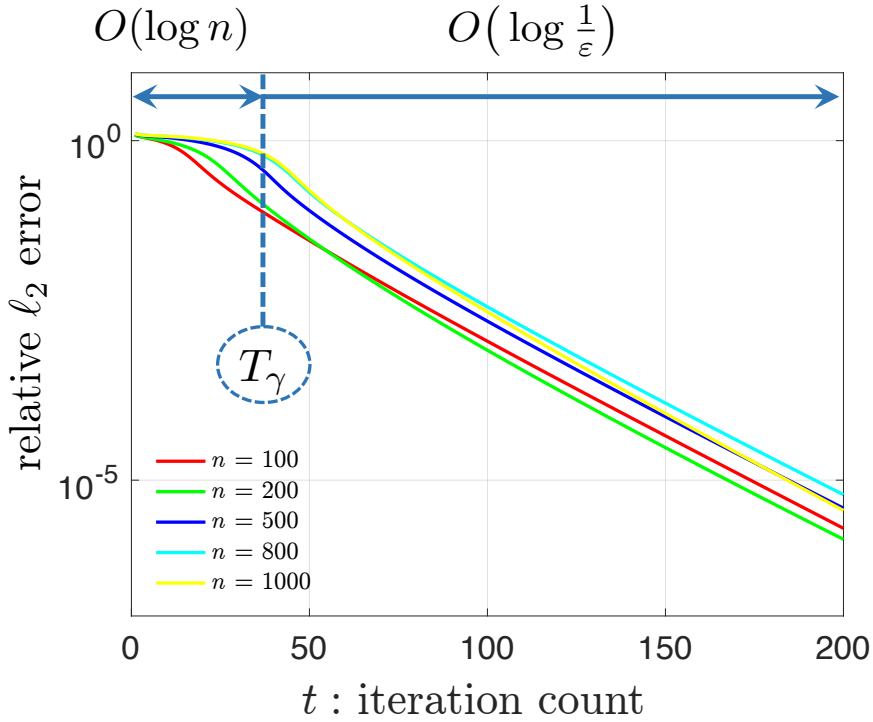
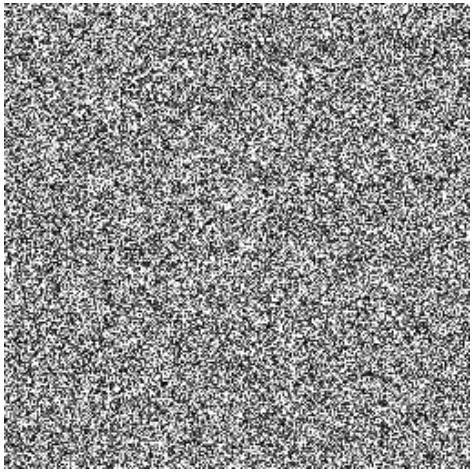


- *near-optimal compututational cost*:
  — $O\big(\log n + \log \frac{1}{\varepsilon}\big)$ iterations to yield ε accuracy
- *near-optimal sample size*: $m \gtrsim n\mathrm{poly}\log m$

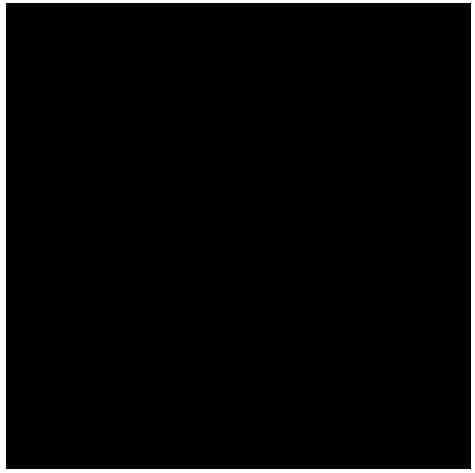
Experiments on images

- coded diffraction patterns
- $\boldsymbol{x}^\star \in \mathbb{R}^{256\times 256}$
- $m/n = 12$

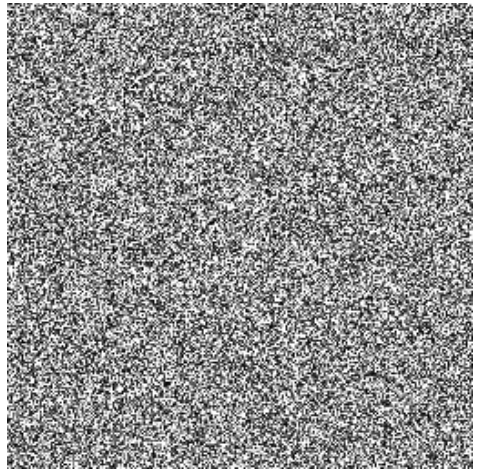
GD with random initialization



$\boldsymbol{x}^t$
GD iterate



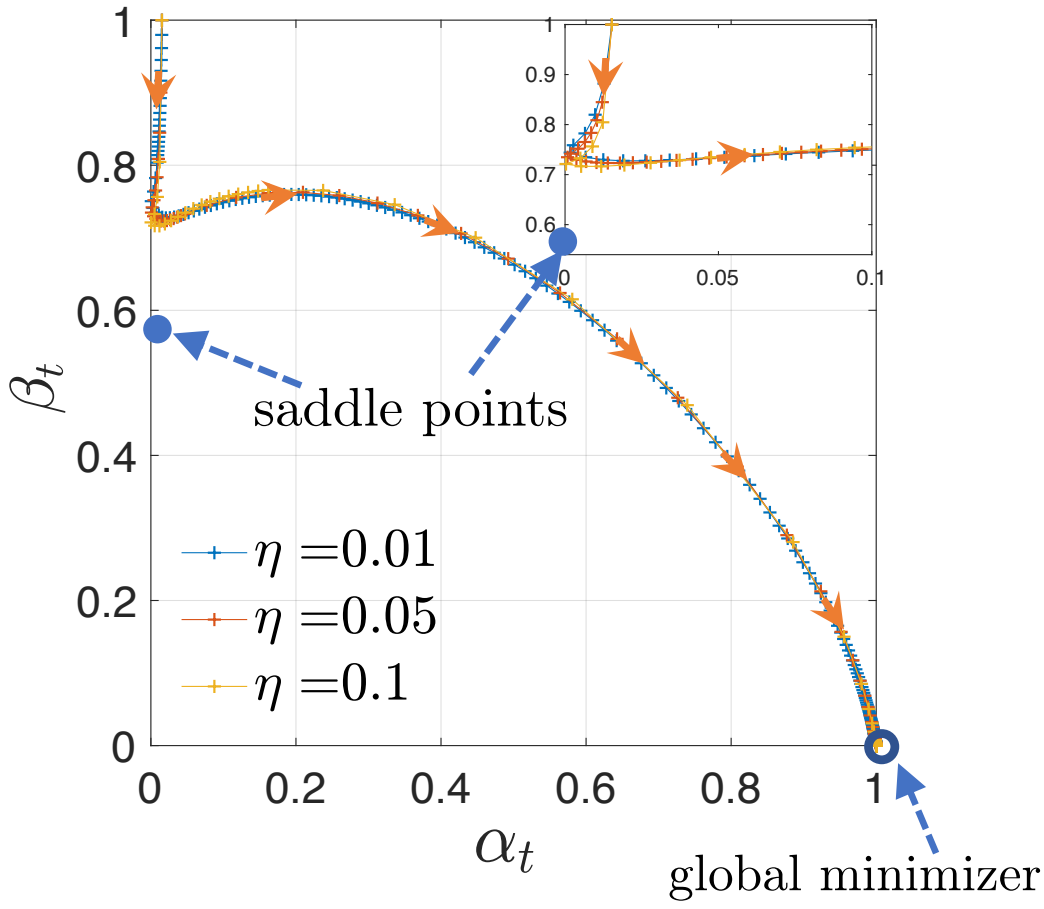
$\langle \boldsymbol{x}^t, \boldsymbol{x}^\star \rangle \boldsymbol{x}^\star$
signal component



$\boldsymbol{x}^t - \langle \boldsymbol{x}^t, \boldsymbol{x}^\star \rangle \boldsymbol{x}^\star$
perpendicular component

use Adobe Acrobat to see animation

Saddle-escaping schemes?



Randomly initialized GD never hits saddle points in phase retrieval!

Other saddle-escaping schemes

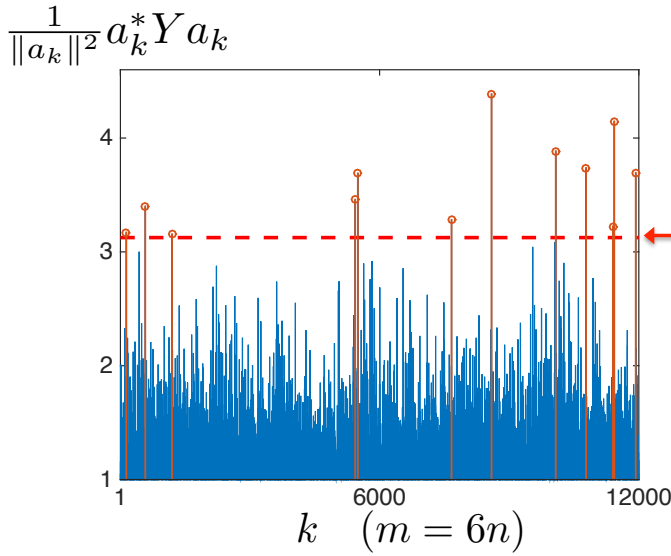
	iteration complexity	**num of iterations needed to escape saddles**	**local iteration complexity**
Trust-region (Sun et al. '16)	$n^7 + \log\log\frac{1}{\varepsilon}$	n^7	$\log\log\frac{1}{\varepsilon}$
Perturbed GD (Jin et al. '17)	$n^3 + n\log\frac{1}{\varepsilon}$	n^3	$n\log\frac{1}{\varepsilon}$
Perturbed accelerated GD (Jin et al. '17)	$n^{2.5} + \sqrt{n}\log\frac{1}{\varepsilon}$	$n^{2.5}$	$\sqrt{n}\log\frac{1}{\varepsilon}$
GD (Chen et al. '18)	$\log n + \log\frac{1}{\varepsilon}$	$\log n$	$\log\frac{1}{\varepsilon}$

Generic optimization theory yields highly suboptimal convergence guarantees

Can we further improve sample complexity?

Truncated spectral initialization

$$\mathbb{E}[\boldsymbol{Y}] := \mathbb{E}\left[\frac{1}{m}\sum_{k=1}^{m} \boldsymbol{y}_k \boldsymbol{a}_k \boldsymbol{a}_k^\top\right] = \|\boldsymbol{x}^\star\|_2^2 \, \boldsymbol{I} + 2\boldsymbol{x}^\star \boldsymbol{x}^{\star\top}$$



problem: unless $m \gg n$, dangerous to use empirical average because large observations $y_k = (\boldsymbol{a}_k^\top \boldsymbol{x}^\star)^2$ bear too much influence

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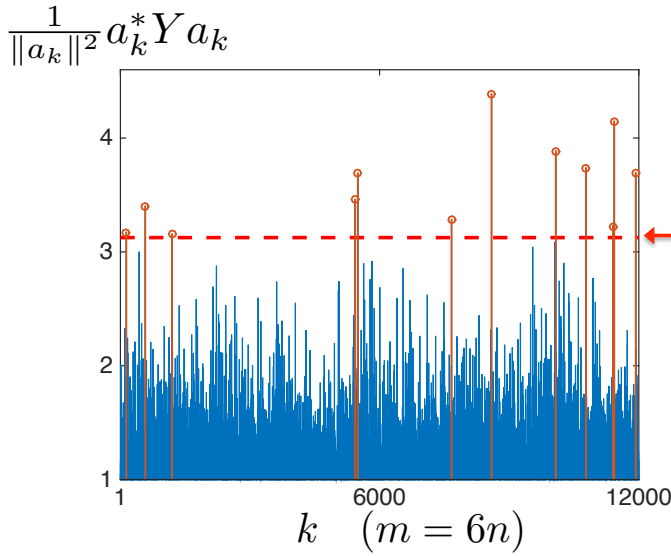
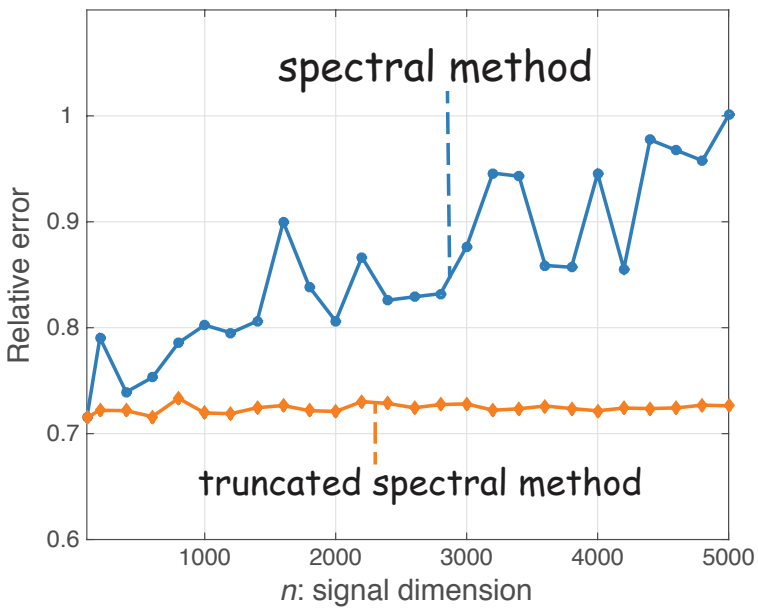


problem: unless $m \gg n$, dangerous to use empirical average because large observations $y_k = (\boldsymbol{a}_k^\top \boldsymbol{x}^\star)^2$ bear too much influence

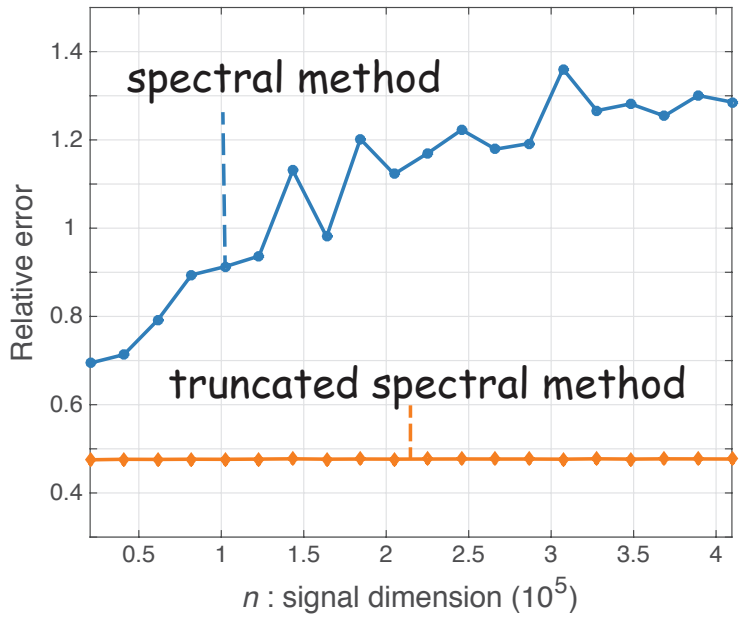
solution: discard high leverage samples and compute leading eigenvector of truncated sum

$$\frac{1}{m}\sum_{k=1}^{m} y_k \boldsymbol{a}_k \boldsymbol{a}_k^\top \cdot \mathbf{1}_{\{|y_k| \le \alpha^2 \, \mathrm{Avg}(|y_j|)\}}$$

Importance of truncated spectral initialization



real Gaussian $m = 6n$

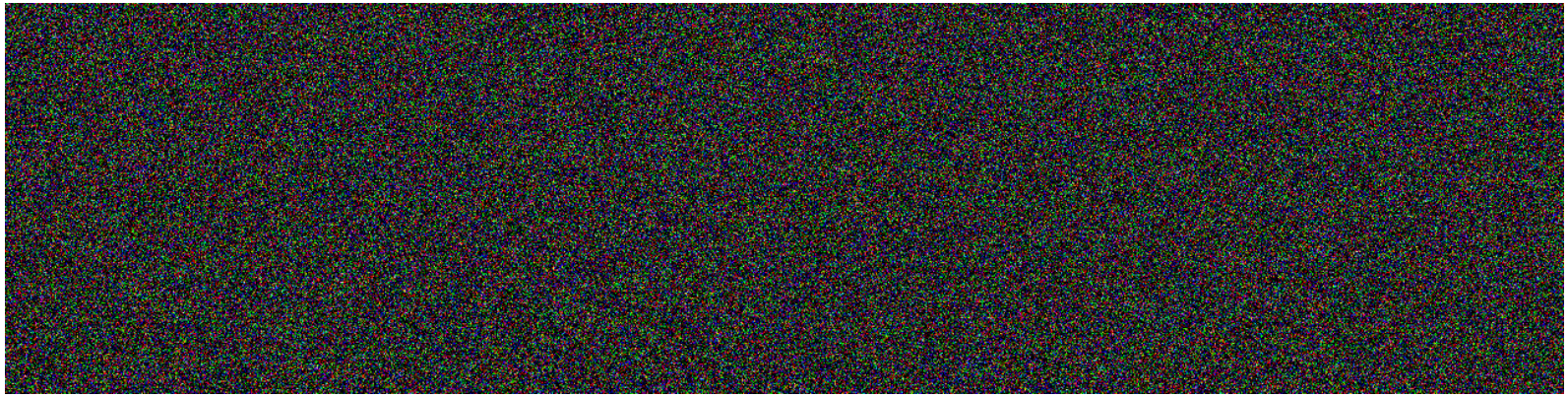


complex CDP $m = 12n$

Importance of truncated spectral initialization

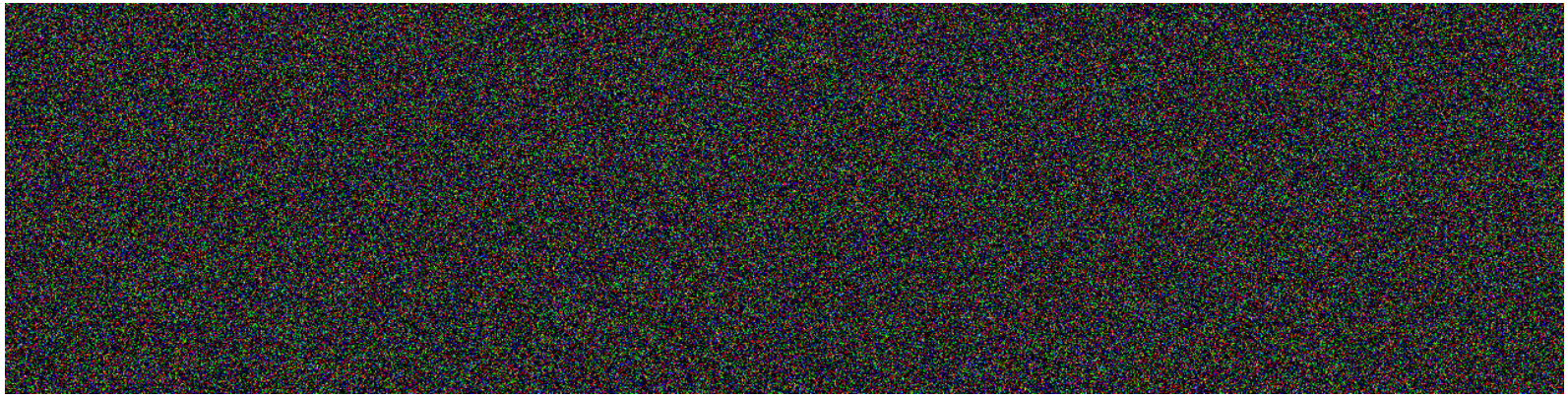
Original image

Importance of truncated spectral initialization



Spectral initialization

Importance of truncated spectral initialization



Spectral initialization

Truncated spectral initialization

Precise asymptotic characterization (Lu, Li '17)

- $m/n \asymp 1$
- i.i.d. Gaussian design

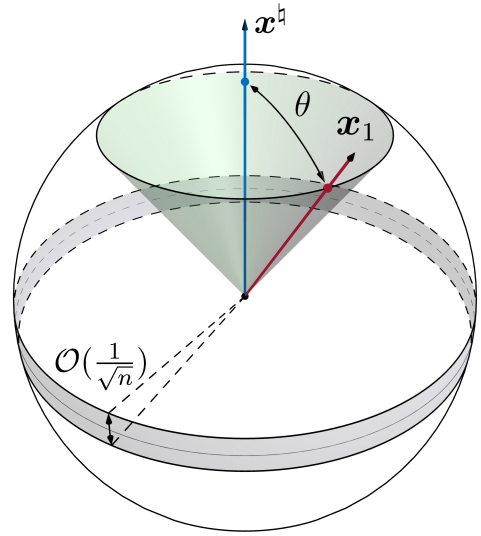


Fig. credit: Lu, Li '17

Theorem 5 (Lu, Li '17, Mondelli, Montanari '17)

There exist analytical formulas $\rho(\cdot)$ *and constants* $\alpha_{\min}$ *and* $\alpha_{\max}$ *s.t.*

$$\underbrace{\frac{\left(\boldsymbol{x}^{\star\top}\boldsymbol{x}^0\right)^2}{\|\boldsymbol{x}^\star\|_2^2\|\boldsymbol{x}^0\|_2^2}}_{\text{cosine similarity}} \longrightarrow \begin{cases} 0, & \text{if } m/n < \alpha_{\min} \\ \rho(m/n), & \text{if } m/n > \alpha_{\max} \end{cases}$$

Theoretical prediction vs. simulations

image size: 64×64

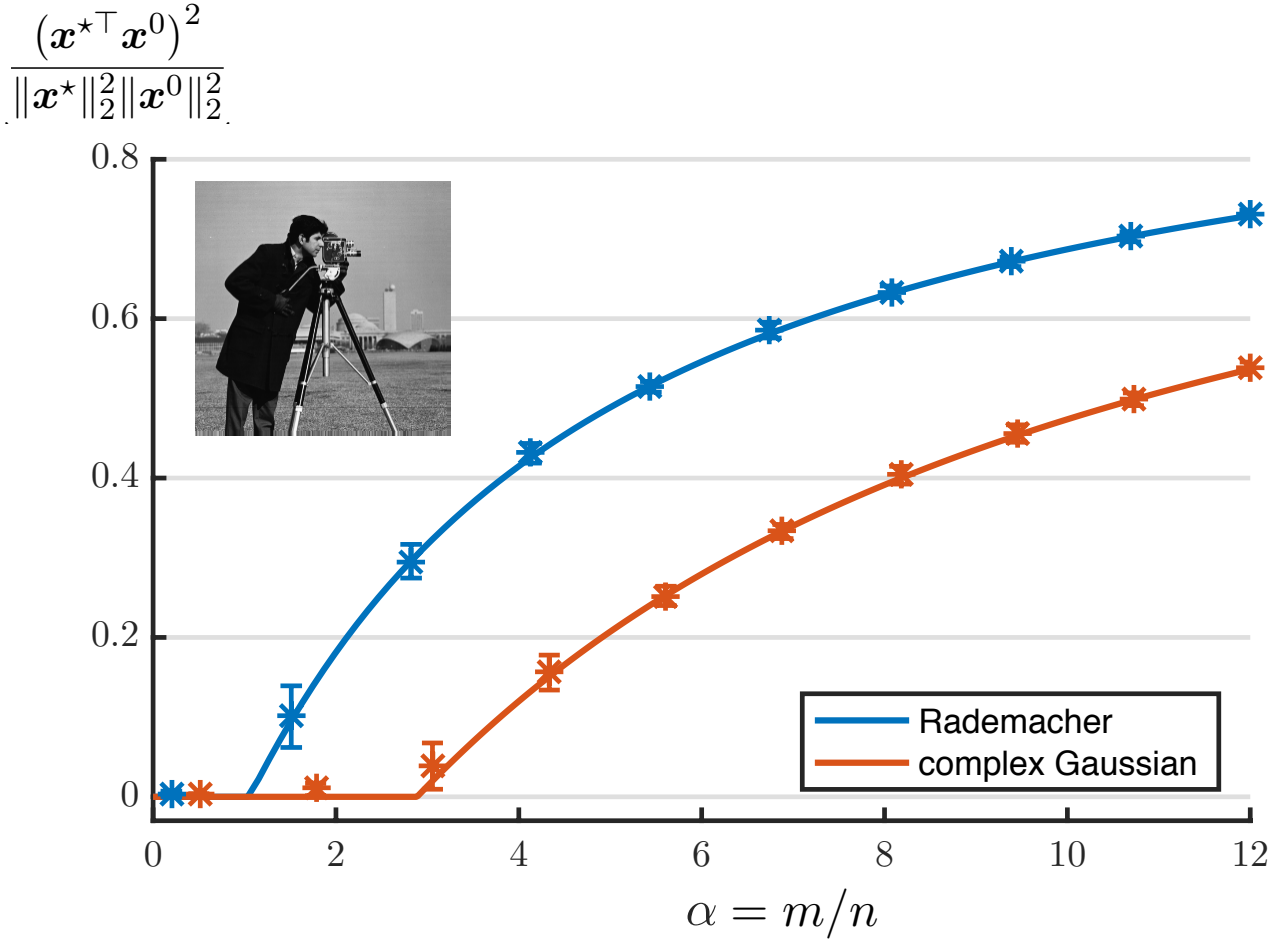


Fig. credit: Lu, Li '17

Improving search directions

WF (GD): $\boldsymbol{x}^{t+1} = \boldsymbol{x}^t - \frac{\eta}{m} \sum_k \nabla f_k(\boldsymbol{x}^t)$

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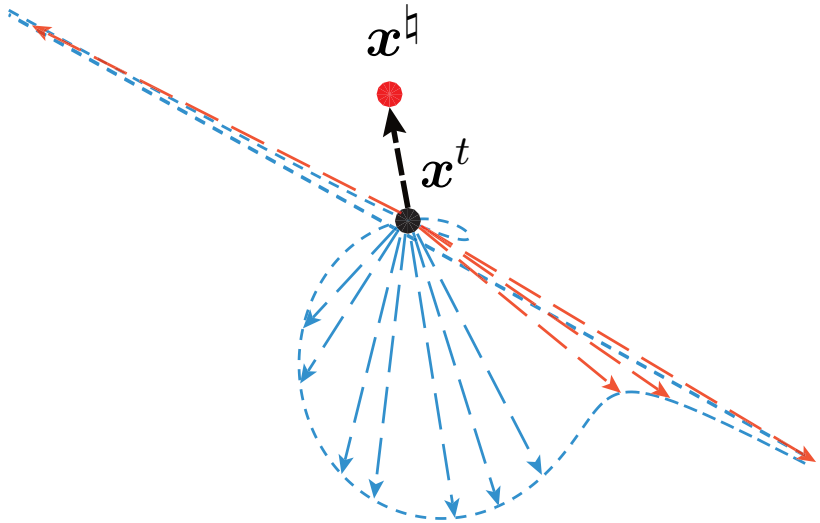

locus of $\{\nabla f_k(\boldsymbol{z})\}$

Improving search directions

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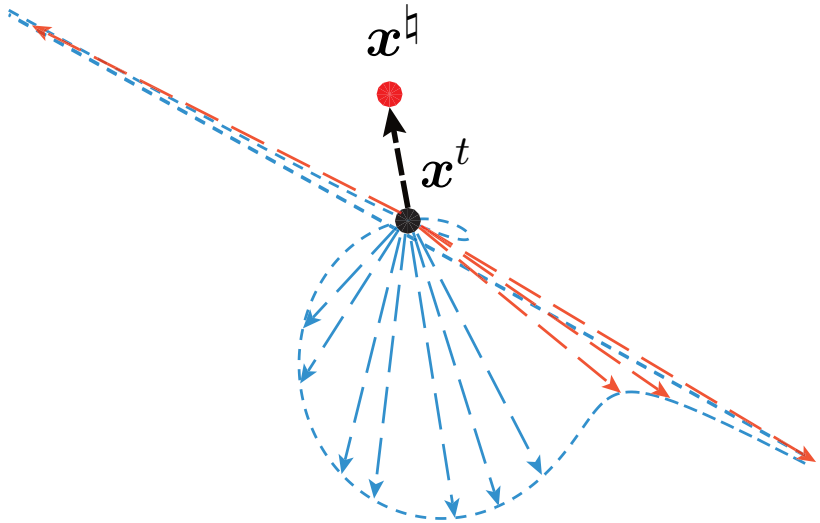
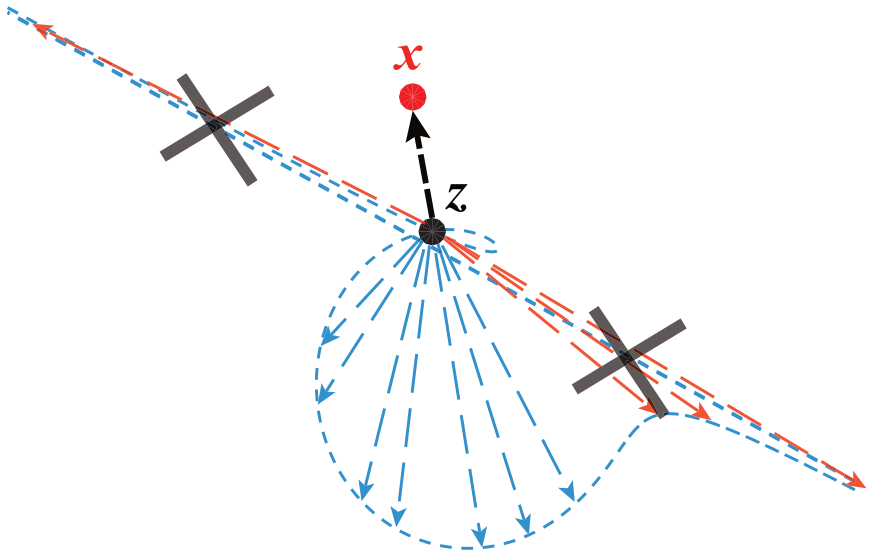


locus of $\{\nabla f_k(\boldsymbol{z})\}$

Problem: descent direction might have large variability

Solution: variance reduction via trimming

More adaptive rule: $\boldsymbol{x}^{t+1} = \boldsymbol{x}^t - \frac{\eta}{m} \sum_{k \in \mathcal{T}_t} \nabla f_k(\boldsymbol{x}^t)$



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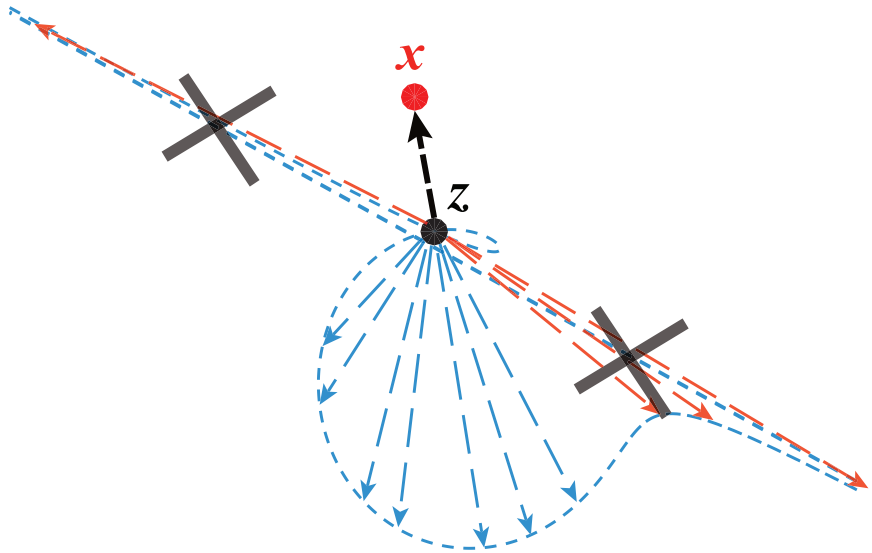


- $\mathcal{T}_t$ trims away excessively large grad components

$$\mathcal{T}_t := \Big\{ k : \quad \big\|\nabla f_k(\boldsymbol{x}^t)\big\|_2 \ \lesssim \ \text{typical-size} \Big\{ \big\|\nabla f_l(\boldsymbol{x}^t)\big\|_2 \Big\}_{1 \le l \le m} \Big\}$$

Slight bias + much reduced variance

Summary: truncated Wirtinger flow

(1) **Regularized spectral initialization:** $\boldsymbol{x}^0 \leftarrow$ principal component of

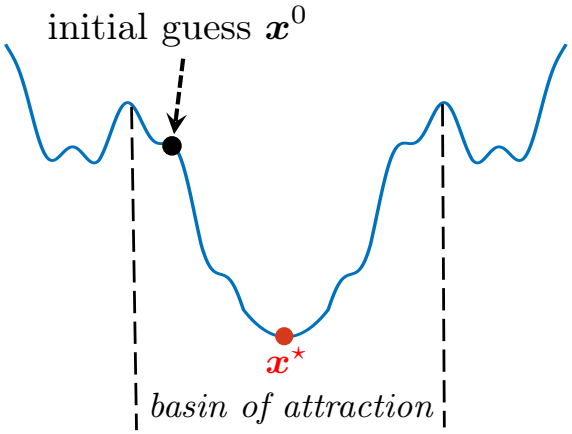
$$\frac{1}{m}\sum\nolimits_{k\in\mathcal{T}_0} y_k\, \boldsymbol{a}_k\boldsymbol{a}_k^\top$$

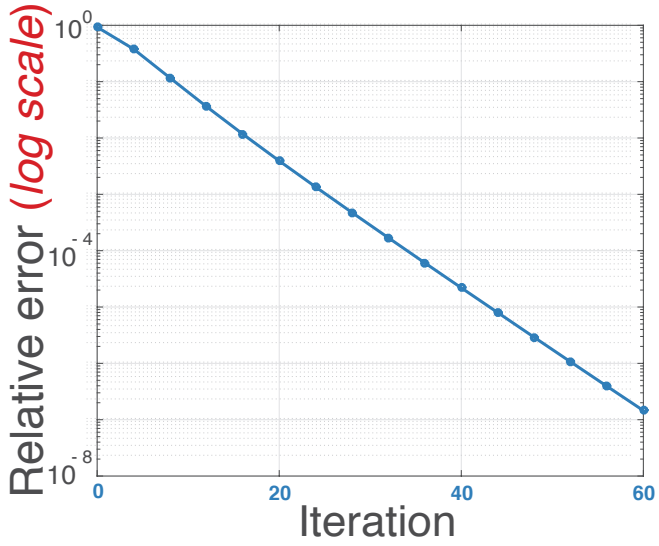
(2) Follow **adaptive gradient descent**

$$\boldsymbol{x}^t = \boldsymbol{x}^t - \frac{\eta_t}{m}\sum\nolimits_{k\in\mathcal{T}_t} \nabla f_k(\boldsymbol{x}^t)$$

Adaptive and iteration-varying rules: discard high-leverage data $\{y_k : k \notin \mathcal{T}_t\}$

Theoretical guarantees (noiseless data)





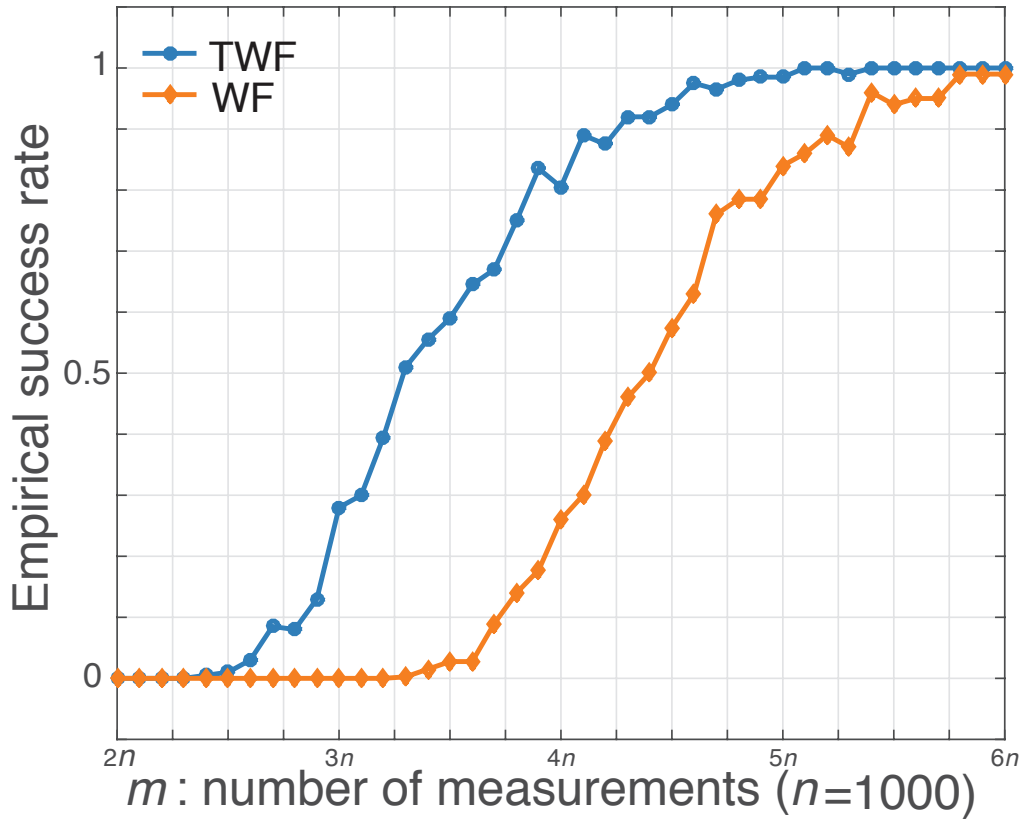
Theorem 6 (Chen, Candès '15)

Suppose $\boldsymbol{a}_k \overset{i.i.d.}{\sim} \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_n)$ *and sample size* $m \gtrsim n$. *With high prob.,*

$$\operatorname{dist}(\boldsymbol{x}^t, \boldsymbol{x}^\star) := \min \|\boldsymbol{x}^t \pm \boldsymbol{x}^\star\|_2 \;\le\; \nu\,(1-\rho)^t\,\|\boldsymbol{x}^\star\|_2$$

where $0 < \nu, \rho < 1$ *are universal constants*

Empirical success rate (noiseless data)



Empirical success rate vs. sample size

Stability vis a vis noise and outliers?

Stability under noisy data

- Noisy data: $y_k = (\boldsymbol{a}_k^\top \boldsymbol{x}^\star)^2 + \eta_k$
- Signal-to-noise ratio:

$$\mathsf{SNR} := \frac{\sum_k (\boldsymbol{a}_k^\top \boldsymbol{x}^\star)^4}{\sum_k \eta_k^2} \approx \frac{3m\|\boldsymbol{x}^\star\|_2^4}{\|\boldsymbol{\eta}\|_2^2}$$

- i.i.d. Gaussian design $\boldsymbol{a}_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_n)$

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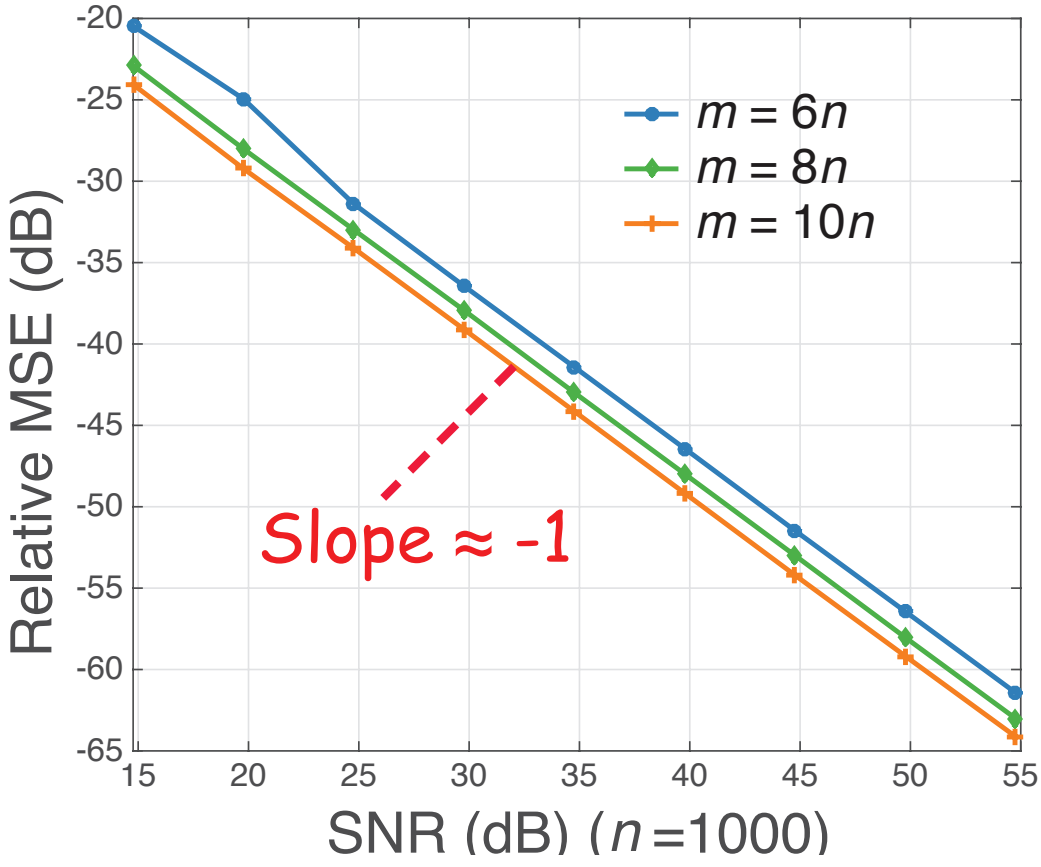
Theorem 7 (Chen, Candès '15)

Relative error of TWF converges to $O(\frac{1}{\sqrt{\mathsf{SNR}}})$

Relative MSE vs. SNR (Poisson data)

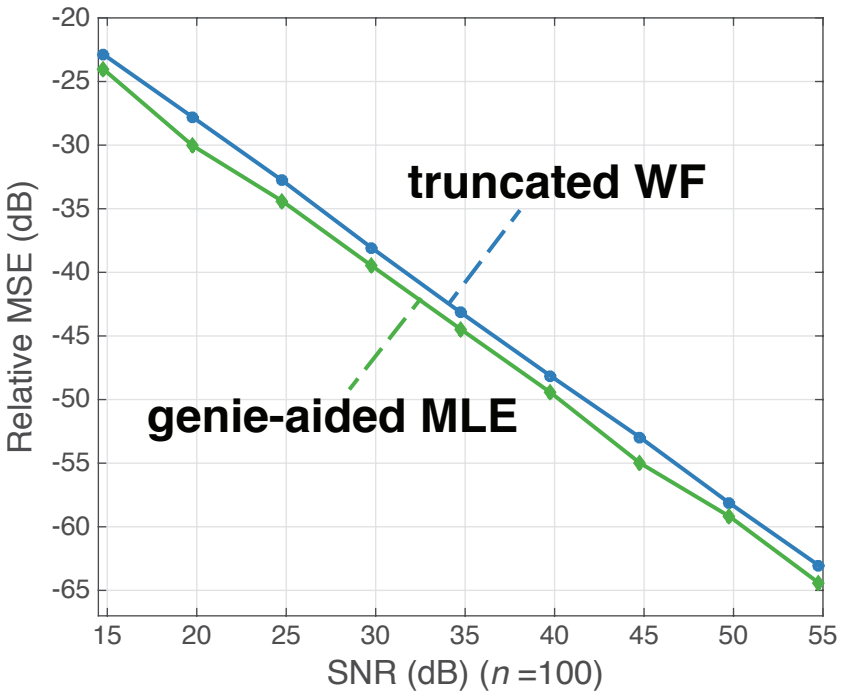


Empirical evidence: relative MSE scales inversely with SNR

This accuracy is nearly un-improvable (empirically)

Comparison with ideal MLE (with phase info. revealed)

ideal knowledge: $y_k \sim \text{Poisson}\big(\,|\boldsymbol{a}_k^\top \boldsymbol{x}^\star|^2\,\big)$ and $\varepsilon_k = \text{sign}\big(\boldsymbol{a}_k^\top \boldsymbol{x}^\star\big)$



Little loss due to missing phases!

This accuracy is nearly un-improvable (theoretically)

- Poisson data: $y_k \overset{\text{ind.}}{\sim} \text{Poisson}(\,|\boldsymbol{a}_k^\top \boldsymbol{x}^\star|^2\,)$
- Signal-to-noise ratio:

$$\text{SNR} \;\approx\; \frac{\sum_k |\boldsymbol{a}_k^\top \boldsymbol{x}^\star|^4}{\sum_k \text{Var}(y_k)} \;\approx\; 3\|\boldsymbol{x}^\star\|_2^2$$

This accuracy is nearly un-improvable (theoretically)

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Theorem 8 (Chen, Candès '15)

Under i.i.d. Gaussian design, for any estimator $\widehat{\boldsymbol{x}}$,

$$\inf_{\widehat{\boldsymbol{x}}} \sup_{\boldsymbol{x}^\star:\ \|\boldsymbol{x}^\star\|_2 \geq \log^{1.5} m} \frac{\mathbb{E}\left[\text{dist}\left(\widehat{\boldsymbol{x}}, \boldsymbol{x}^\star\right) \mid \{\boldsymbol{a}_k\}\right]}{\|\boldsymbol{x}^\star\|_2} \;\gtrsim\; \frac{1}{\sqrt{\text{SNR}}},$$

provided that sample size $m \asymp n$

Robust recovery vis a vis outliers

Consider now two sources of corruption: *sparse outliers* and *bounded noise*

$$y_i = |\boldsymbol{a}_i^\top \boldsymbol{x}^\star|^2 + \eta_i + w_i, \quad i = 1, \dots, m,$$

- $\|\boldsymbol{\eta}\|_0 \leq s \cdot m$: sparse outlier, where $0 \leq s < 1$ is fraction of outliers
- $\boldsymbol{w}$: bounded noise

Motivation: outliers happen with sensor failures, malicious attacks ...

Robust recovery vis a vis outliers

Goal: develop algorithms that are *oblivious* to outliers, and statistically and computationally efficient

- performs equally well regardless of existence of outliers
- small sample size: ideally $m \asymp n$
- large fraction of outliers: ideally $s \asymp 1$
- low computational complexity and easy to implement

Existing approaches are not robust in the presence of arbitrary outliers

- **Spectral initialization would fail:** leading eigenvector of $\boldsymbol{Y}$ can be arbitrarily perturbed

$$\boldsymbol{Y} = \frac{1}{m}\sum_{i=1}^{m} y_i \boldsymbol{a}_i \boldsymbol{a}_i^\top \qquad \text{(WF)}$$

$$\text{or} \quad \boldsymbol{Y} = \frac{1}{m}\sum_{i=1}^{m} y_i \boldsymbol{a}_i \boldsymbol{a}_i^\top \mathbb{1}_{\{|y_i| \lesssim \text{mean}(\{y_i\})\}} \qquad \text{(TWF)}$$

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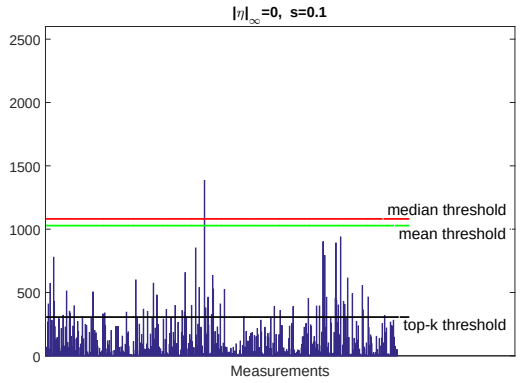
- **GD would fail:** search directions can be arbitrarily perturbed

$$\boldsymbol{x}^{t+1} = \boldsymbol{x}^t - \frac{\eta}{m}\sum_{i=1}^{m} \nabla f_k(\boldsymbol{x}^t)$$

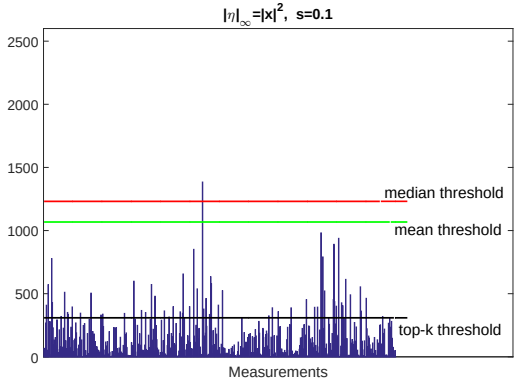
Solution: median truncation

Median is often more stable for various levels of outliers

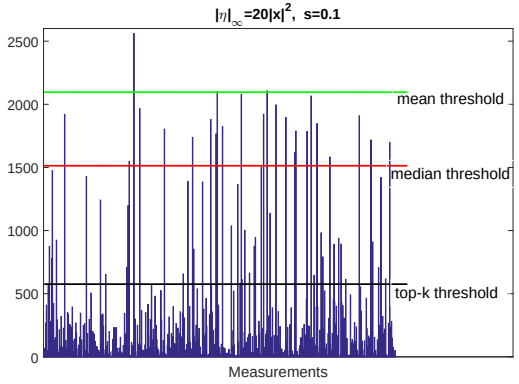
- well-known in robust statistics to be outlier-resilient



no outliers



small outlier magnitudes



large outlier magnitudes

Solution: median truncation

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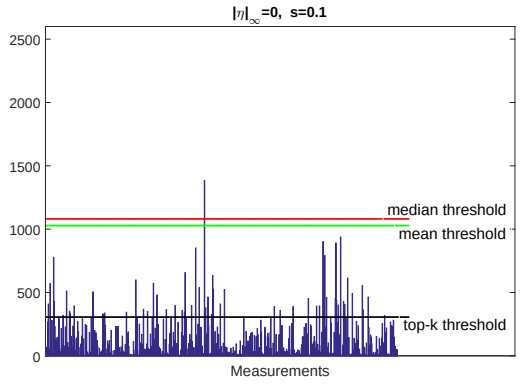
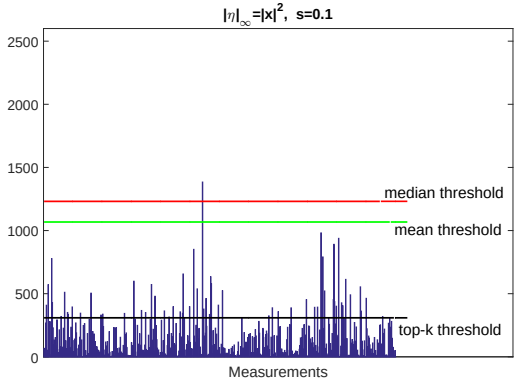
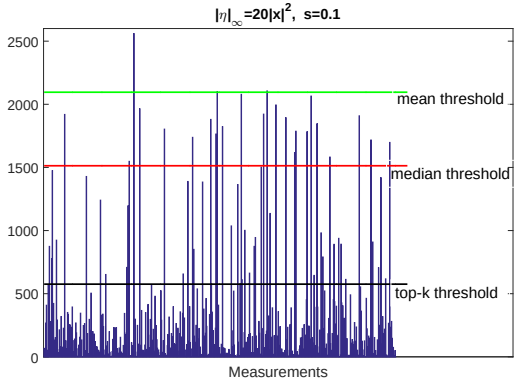


no outliers



small outlier magnitudes



large outlier magnitudes

Key idea: "median-truncation" — discard samples *adaptively* based on how large sample gradients / values deviate from median

Median-truncated gradient descent

(1) **Median-truncated spectral initialization:** $\boldsymbol{x}^0 \leftarrow$ leading eigenvector of

$$\boldsymbol{Y} = \frac{1}{m}\sum_{i=1}^{m} y_i \boldsymbol{a}_i \boldsymbol{a}_i^\top \mathbb{1}_{\{|y_i| \lesssim \text{median}(\{y_i\})\}}$$

(2) **Median-truncated gradient descent:**

$$\boldsymbol{x}^{t+1} = \boldsymbol{x}^t - \frac{\eta}{m}\sum_{k\in\mathcal{T}_t} \nabla f_k(\boldsymbol{x}^t),$$

where

$$\mathcal{T}_t = \left\{k: \ \left|y_k - |\boldsymbol{a}_k^\top \boldsymbol{x}^t|\right| \lesssim \text{median}\left(\left\{\left|y_k - |\boldsymbol{a}_k^\top \boldsymbol{x}^t|\right|\right\}\right)\right\}$$

Performance guarantees

Theorem 9 (Zhang, Chi and Liang '16)

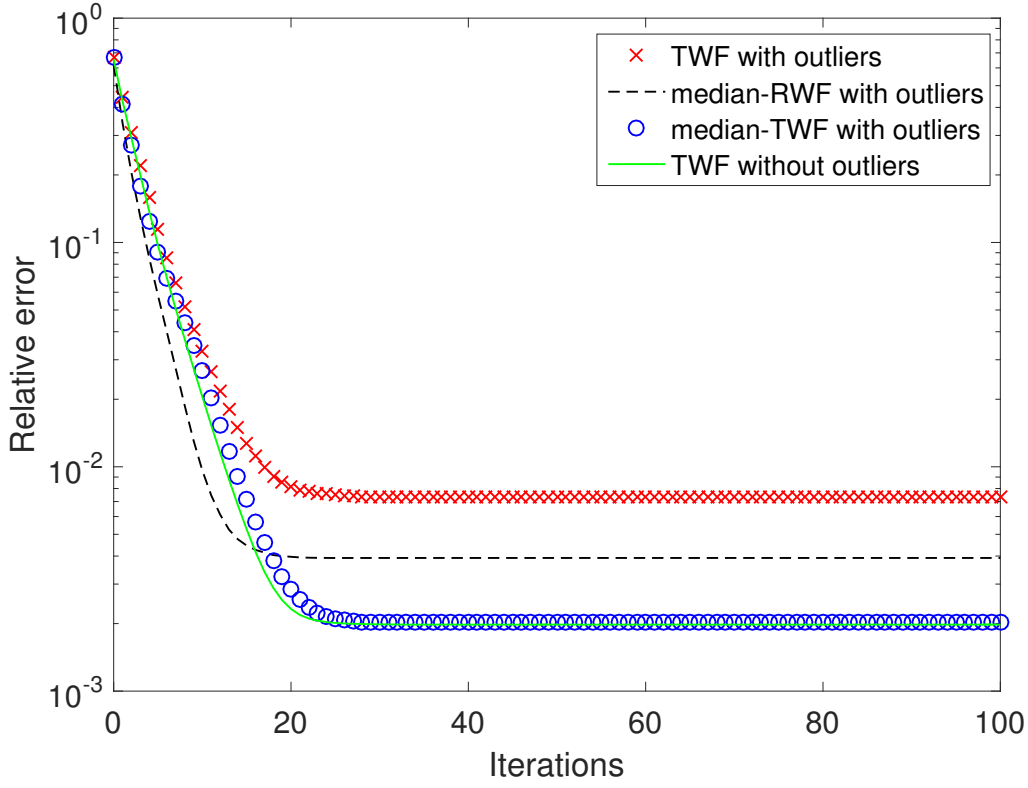
Assume $\|\boldsymbol{w}\|_\infty \le c_1\|\boldsymbol{x}^\star\|_2^2$, *and* $\boldsymbol{a}_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\boldsymbol{0}, \boldsymbol{I}_n)$. *If* $m \gtrsim n\log n$ *and* $s \lesssim s_0$, *then with high prob., median-TWF / RWF yields*

$$\text{dist}(\boldsymbol{x}^t, \boldsymbol{x}^\star) \lesssim \frac{\|\boldsymbol{w}\|_\infty}{\|\boldsymbol{x}^\star\|_2} + (1-\rho)^t\|\boldsymbol{x}^\star\|_2, \quad t = 0, 1, \cdots$$

for some constants $0 < \rho, s_0 < 1$

- **Exact recovery** when $\boldsymbol{w} = \boldsymbol{0}$ but with a constant fraction of outliers $s \asymp 1$
- **Stable recovery** with additional bounded noise
- Resist outliers **obliviously**: no prior knowledge of outliers (except sparsity)

Numerical experiment with both dense noise and sparse outliers



Median-TWF with outliers achieves almost identical accuracy as TWF without outliers

Outline

- Part I: Overview
- Part II: Phase retrieval: a case study
  - Spectral initialization
  - Local refinement: algorithm and analysis
- Part III: Low-rank matrix estimation
- Part IV: Closing remarks

Motivation

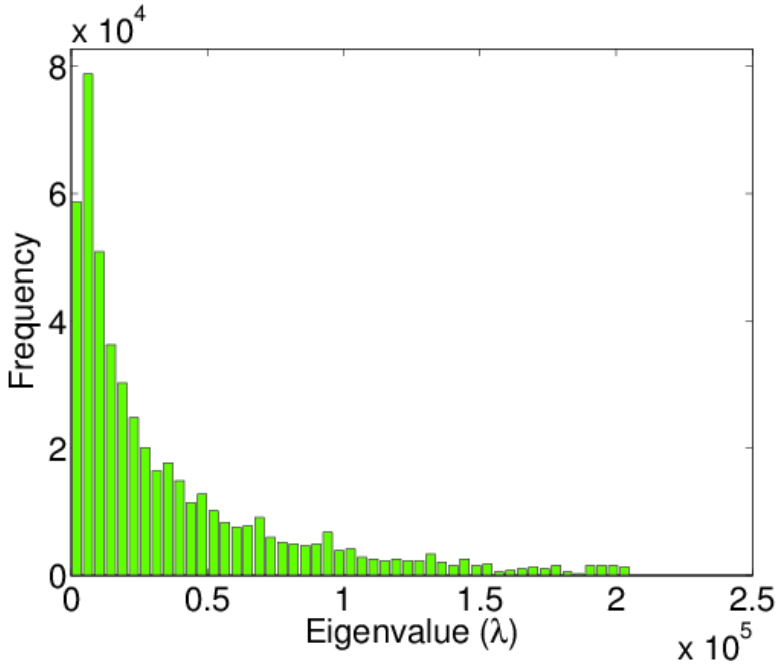
Low-rank matrix estimation problems arise in many applications

A popular example is **recommendation systems**: how to predict unseen user ratings for movies?

figure credit: E. Candès

Low-rank modeling

figure credit: E. Candès



A few factors explain most of the data

Low-rank modeling

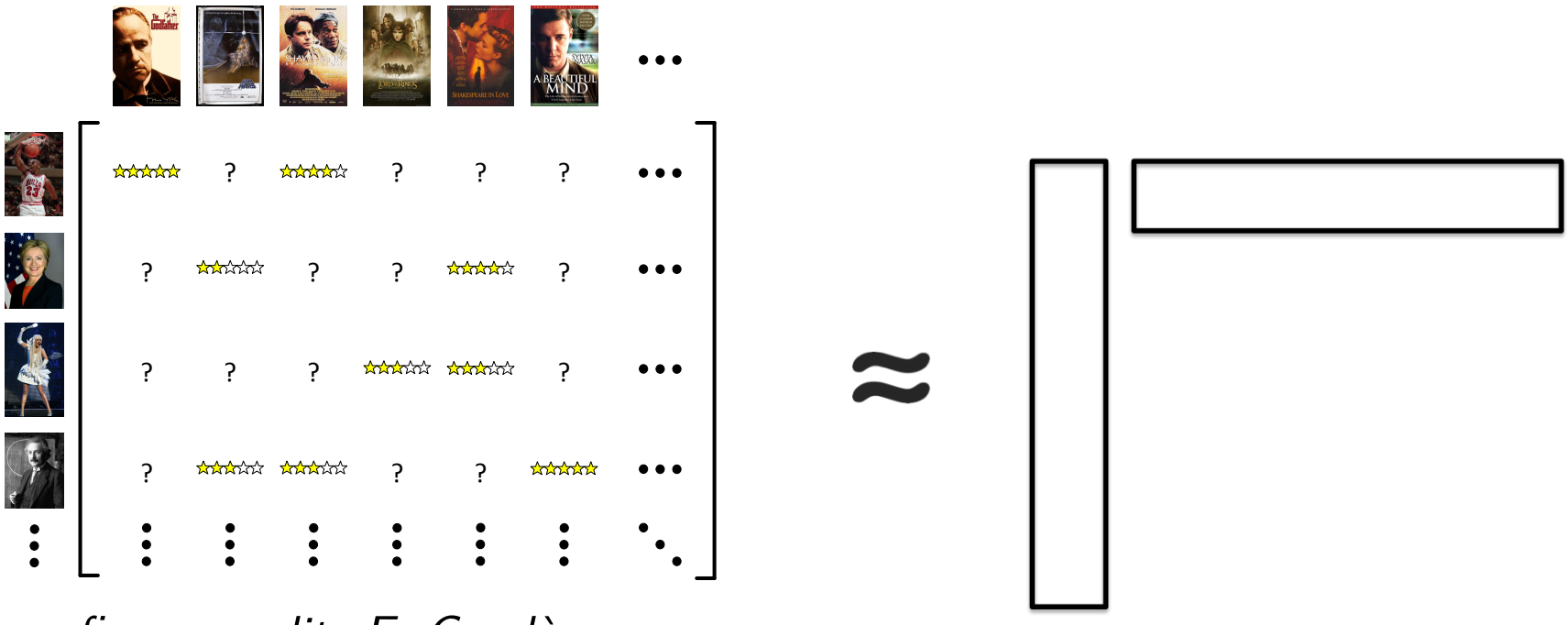


figure credit: E. Candès

A few factors explain most of the data $\longrightarrow$ low-rank approximation

How to exploit (approx.) low-rank structure in prediction?

Other problems with low-rank matrices

- sensor network localization
- structure from motion
- system identification and time series analysis
- spatial-temporal data modeling, e.g. video, network traffic, ..
- face recognition
- quantum state tomography
- community detection
- ...

Rank-constrained optimization

Rank-constrained optimization:

$$\text{minimize}_{\boldsymbol{M} \in \mathbb{R}^{n \times n}} \quad F(\boldsymbol{M}) \quad \text{s.t.} \quad \text{rank}(\boldsymbol{M}) \leq r,$$

where $F(\boldsymbol{M})$ is convex in $\boldsymbol{M}$, and $r \ll n$

- useful model for many low-rank estimation problems;
- computationally intractable.

Convex relaxation

Convex relaxation:

$$\text{minimize}_{\boldsymbol{M} \in \mathbb{R}^{n \times n}} \quad F(\boldsymbol{M}) \quad \text{s.t.} \quad \|\boldsymbol{M}\|_* \leq \zeta$$

where $\|\cdot\|_*$ is nuclear norm — convex relaxation of rank

- **Pros:** mature theory; versatile to incorporate other constraints
- **Cons:** run-time in $O(n^3)$; even $\boldsymbol{M}$ itself takes $O(n^2)$ storage

Question: can we develop algorithms that work with computational and memory complexities nearly linear in n?

Burer-Monteiro factorization

Matrix factorization:

$$\text{minimize}_{\boldsymbol{U},\boldsymbol{V}} \; f(\boldsymbol{U},\boldsymbol{V}) := F(\boldsymbol{U}\boldsymbol{V}^\top)$$

where $\boldsymbol{M} = \boldsymbol{U}\boldsymbol{V}^\top$, where $\boldsymbol{U},\boldsymbol{V} \in \mathbb{R}^{n\times r}$.

- pioneered by Burer, Monteiro '03
- highly non-convex
- global ambiguity: for any orthonormal $\boldsymbol{R} \in \mathbb{R}^{r\times r}$ and $\alpha \neq 0$,

$$\boldsymbol{U}\boldsymbol{V}^\top = (\alpha\boldsymbol{U}\boldsymbol{R})(\alpha^{-1}\boldsymbol{V}\boldsymbol{R})^\top$$

  i.e. if $(\boldsymbol{U},\boldsymbol{V})$ is a global minimizer, so does $(\alpha\boldsymbol{U}\boldsymbol{R}, \alpha^{-1}\boldsymbol{V}\boldsymbol{R})$

Revisiting PCA

Given PSD $\boldsymbol{M} \in \mathbb{R}^{n\times n}$ (not necessarily low-rank), solve *low-rank approximation problem* (best rank-r approximation):

$$\underbrace{\widehat{\boldsymbol{M}} = \mathsf{argmin}_{\boldsymbol{Z}} \ \|\boldsymbol{Z}-\boldsymbol{M}\|_{\mathrm{F}}^2 \quad \text{s.t.} \quad \mathsf{rank}(\boldsymbol{Z}) \le r}_{\text{nonconvex optimization!}}$$

Solution is truncated eigen-decomposition (Eckart-Young theorem)

- let $\boldsymbol{M} = \sum_{i=1}^n \sigma_i \boldsymbol{u}_i \boldsymbol{u}_i^\top$ be EVD of $\boldsymbol{M}$ ($\sigma_1 \ge \cdots \ge \sigma_n$), then

$$\widehat{\boldsymbol{M}} = \sum_{i=1}^r \sigma_i \boldsymbol{u}_i \boldsymbol{u}_i^\top$$

— *nonconvex, but tractable*

Optimization viewpoint

Factorize $\boldsymbol{Z} = \boldsymbol{X}\boldsymbol{X}^\top$ with $\boldsymbol{X} \in \mathbb{R}^{n \times r}$. We're interested in the landscape of

$$f(\boldsymbol{X}) := \frac{1}{4}\|\boldsymbol{X}\boldsymbol{X}^\top - \boldsymbol{M}\|_{\mathrm{F}}^2$$

Optimization viewpoint

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$$f(\boldsymbol{X}) := \frac{1}{4}\|\boldsymbol{X}\boldsymbol{X}^\top - \boldsymbol{M}\|_{\mathrm{F}}^2$$

To simplify exposition: set $r = 1$.

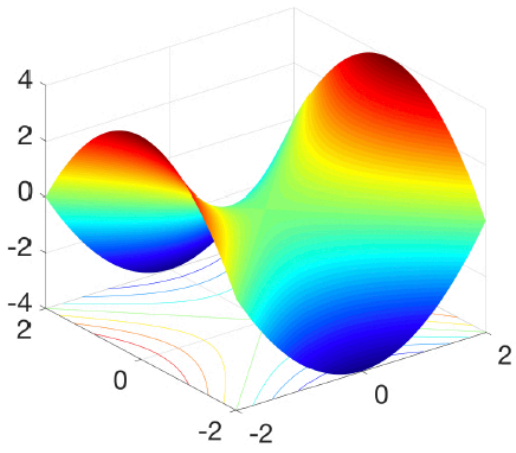
$$f(\boldsymbol{x}) = \frac{1}{4}\|\boldsymbol{x}\boldsymbol{x}^\top - \boldsymbol{M}\|_{\mathrm{F}}^2$$

Definition 10 (critical points)

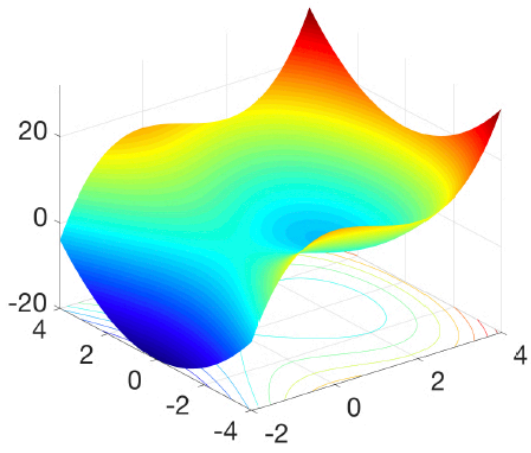
A first-order critical point (stationary point) of f satisfies

$$\nabla f(\boldsymbol{x}) = \boldsymbol{0}$$

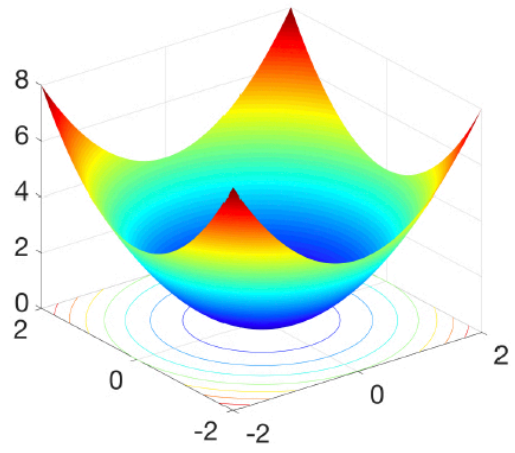
Several types of critical points



(a) strict saddle



(b) local minimum



(c) global minimum

figure credit: Li et al. '16

Critical points of $f(\boldsymbol{x})$

$\boldsymbol{x}$ is critical point, i.e. $\nabla f(\boldsymbol{x}) = (\boldsymbol{x}\boldsymbol{x}^\top - \boldsymbol{M})\boldsymbol{x} = \boldsymbol{0}$

$$\Updownarrow$$

$$\boldsymbol{M}\boldsymbol{x} = \|\boldsymbol{x}\|_2^2 \boldsymbol{x}$$

$$\Updownarrow$$

$\boldsymbol{x}$ aligns with eigenvectors of $\boldsymbol{M}$ or $\boldsymbol{x} = \boldsymbol{0}$

Since $\boldsymbol{M}\boldsymbol{u}_i = \sigma_i \boldsymbol{u}_i$, set of critical points is given by

$$\{\boldsymbol{0}\} \cup \{\sqrt{\sigma_i}\boldsymbol{u}_i,\ i = 1, \ldots, n\}$$

Categorization of critical points

Critical points can be further categorized based on **Hessians**:

$$\nabla^2 f(\boldsymbol{x}) := 2\boldsymbol{x}\boldsymbol{x}^\top + \|\boldsymbol{x}\|_2^2 \boldsymbol{I} - \boldsymbol{M}$$

- For any non-zero critical points $\boldsymbol{x}_k := \sqrt{\sigma_k}\boldsymbol{u}_k$:

$$\begin{aligned}
\nabla^2 f(\boldsymbol{x}_k) &= 2\sigma_k \boldsymbol{u}_k \boldsymbol{u}_k^\top + \sigma_k \boldsymbol{I} - \boldsymbol{M} \\
&= 2\sigma_k \boldsymbol{u}_k \boldsymbol{u}_k^\top + \sigma_k \left(\sum_{i=1}^n \boldsymbol{u}_i \boldsymbol{u}_i^\top \right) - \sum_{i=1}^n \sigma_i \boldsymbol{u}_i \boldsymbol{u}_i^\top \\
&= \sum_{i:i\neq k} (\sigma_k - \sigma_i) \boldsymbol{u}_i \boldsymbol{u}_i^\top + 2\sigma_k \boldsymbol{u}_k \boldsymbol{u}_k^\top
\end{aligned}$$

Categorization of critical points

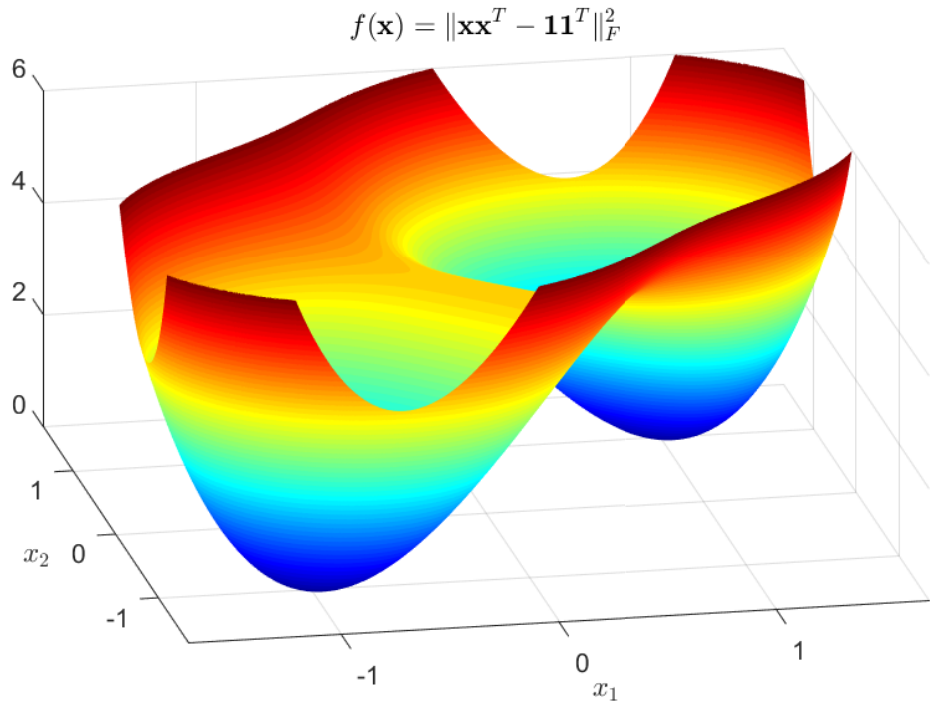
Critical points can be further categorized based on **Hessians**:

$$\nabla^2 f(\boldsymbol{x}) := 2\boldsymbol{x}\boldsymbol{x}^\top + \|\boldsymbol{x}\|_2^2 \boldsymbol{I} - \boldsymbol{M}$$

- If $\sigma_1 > \sigma_2 \geq \ldots \geq \sigma_n \geq 0$, then
  - $k = 1$: $\nabla^2 f(\boldsymbol{x}_1) \succ \boldsymbol{0} \quad \rightarrow$ local minima
  - $1 < k \leq n$: $\lambda_{\min}(\nabla^2 f(\boldsymbol{x}_k)) < 0$, $\lambda_{\max}(\nabla^2 f(\boldsymbol{x}_k)) > 0$ $\rightarrow$ strict saddle
  - $\boldsymbol{x} = \boldsymbol{0}$: $\nabla^2 f(\boldsymbol{0}) \preceq \boldsymbol{0} \quad \rightarrow$ local maxima (or strict saddle)

Good news: benign landscape

For example, for 2-dimensional case $f(\boldsymbol{x}) = \left\| \boldsymbol{x}\boldsymbol{x}^\top - \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\|_{\mathrm{F}}^2$



global minima $\boldsymbol{x} = \pm \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ & strict saddle $\boldsymbol{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and $\pm \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

— No "spurious" local minima!

Key messages from landscape analysis

$$f(\boldsymbol{X}) := \frac{1}{4}\|\boldsymbol{X}\boldsymbol{X}^\top - \boldsymbol{M}\|_{\mathrm{F}}^2, \qquad \boldsymbol{X} \in \mathbb{R}^{n\times r}$$

If $\sigma_r > \sigma_{r+1}$:

- **all local minima are global:** $\boldsymbol{X}$ contains top-r eigenvectors (up to orthonormal transformation)
- **strict saddle points:** all stationary points are saddle points except global optimum

Low-rank recovery with few measurements

Consider linear measurements:

$$\boldsymbol{y} = \mathcal{A}(\boldsymbol{M}), \quad \boldsymbol{y} \in \mathbb{R}^m, \quad m \ll n^2$$

where $\boldsymbol{M} \in \mathbb{R}^{n \times n}$ is rank-r ($r \ll n$) and PSD (for simplicity).

- Consider least-squares loss function:

$$f(\boldsymbol{X}) := \frac{1}{4}\|\mathcal{A}(\boldsymbol{X}\boldsymbol{X}^\top - \boldsymbol{M})\|_{\mathrm{F}}^2$$

- If $\mathcal{A}$ is isotropic (i.e. $\mathbb{E}[\mathcal{A}^*\mathcal{A}] = \mathcal{I}$), then

$$\mathbb{E}[f(\boldsymbol{X})] = \frac{1}{4}\|\boldsymbol{X}\boldsymbol{X}^\top - \boldsymbol{M}\|_{\mathrm{F}}^2$$

- Does $f(\boldsymbol{X})$ inherit benign landscape?

Landscape preserving under RIP

Definition 11

Rank-r restricted isometry constants δ_r is smallest quantity obeying

$$(1-\delta_r)\|\boldsymbol{M}\|_{\mathsf{F}}^2 \leq \|\mathcal{A}(\boldsymbol{M})\|_{\mathsf{F}}^2 \leq (1+\delta_r)\|\boldsymbol{M}\|_{\mathsf{F}}^2,\ \forall \boldsymbol{M} : \mathsf{rank}(\boldsymbol{M}) \leq r$$

Landscape preserving under RIP

Definition 11

Rank-r restricted isometry constants δ_r is smallest quantity obeying

$$(1-\delta_r)\|\boldsymbol{M}\|_{\mathsf{F}}^2 \leq \|\mathcal{A}(\boldsymbol{M})\|_{\mathsf{F}}^2 \leq (1+\delta_r)\|\boldsymbol{M}\|_{\mathsf{F}}^2,\ \forall \boldsymbol{M} : \mathsf{rank}(\boldsymbol{M}) \leq r$$

Key message: benign landscape is preserved when $\mathcal{A}$ satisfies RIP
e.g., when $\mathcal{A}$ follows the Gaussian design

Landscape preserving under RIP

Definition 11

Rank-r restricted isometry constants δ_r is smallest quantity obeying

$$(1-\delta_r)\|\boldsymbol{M}\|_{\mathsf{F}}^2 \le \|\mathcal{A}(\boldsymbol{M})\|_{\mathsf{F}}^2 \le (1+\delta_r)\|\boldsymbol{M}\|_{\mathsf{F}}^2,\ \forall \boldsymbol{M} : \mathsf{rank}(\boldsymbol{M}) \le r$$

Theorem 12 (Bhojanapalli et al. '16, Ge et al. '17)

If $\mathcal{A}$ satisfies RIP with $\delta_{2r} < \frac{1}{10}$, then

- *all local min are global: any local minimum $\boldsymbol{X}$ of $f(\cdot)$ satisfies $\boldsymbol{X}\boldsymbol{X}^\top = \boldsymbol{M}$*
- *strict saddle points: any non-local min critical point $\boldsymbol{X}$ of $f(\cdot)$ satisfies $\lambda_{\min}[\nabla^2 f(\boldsymbol{X})] \le -\frac{2}{5}\sigma_r$*

Landscape without RIP

Matrix completion:

Complete $\boldsymbol{M}$ from partial entries $M_{i,j},\ (i,j) \in \Omega$

where (i,j) is included in Ω independently with prob. p

find low-rank $\widehat{\boldsymbol{M}}$ s.t. $\mathcal{P}_\Omega(\widehat{\boldsymbol{M}}) = \mathcal{P}_\Omega(\boldsymbol{M})$

In matrix completion, RIP does not hold

$\rightarrow$ need to regularize loss function by promoting **incoherent** solutions

Incoherence for matrix completion

Definition 13 (Incoherence for matrix completion)

A rank-r matrix $\boldsymbol{M}$ with eigendecomposition $\boldsymbol{M} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{U}^\top$ is said to be μ-incoherent if

$$\|\boldsymbol{U}\|_{2,\infty} \leq \sqrt{\frac{\mu}{n}}\|\boldsymbol{U}\|_{\mathrm{F}} = \sqrt{\frac{\mu r}{n}}.$$

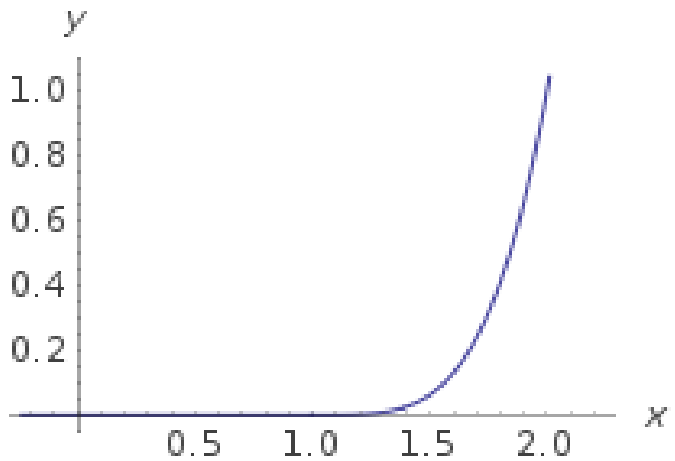
e.g. $\underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}}_{\text{hard } \mu=n}$ vs. $\underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & & \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix}}_{\text{easy } \mu=1}$

Regularization

One possible regularizer:

$$Q(\boldsymbol{X}) = \sum_{i=1}^{n} (\underbrace{\|\boldsymbol{e}_i^\top \boldsymbol{X}\|_2}_{\text{row norm}} - \alpha)_+^4 := \sum_{i=1}^{n} Q_i(\|\boldsymbol{e}_i^\top \boldsymbol{X}\|_2)$$

where α is regularization parameter, and $z_+ = \max\{z, 0\}$



(x from −0.2 to 2.2)

MC has no spurious local minima under proper regularization

Consider *regularized* loss function

$$f_{\text{reg}}(\boldsymbol{X}) = \frac{1}{p}\|\mathcal{P}_{\Omega}(\boldsymbol{X}\boldsymbol{X}^{\top} - \boldsymbol{M})\|_{\text{F}}^2 + \underbrace{\lambda Q(\boldsymbol{X})}_{\text{promote incoherence}}$$

where λ: regularization parameter

Theorem 14 (Ge et al, 2016)

If sample size $n^2 p \gtrsim \mu^4 n r^6 \log n$ and if α and λ are chosen properly, then with high prob.,

- *all local min are global: any local minimum $\boldsymbol{X}$ of $f_{\text{reg}}(\cdot)$ satisfies $\boldsymbol{X}\boldsymbol{X}^{\top} = \boldsymbol{M}$*
- *saddle points that are not local minima are strict saddles*

Initialization-free theory

Implications:

- Under benign landscape, local search algorithms that can find local minima are often sufficient, *regardless of initialization*
- Key algorithm issue: how to escape saddle points

Saddle-point escaping algorithms

- *Vanilla GD with random initialization*: converges to global minimizers almost surely, but no rates are known (Lee et al. '16)
- *Second-order algorithms (Hessian-based)*: trust-region methods, ... (Sun et al. '16)
- *First-order algorithms*: (perturbed) gradient descent, stochastic gradient descent, ... (Jin et al. '17)

Open problem: does MC converge fast with random initialization?

Gradient descent for matrix completion

Let $\boldsymbol{M} = \boldsymbol{X}^\star \boldsymbol{X}^{\star\top}$. Observe

$$Y_{i,j} = M_{i,j} + E_{i,j}, \quad (i,j) \in \Omega$$

where $\mathbb{P}\left((i,j) \in \Omega\right) = p$ and $E_{i,j} \sim \mathcal{N}(0, \sigma^2)$[1]

$$\text{minimize} \left\| \mathcal{P}_\Omega(\widehat{\boldsymbol{M}} - \boldsymbol{Y}) \right\|_{\mathrm{F}}^2 \quad \text{s.t.} \quad \text{rank}(\widehat{\boldsymbol{M}}) \le r$$

[1] can be relaxed to sub-Gaussian noise and asymmetric case

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$$\text{minimize}_{\boldsymbol{X} \in \mathbb{R}^{n \times r}} \quad \underbrace{f(\boldsymbol{X}) = \sum_{(j,k) \in \Omega} \left(\boldsymbol{e}_j^\top \boldsymbol{X} \boldsymbol{X}^\top \boldsymbol{e}_k - Y_{j,k} \right)^2}_{\text{unregularized least-squares loss}}$$

[1] can be relaxed to sub-Gaussian noise and asymmetric case

Gradient descent for matrix completion

(1) **Spectral initialization**: let $\boldsymbol{U}^0\boldsymbol{\Sigma}^0\boldsymbol{U}^{0\top}$ be rank-r eigendecomposition of

$$\frac{1}{p}\mathcal{P}_\Omega(\boldsymbol{Y}).$$

and set $\boldsymbol{X}^0 = \boldsymbol{U}^0\left(\boldsymbol{\Sigma}^0\right)^{1/2}$

(2) **Gradient descent updates:**

$$\boldsymbol{X}^{t+1} = \boldsymbol{X}^t - \eta_t \nabla f\left(\boldsymbol{X}^t\right), \qquad t = 0, 1, \cdots$$

Gradient descent for matrix completion

Define optimal transform from the tth iterate $\boldsymbol{X}^t$ to $\boldsymbol{X}^\star$ as

$$\boldsymbol{Q}^t := \mathrm{argmin}_{\boldsymbol{R}\in\mathcal{O}^{r\times r}} \left\|\boldsymbol{X}^t\boldsymbol{R} - \boldsymbol{X}^\star\right\|_{\mathrm{F}}$$

Theorem 15 (Noiseless MC, Ma, Wang, Chi, Chen '17)

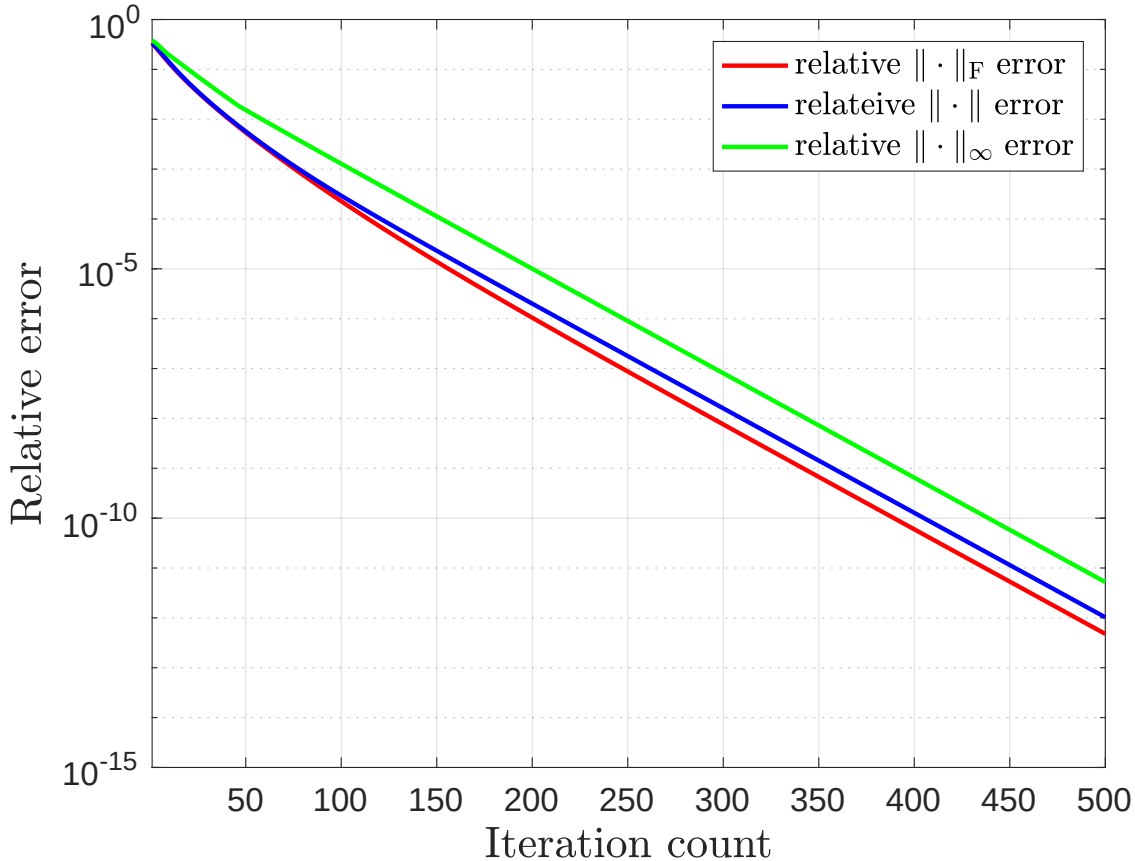
Suppose $\boldsymbol{M} = \boldsymbol{X}^\star\boldsymbol{X}^{\star\top}$ is rank-r, incoherent and well-conditioned. Vanilla GD (with spectral initialization) achieves

- $\left\|\boldsymbol{X}^t\boldsymbol{Q}^t - \boldsymbol{X}^\star\right\|_{\mathrm{F}} \lesssim \rho^t \mu r \frac{1}{\sqrt{np}} \|\boldsymbol{X}^\star\|_{\mathrm{F}}$,
- $\left\|\boldsymbol{X}^t\boldsymbol{Q}^t - \boldsymbol{X}^\star\right\| \lesssim \rho^t \mu r \frac{1}{\sqrt{np}} \|\boldsymbol{X}^\star\|$, *(spectral)*
- $\left\|\boldsymbol{X}^t\boldsymbol{Q}^t - \boldsymbol{X}^\star\right\|_{2,\infty} \lesssim \rho^t \mu r \sqrt{\frac{\log n}{np}} \|\boldsymbol{X}^\star\|_{2,\infty}$, *(incoherence)*

where $0 < \rho < 1$, if step size $\eta \asymp 1/\sigma_{max}$ and sample complexity $n^2 p \gtrsim \mu^3 n r^3 \log^3 n$

- vanilla gradient descent converges linearly for matrix completion!

Numerical evidence for noiseless data



Relative error of $\boldsymbol{X}^t \boldsymbol{X}^{t\top}$ (measured by $\|\cdot\|_{\mathrm{F}}, \|\cdot\|, \|\cdot\|_{\infty}$) vs. iteration count for MC, where $n = 1000$, $r = 10$, $p = 0.1$, and $\eta_t = 0.2$

Noisy matrix completion

Theorem 16 (Noisy MC, Ma, Wang, Chi, Chen '17)

Under sample complexity of Theorem 15, if noise satisfies $\sigma\sqrt{\frac{n}{p}} \ll \frac{\sigma_{\min}}{\sqrt{\kappa^3 \mu r \log^3 n}}$, *then GD iterates satisfy*

$$\left\|\boldsymbol{X}^t\boldsymbol{Q}^t - \boldsymbol{X}^\star\right\|_{\mathrm{F}} \lesssim \left(\rho^t \mu r \frac{1}{\sqrt{np}} + \frac{\sigma}{\sigma_{\min}}\sqrt{\frac{n}{p}}\right)\left\|\boldsymbol{X}^\star\right\|_{\mathrm{F}},$$

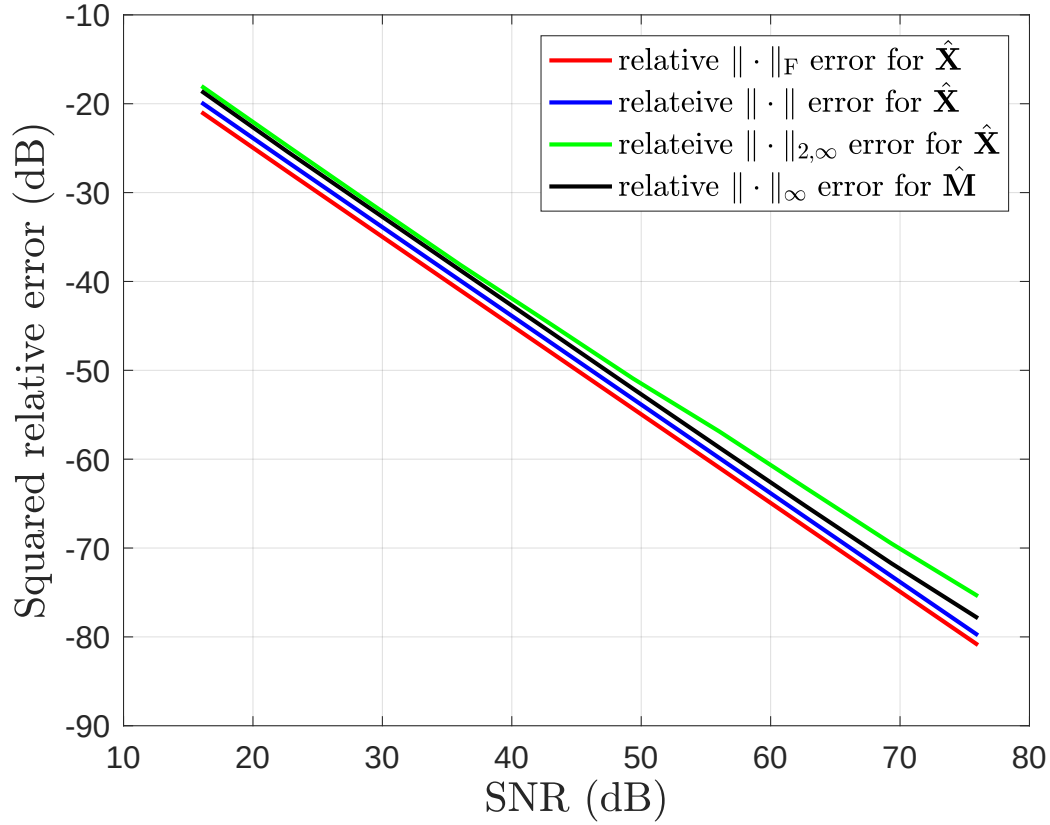
$$\left\|\boldsymbol{X}^t\boldsymbol{Q}^t - \boldsymbol{X}^\star\right\|_{2,\infty} \lesssim \left(\rho^t \mu r \sqrt{\frac{\log n}{np}} + \frac{\sigma}{\sigma_{\min}}\sqrt{\frac{n\log n}{p}}\right)\left\|\boldsymbol{X}^\star\right\|_{2,\infty},$$

$$\left\|\boldsymbol{X}^t\boldsymbol{Q}^t - \boldsymbol{X}^\star\right\| \lesssim \left(\rho^t \mu r \frac{1}{\sqrt{np}} + \frac{\sigma}{\sigma_{\min}}\sqrt{\frac{n}{p}}\right)\left\|\boldsymbol{X}^\star\right\|$$

- *minimax entrywise error control in* $\left\|\boldsymbol{X}^t\boldsymbol{X}^{t\top} - \boldsymbol{X}^\star\boldsymbol{X}^{\star\top}\right\|_\infty$

Numerical evidence for noisy data

Set SNR $:= \frac{\|\boldsymbol{M}\|_{\mathrm{F}}^2}{n^2\sigma^2}$



Squared relative error of the estimate $\widehat{\boldsymbol{X}}$ (measured by $\|\cdot\|_{\mathrm{F}}, \|\cdot\|, \|\cdot\|_{2,\infty}$) and $\widehat{\boldsymbol{M}} = \widehat{\boldsymbol{X}}\widehat{\boldsymbol{X}}^\top$ (measured by $\|\cdot\|_{\infty}$) vs. SNR, where $n = 500$, $r = 10$, $p = 0.1$, and $\eta_t = 0.2$

Related theory

$$\text{minimize}_{\boldsymbol{X} \in \mathbb{R}^{n \times r}} \quad f(\boldsymbol{X}) = \sum_{(j,k) \in \Omega} \left(\boldsymbol{e}_j^\top \boldsymbol{X} \boldsymbol{X}^\top \boldsymbol{e}_k - Y_{j,k} \right)^2$$

Related theory promotes incoherence explicitly:

Related theory

$$\text{minimize}_{\boldsymbol{X} \in \mathbb{R}^{n \times r}} \quad f(\boldsymbol{X}) = \sum_{(j,k) \in \Omega} \left(\boldsymbol{e}_j^\top \boldsymbol{X} \boldsymbol{X}^\top \boldsymbol{e}_k - Y_{j,k} \right)^2$$

Related theory promotes incoherence explicitly:

- regularized loss (solve $\min_{\boldsymbol{X}} \; f(\boldsymbol{X}) + Q(\boldsymbol{X})$ instead)
  - e.g. Keshavan, Montanari, Oh '10, Sun, Luo '14, Ge, Lee, Ma '16

Related theory

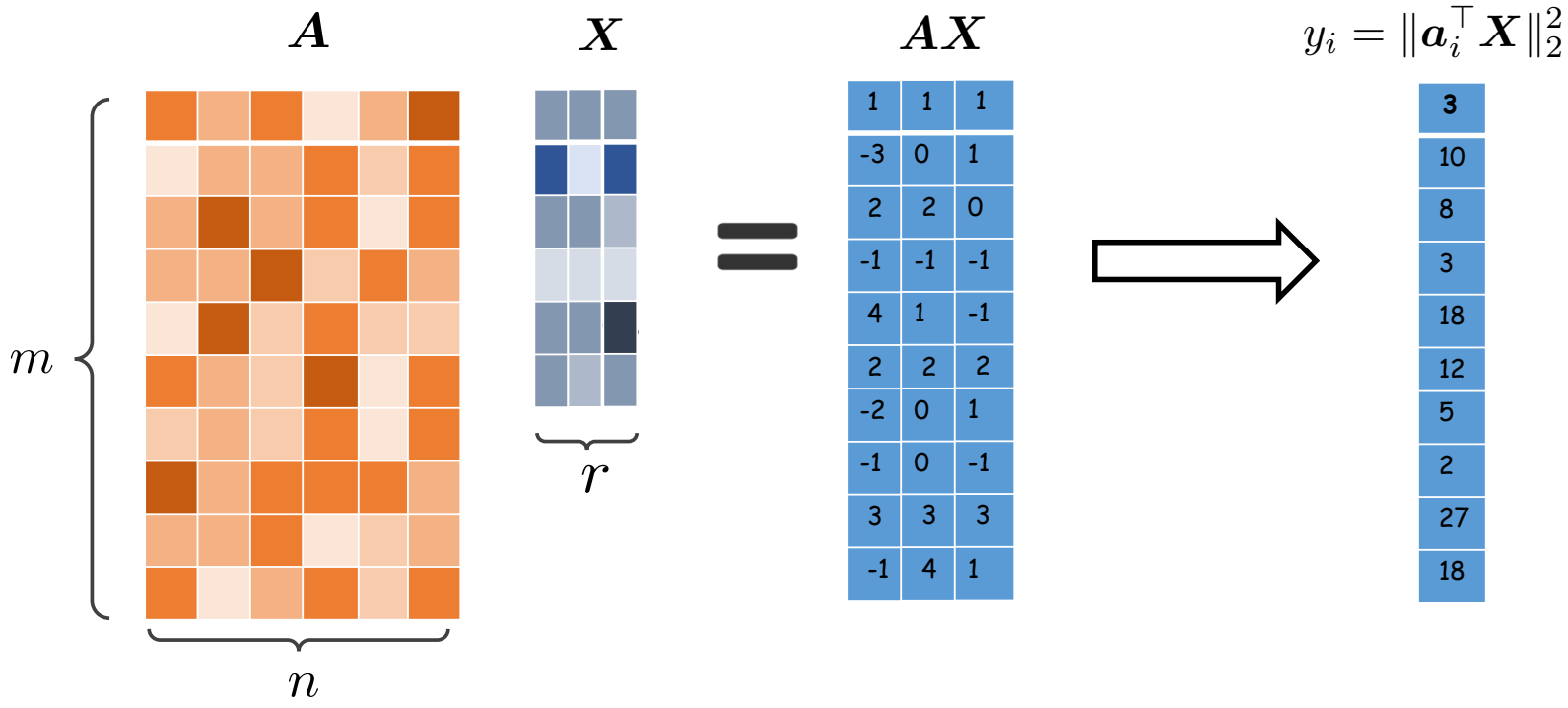
$$\text{minimize}_{\boldsymbol{X} \in \mathbb{R}^{n \times r}} \quad f(\boldsymbol{X}) = \sum_{(j,k) \in \Omega} \left(\boldsymbol{e}_j^\top \boldsymbol{X} \boldsymbol{X}^\top \boldsymbol{e}_k - Y_{j,k} \right)^2$$

Related theory promotes incoherence explicitly:

- regularized loss (solve $\min_{\boldsymbol{X}} \; f(\boldsymbol{X}) + Q(\boldsymbol{X})$ instead)
  - e.g. Keshavan, Montanari, Oh '10, Sun, Luo '14, Ge, Lee, Ma '16
- projection onto set of incoherent matrices
  - e.g. Chen, Wainwright '15, Zheng, Lafferty '16

$$\boldsymbol{X}^{t+1} = \mathcal{P}_{\mathcal{C}} \left(\boldsymbol{X}^t - \eta_t \nabla f\left(\boldsymbol{X}^t\right) \right), \qquad t = 0, 1, \cdots$$

Quadratic sampling



Recover $\boldsymbol{X}^\star \in \mathbb{R}^{n \times r}$ from m random quadratic measurements

$$y_i \quad = \quad \|\boldsymbol{a}_i^\top \boldsymbol{X}^\star\|_2^2, \qquad i = 1, \ldots, m$$

Applications: quantum state tomography, covariance sketching, ...

Gradient descent with spectral initialization

$$\text{minimize}_{\boldsymbol{X} \in \mathbb{R}^{n \times r}} \quad f(\boldsymbol{X}) = \frac{1}{4m} \sum_{k=1}^{m} \left(\left\| \boldsymbol{a}_k^\top \boldsymbol{X} \right\|_2^2 - y_k \right)^2$$

Gradient descent with spectral initialization

$$\text{minimize}_{\boldsymbol{X}\in\mathbb{R}^{n\times r}} \quad f(\boldsymbol{X}) = \frac{1}{4m}\sum_{k=1}^{m}\left(\|\boldsymbol{a}_k^\top \boldsymbol{X}\|_2^2 - y_k\right)^2$$

Theorem 17 (Quadratic sampling)

Under i.i.d. Gaussian designs $\boldsymbol{a}_i \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\boldsymbol{0}, \boldsymbol{I})$*, GD (with spectral initialization) achieves*

- $\max_l \left\|\boldsymbol{a}_l^\top(\boldsymbol{X}^t\boldsymbol{Q}^t - \boldsymbol{X}^\star)\right\|_2 \lesssim \sqrt{\log n}\,\frac{\sigma_r^2(\boldsymbol{X}^\star)}{\|\boldsymbol{X}^\star\|_{\mathrm{F}}}$ *(incoherence)*
- $\|\boldsymbol{X}^t\boldsymbol{Q}^t - \boldsymbol{X}^\star\|_{\mathrm{F}} \lesssim \left(1 - \frac{\sigma_r^2(\boldsymbol{X}^\star)\eta}{2}\right)^t \|\boldsymbol{X}^\star\|_{\mathrm{F}}$ *(linear convergence)*

provided that $\eta \asymp \frac{1}{(\log n \vee r)^2\sigma_r^2(\boldsymbol{X}^\star)}$ *and* $m \gtrsim nr^4\log n$

Demixing sparse and low-rank matrices

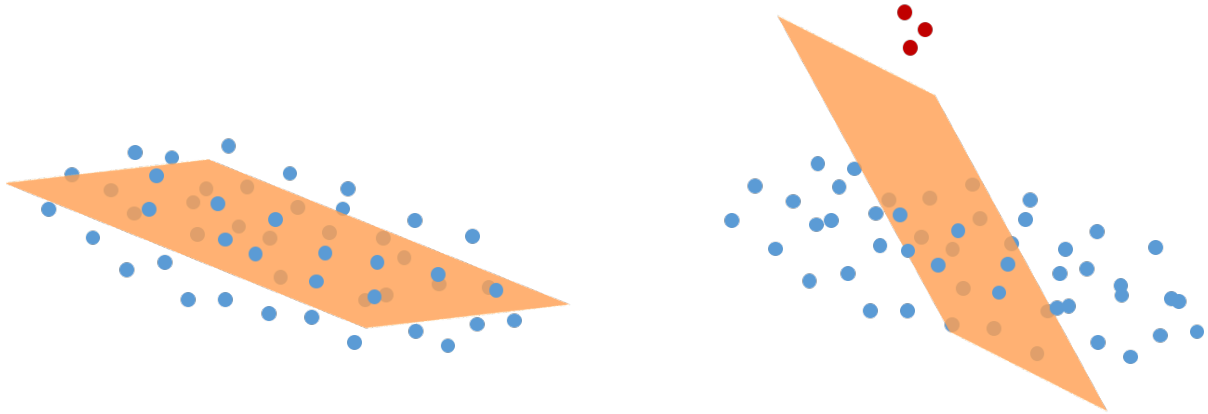
Suppose we are given a matrix

$$\boldsymbol{M} = \underbrace{\boldsymbol{L}}_{\text{low-rank}} + \underbrace{\boldsymbol{S}}_{\text{sparse}} \in \mathbb{R}^{n \times n}$$

Question: can we hope to recover both $\boldsymbol{L}$ and $\boldsymbol{S}$ from $\boldsymbol{M}$?

Applications

- Robust PCA



- Video surveillance: separation of background and foreground

Nonconvex approach

- $\text{rank}(\boldsymbol{L}) \leq r$; if we write the SVD of $\boldsymbol{L} = \boldsymbol{U}\boldsymbol{\Sigma}\boldsymbol{V}^\top$, set

$$\boldsymbol{X}^\star = \boldsymbol{U}_L \boldsymbol{\Sigma}^{1/2}; \quad \boldsymbol{Y}^\star = \boldsymbol{V}\boldsymbol{\Sigma}^{1/2}$$

- non-zero entries of $\boldsymbol{S}$ are "spread out" (no more than s fraction of non-zeros per row/column), but otherwise arbitrary

$$\mathcal{S}_s = \{\boldsymbol{S} \in \mathbb{R}^{n\times n} : \quad \|\boldsymbol{S}_{i,:}\|_0 \leq s \cdot n;\ \|\boldsymbol{S}_{:,j}\|_0 \leq s \cdot n\}$$

$$\underset{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{S}\in\mathcal{S}_s}{\text{minimize}}\ F(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{S}) := \underbrace{\|\boldsymbol{M} - \boldsymbol{X}\boldsymbol{Y}^\top - \boldsymbol{S}\|_{\mathrm{F}}^2}_{\text{least-squares loss}} + \underbrace{\frac{1}{4}\|\boldsymbol{X}^\top\boldsymbol{X} - \boldsymbol{Y}^\top\boldsymbol{Y}\|_{\mathrm{F}}^2}_{\text{fix scaling ambiguity}}$$

where $\boldsymbol{X}, \boldsymbol{Y} \in \mathbb{R}^{n\times r}$.

Gradient descent and hard thresholding

$$\text{minimize}_{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{S}\in\mathcal{S}_s} \quad F(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{S})$$

- **Spectral initialization**: Set $\boldsymbol{S}^0 = \mathcal{H}_{\gamma s}(\boldsymbol{M})$. Let $\boldsymbol{U}^0\boldsymbol{\Sigma}^0\boldsymbol{V}^{0\top}$ be rank-r SVD of $\boldsymbol{M}^0 := \mathcal{P}_\Omega(\boldsymbol{M}-\boldsymbol{S}^0)$; set $\boldsymbol{X}^0 = \boldsymbol{U}^0\left(\boldsymbol{\Sigma}^0\right)^{1/2}$ and $\boldsymbol{Y}^0 = \boldsymbol{V}^0\left(\boldsymbol{\Sigma}^0\right)^{1/2}$

Gradient descent and hard thresholding

$$\text{minimize}_{\boldsymbol{X},\boldsymbol{Y},\boldsymbol{S}\in\mathcal{S}_s} \quad F(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{S})$$

- **Spectral initialization**: Set $\boldsymbol{S}^0 = \mathcal{H}_{\gamma s}(\boldsymbol{M})$. Let $\boldsymbol{U}^0\boldsymbol{\Sigma}^0\boldsymbol{V}^{0\top}$ be rank-r SVD of $\boldsymbol{M}^0 := \mathcal{P}_\Omega(\boldsymbol{M}-\boldsymbol{S}^0)$; set $\boldsymbol{X}^0 = \boldsymbol{U}^0\left(\boldsymbol{\Sigma}^0\right)^{1/2}$ and $\boldsymbol{Y}^0 = \boldsymbol{V}^0\left(\boldsymbol{\Sigma}^0\right)^{1/2}$
- **for** $t = 0, 1, 2, \cdots$
  - **Hard thresholding**: $\boldsymbol{S}^{t+1} = \mathcal{H}_{\gamma s}(\boldsymbol{M} - \boldsymbol{X}^t\boldsymbol{Y}^{t\top})$
  - **Gradient updates**:

$$\boldsymbol{X}^{t+1} = \boldsymbol{X}^t - \eta\nabla_{\boldsymbol{X}}F\left(\boldsymbol{X}^t,\boldsymbol{Y}^t,\boldsymbol{S}^{t+1}\right)$$
$$\boldsymbol{Y}^{t+1} = \boldsymbol{Y}^t - \eta\nabla_{\boldsymbol{Y}}F\left(\boldsymbol{X}^t,\boldsymbol{Y}^t,\boldsymbol{S}^{t+1}\right)$$

Efficient nonconvex recovery

Theorem 18 (Nonconvex RPCA, Yi et al. '16)

Set $\gamma = 2$ *and* $\eta = 1/(36\sigma_{\max})$. *Suppose that*

$$s \lesssim \min\left\{\frac{1}{\mu\sqrt{\kappa r^3}}, \frac{1}{\mu\kappa^2 r}\right\}$$

Then GD+HT satisfies

$$\|\boldsymbol{X}^t\boldsymbol{Y}^{t\top} - \boldsymbol{L}\|_{\mathrm{F}}^2 \lesssim \left(1 - \frac{1}{288\kappa}\right)^t \mu^2\kappa r^3 s^2 \sigma_{\max}$$

- $O(\kappa \log \frac{1}{\varepsilon})$ iterations to reach ε accuracy
- for adversarial outliers, optimal fraction is $s = O(1/\mu r)$; Theorem 18 is suboptimal by a factor of $\sqrt{r}$
- extendable to partial observation models

Outline

- Part I: Overview
- Part II: Phase retrieval: a case study
  - Spectral initialization
  - Local refinement: algorithm and analysis
- Part III: Low-rank matrix estimation
- Part IV: Closing remarks

A growing list of "benign" nonconvex problems

- blind deconvolution / self-calibration
- dictionary learning
- tensor decomposition
- robust PCA
- mixed linear regression
- Gaussian mixture models
- etc...

Topics we did not cover

- **other algorithms:** alternating minimization, stochastic gradient descent, mirror descent, singular value projection, etc...
- **additional structures:** e.g. sparsity, piece-wise smoothness
- **saddle-point escaping algorithms**

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Thanks!