Sparse MIMO Radar via Structured Matrix Completion

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***Abstract*—We explore sub-Nyquist sampling strategies in a bistatic MIMO radar with M transmit and N receive antennas to reconstruct the sparse scatter scene with $K \ll MN$ targets. We develop a front-end with a matched filter bank at each receive antenna and sample the branch output at random with a total of L samples per pulse. Sparse recovery is then obtained via enhanced matrix completion techniques that make no grid assumptions over the target scene. We demonstrate that as long as L is on the order of $O(K\log^2(MN))$, it is possible to recover the target scene under a mild condition with high probability, thus greatly reducing the sampling complexity from the Nyquist rate MN samples per pulse. The performance is numerically examined with comparison against compressive sensing approaches. The framework can also be explored to reduce the size of filter banks at the front-end.**

I. Introduction

The challenge of radar imaging is to invert the locations of targets unambiguously with high resolution from a limited number of samples of the scatter scene. Multiple-Input Multiple-Output (MIMO) radar systems [1], unlike phased array radars, enable simultaneous transmission of independent waveforms across multiple antennas and potentially can achieve better spatial resolution.

In this paper, we consider estimating the direction-of-arrival (DOA) and the direction-of-departure (DOD) of each target by processing a single pulse in a bistatic MIMO radar with M transmit and N receive antennas in a nondispersive propagation environment. The resolution depends on the degree of freedom of the MIMO system, given as MN if the antennas are arranged as a unitary linear array (ULA) with half-wavelength equal spacing [2]. Conventional processing based on matched filtering requires the number of samples, or the size of the matched filter bank, to scale linearly with MN, thus posing great challenge on energy consumption for sampling as well as front-end hardware costs.

Pioneered by the work of Herman and Strohmer [3], Compressive Sensing (CS) becomes an appealing technique for radars to reduce sampling complexity or alternatively to improve resolution by exploiting the sparsity of the scatter scene. In particular, [4]–[6] explored random antenna arrays to achieve the same high resolution of a fully implemented ULA with a smaller array size in MIMO radars. In particular, [6] showed that the product of the number of transmit and receive antennas only needs to be on the order of $O(K\log^2(MN))$, where K is the number of targets. In this framework, random antenna array is necessary to obtain the desired isometry properties of the sensing matrix. However, one caveat in most existing literature is that the targets are approximated to lie on a discretized grid, which is not satisfied by the physics of scattering and may degenerate the performance severely [7].

In this paper, we assume a ULA is implemented at both the transmitter and the receiver, however, we look to complexity reduction by reducing the required number of samples per pulse. We first develop a sampling front-end consisting of a matched filter bank with respect to transmit waveforms at each receive antenna, and then sample the output uniformly at random at the pulse rate. Sparse recovery is performed via convex optimization based on Enhanced Matrix Completion (EMaC) [8], [9], which is recently proposed to recover multi-dimensional frequency models by nuclear norm minimization of a low-rank enhanced matrix pencil constructed from the data. Under mild coherence conditions, the EMaC algorithm succeeds with high probability with a random sample size of $O(K\log^2(MN))$, which is much smaller than the Nyquist rate MN samples per pulse, and comparable to the complexity of random array schemes [6]. Finally, numerical examples are provided with comparison against CS approaches.

The rest of the paper is organized as follows. Section II presents the MIMO radar signal model, and proposes a novel front-end with sub-Nyquist sampling of the output of the matched filter bank. Section III discusses recovery strategies using EMaC, along with its performance guarantee. Section IV presents numerical simulations and we conclude in Section V.

II. MIMO Radar Model and Front-End

We consider a bistatic MIMO radar model with M transmit antennas and N receive antennas, where the spacing between antennas is $\lambda/2$, λ being the wavelength of the carrier signal. Let T be the pulse repetition interval (PRI). Let $s_m(t)$ be the narrow-band pulse waveform transmitted from the mth transmit antenna, $1 \leq m \leq M$, and $s_m(t)$'s are orthogonal across different antennas, i.e.

$$\langle s_m(t), s_{m'}(t)\rangle = \int_0^T s_m(t)s_{m'}(t)dt = \delta_{m,m'}. \qquad (1)$$

Assume there are K targets, where θ_k is the DOD, ϕ_k is the DOA, β_k is the fading coefficient, respectively, of the kth target, $1 \leq k \leq K$. In a single pulse model, assume there is no clutter and all synchronizations are perfect, we can write the received signal at the nth receive antenna as [2]

$$r_n(t) = \sum_{k=1}^{K}\sum_{m=1}^{M} \beta_k e^{j\pi(n-1)\sin(\phi_k)} e^{j\pi(m-1)\sin(\theta_k)} s_m(t) + w_n(t), \qquad (2)$$

where $w_n(t)$ is an additive white Gaussian noise with variance σ^2, i.e. $\mathbb{E}[w_n(t)w_n(t')] = \sigma^2\delta_{t,t'}$.

A. Nyquist-Rate Sampling Front-end

At the $n$the receive antenna, if we match filter the received signal $r_n(t)$ with the transmitted waveform $s_m(t)$ from the mth antenna, and sample the output at the pulse rate $1/T$, the sampled output can be written as

$$\begin{aligned} y_{m,n} &= \langle r_n(t), s_m(t)\rangle \\ &= \sum_{k=1}^{K}\sum_{m'=1}^{M} \beta_k e^{j\pi(n-1)\sin(\phi_k)} e^{j\pi(m'-1)\sin(\theta_k)} \cdot \\ &\qquad \langle s_{m'}(t), s_m(t)\rangle + \langle w_n(t), s_m(t)\rangle \\ &= \sum_{k=1}^{K} \beta_k e^{j\pi(n-1)\sin(\phi_k)} e^{j\pi(m-1)\sin(\theta_k)} + z_{m,n}, \end{aligned} \quad (3)$$

where $z_{m,n} = \langle w_n(t), s_m(t)\rangle$, and (3) follows from (1). Write all samples in a matrix form we obtain

$$\mathbf{Y} = \mathbf{B}\boldsymbol{\Sigma}\mathbf{A}^T + \mathbf{Z} \in \mathbb{C}^{M\times N}, \quad (4)$$

where $\mathbf{Y} = [y_{m,n}]$, $\mathbf{Z} = [z_{m,n}]$, and $\boldsymbol{\Sigma} = \mathrm{diag}[\beta_1, \ldots, \beta_K]$. The matrices $\mathbf{A} = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_K)] \in \mathbb{C}^{M\times K}$ and $\mathbf{B} = [\mathbf{b}(\phi_1), \ldots, \mathbf{b}(\phi_K)] \in \mathbb{C}^{N\times K}$ are the transmit steering matrix and the receive steering matrix, respectively, where $\mathbf{a}(\theta_i)$ is given as

$$\mathbf{a}(\theta_i) = [1, e^{j\pi\sin(\theta_i)}, \ldots, e^{j\pi(M-1)\sin(\theta_i)}], \quad (5)$$

and $\mathbf{b}(\phi_i)$ is given as

$$\mathbf{b}(\phi_i) = [1, e^{j\pi\sin(\phi_i)}, \ldots, e^{j\pi(N-1)\sin(\phi_i)}]. \quad (6)$$

We can confirm $\mathbf{Z}$ is additive Gaussian with i.i.d. entries by

$$\begin{aligned} &\mathbb{E}[z_{n,m} z_{n',m'}] = \mathbb{E}[\langle w_n(t), s_m(t)\rangle \langle w_{n'}(t), s_{m'}(t)\rangle] \\ &= \mathbb{E}\int_0^T\int_0^T s_m(t_1)s_{m'}(t_2)w_n(t_1)w_{n'}(t_2)dt_1dt_2 \\ &= \int_0^T\int_0^T s_m(t_1)s_{m'}(t_2)\delta_{n,n'}\delta_{t_1,t_2}\sigma^2 dt_1dt_2 \\ &= \sigma^2\delta_{n,n'}\delta_{m,m'}. \end{aligned}$$

It is then possible to recover the target parameters from $\mathbf{Y}$ based on (4) using conventional spectrum estimation methods such as ESPRIT [10]. As the product MN gets large, our goal is to reduce the sampling complexity by only observing a small set of entries of $\mathbf{Y}$.

B. Sub-Nyquist Rate Sampling Front-End

At each receive antenna, we sample the output of each branch of the matched filter bank at the pulse rate $1/T$ uniformly at random, as in Figure 1, where $\delta_{m,n} = 1$ if $y_{m,n}$ is sampled, and $\delta_{m,n} = 0$ if otherwise. Let $\Omega = \{(m,n)|\delta_{m,n} = 1\}$ denote the set of sampled branches, and $\mathcal{P}_\Omega$ be the orthogonal projection onto the linear space of matrices supported on Ω. Then, the set of samples can be given as $\mathcal{P}_\Omega(\mathbf{Y})$, and the number of samples is given as $L = |\Omega|$.

Our goal is thus to first recover the missing entries of $\mathbf{Y}$ given the observation $\mathcal{P}_\Omega(\mathbf{Y})$. In general this problem is ill-posed, but we will show that with the prior knowledge that the number of targets K is small, the problem becomes tractable as soon as L is on the order of $O(K\log^2(MN))$. Once the whole matrix $\mathbf{Y}$ is recovered, we can assume conventional approaches to estimate the target parameters.

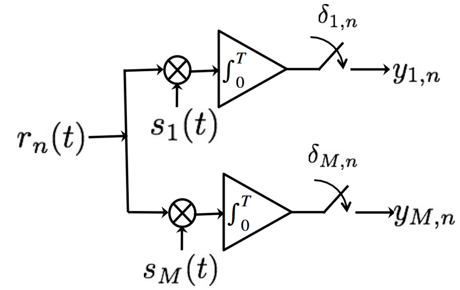

Fig. 1. The proposed sampling front-end with a matched filter bank at each receive antenna.

By writing (4) in a vector we have

$$\mathbf{y} = (\mathbf{A}\otimes\mathbf{B})\mathrm{vec}(\boldsymbol{\Sigma}) + \mathrm{vec}(\mathbf{Z}) = (\mathbf{A}\otimes\mathbf{B})\boldsymbol{\beta} + \mathbf{z}, \quad (7)$$

where $\mathbf{y} = \mathrm{vec}(\mathbf{Y})$, $\mathbf{z} = \mathrm{vec}(\mathbf{Z})$ and $\boldsymbol{\beta} = \mathrm{vec}(\boldsymbol{\Sigma})$. If we approximate $\mathbf{A}$ and $\mathbf{B}$ by a DFT basis or a DFT frame, respectively as $\mathbf{D}_M$ and $\mathbf{D}_N$, then $\mathbf{y}$ can be approximated as a sparse vector in $\mathbf{D}_M\otimes\mathbf{D}_N$, and it is possible to recover $\mathbf{y}$ by Basis Pursuit (BP) [11] or other CS recovery algorithms from $\mathcal{P}_\Omega(\mathbf{Y})$ using (7). However, the grid assumption incurred by assuming a prior basis may not be satisfied in practice and cause severe performance degeneration as discussed in details in [7].

III. Enhanced Matrix Completion

In this paper, we consider the recently proposed Enhanced Matrix Completion (EMaC) to recover the matrix $\mathbf{Y}$ without any grid assumptions. For simplicity we focus on the noise-free scenario. We assume K is much smaller than MN, i.e. $K \ll MN$. However, K can be on the same scale as M or N, hence $\mathbf{Y}$ itself is not low-rank and it is not appropriate to apply naive matrix completion [12] directly on $\mathcal{P}_\Omega(\mathbf{Y})$. However, the harmonic structure of $\mathbf{Y}$ allows us to formulate an enhanced matrix form where the low-rank structure is evident in a matrix of ambient dimension $\Theta(MN)$.

For a given matrix $\boldsymbol{\Phi} = [\phi_{l,k}] \in \mathbb{C}^{M\times N}$, we first define an enhanced form $\mathcal{H}(\boldsymbol{\Phi})$ as a $k_1 \times (M - k_1 + 1) = k_1 \times k_3$ block Hankel matrix:

$$\mathcal{H}(\boldsymbol{\Phi}) = \begin{bmatrix} \boldsymbol{\Phi}_0 & \boldsymbol{\Phi}_1 & \cdots & \boldsymbol{\Phi}_{M-k_1} \\ \boldsymbol{\Phi}_1 & \boldsymbol{\Phi}_2 & \cdots & \boldsymbol{\Phi}_{M-k_1+1} \\ \vdots & \vdots & \vdots & \vdots \\ \boldsymbol{\Phi}_{k_1-1} & \boldsymbol{\Phi}_{k_1} & \cdots & \boldsymbol{\Phi}_{M-1} \end{bmatrix}, \quad (8)$$

where each block is a $k_2 \times (N - k_2 + 1) = k_2 \times k_4$ Hankel matrix defined as

$$\boldsymbol{\Phi}_l = \begin{bmatrix} \phi_{l,0} & \phi_{l,1} & \cdots & \phi_{l,N-k_2} \\ \phi_{l,1} & \phi_{l,2} & \cdots & \phi_{l,N-k_2+1} \\ \vdots & \vdots & \vdots & \vdots \\ \phi_{l,k_2-1} & \phi_{l,k_2} & \cdots & \phi_{l,N-1} \end{bmatrix}.$$

for $0 \leq l \leq M-1$, k_1 and k_2 are called the pencil parameters. If $\mathbf{\Phi} = \mathbf{Y}$, from [9], the rank of $\mathcal{H}(\mathbf{\Phi})$ is always upper bounded as $\text{rank}\left(\mathcal{H}(\mathbf{\Phi})\right) \leq r$. Further, if we choose the pencil parameter as $k_1 = \lceil \frac{M}{2} \rceil$ and $k_2 = \lceil \frac{N}{2} \rceil$, the size of $\mathcal{H}(\mathbf{\Phi})$ is on the scale of $\Theta(MN)$, hence $\mathcal{H}(\mathbf{\Phi})$ is indeed low-rank as hoped. We aim to find the enhanced matrix $\mathcal{H}(\mathbf{\Phi})$ with the smallest rank that satisfies the measurements, i.e.

$$\min_{\mathbf{\Phi} \in \mathbb{C}^{M\times N}} \text{rank}(\mathcal{H}(\mathbf{\Phi})) \quad \text{subject to } \mathcal{P}_\Omega\left(\mathbf{\Phi}\right) = \mathcal{P}_\Omega\left(\mathbf{Y}\right).$$

As proposed in [9], we seek to solve a convex relaxation of the rank minimization problem, called Enhanced Matrix Completion (EMaC):

$$\min_{\mathbf{\Phi} \in \mathbb{C}^{M\times N}} \quad \left\|\mathcal{H}(\mathbf{\Phi})\right\|_* \quad \text{subject to } \mathcal{P}_\Omega\left(\mathbf{\Phi}\right) = \mathcal{P}_\Omega\left(\mathbf{Y}\right).$$

To state the performance guarantee of EMaC, we first need to define a proper notion of coherence. Denote $\Omega_{\mathcal{H}}(k,l)$ as the set of locations in $\mathcal{H}(\mathbf{\Phi})$ containing copies of $\phi_{k,l}$, and $\mathbf{A}_{k,l}$ to denote a *basis* matrix that extracts the average of all entries in $\Omega_{\mathcal{H}}(k,l)$, i.e.

$$\mathbf{A}_{k,l}(\alpha,\beta) = \begin{cases} \frac{1}{\sqrt{|\Omega_{\mathcal{H}}(k,l)|}} & \text{if } (\alpha,\beta) \in \Omega_{\mathcal{H}}(k,l), \\ 0 & \text{otherwise.} \end{cases}$$

We further define $\mathbf{G}_{\text{L}}$ and $\mathbf{G}_{\text{R}}$ as the $K \times K$ Gram matrices as

$$\mathbf{G}_{\text{L}}(i_1,i_2) = \frac{1}{k_1 k_2} \frac{1 - e^{j2\pi k_1 (f_{1i_1} - f_{1i_2})}}{1 - e^{j2\pi (f_{1i_1} - f_{1i_2})}} \frac{1 - e^{j2\pi k_2 (f_{2i_1} - f_{2i_2})}}{1 - e^{j2\pi (f_{2i_1} - f_{2i_2})}},$$

$$\mathbf{G}_{\text{R}}(i_1,i_2) = \frac{1}{k_3 k_4} \frac{1 - e^{j2\pi k_3 (f_{1i_1} - f_{1i_2})}}{1 - e^{j2\pi (f_{1i_1} - f_{1i_2})}} \frac{1 - e^{j2\pi k_4 (f_{2i_1} - f_{2i_2})}}{1 - e^{j2\pi (f_{2i_1} - f_{2i_2})}},$$

with $\boldsymbol{G}_{\text{L}}(i,i) = \boldsymbol{G}_{\text{R}}(i,i) = 1$. The entries of $\boldsymbol{G}_{\text{L}}$ and $\boldsymbol{G}_{\text{R}}$ can be obtained via sampling the two-dimensional Dirichlet kernel. The coherence condition is defined as below.

Definition 1 (**Incoherence**). *Let the SVD of* $\mathcal{H}(\mathbf{Y})$ *be* $\mathcal{H}(\mathbf{Y}) = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^*$*, then the coherence of* $\mathbf{Y}$ *is defined as the smallest quantities such that*

$$\sigma_{\min}\left(\mathbf{G}_{\text{L}}\right) \geq \frac{1}{\mu_1}, \quad \sigma_{\min}\left(\mathbf{G}_{\text{R}}\right) \geq \frac{1}{\mu_1}; \tag{9}$$

$$\max_{(k,l)\in[M]\times[N]} \frac{\left|\langle \mathbf{U}\mathbf{V}^*, \mathbf{A}_{k,l} \rangle\right|^2}{|\Omega_{\mathcal{H}}(k,l)|} \leq \frac{\mu_2 K}{(MN)^2}; \tag{10}$$

and $\forall (k',l') \in [M] \times [N]$*:*

$$\sum_{(k,l)\in[M]\times[N]} \left| \left\langle \mathbf{U}\mathbf{U}^* \mathbf{A}_{k',l'} \mathbf{V}\mathbf{V}^*, \sqrt{\Omega_{\mathcal{H}}(k,l)} \mathbf{A}_{k,l} \right\rangle \right|^2 \leq \frac{\mu_3 K}{MN} |\Omega_{\mathcal{H}}(k',l')|. \tag{11}$$

We denote condition (9) as the *weak incoherence condition*, which only requires that the locations of targets are not too close with respect to the Rayleigh limit, such that the smallest eigenvalue of the Gram matrix is lower bounded. Conditions (10) and (11) further depend on the SVD structure of the form $\mathcal{H}(\mathbf{Y})$, which depends on both the locations and amplitudes of the targets. We denote conditions (9), (10) and (11) as the *strong incoherence condition*. It is expected that these conditions are satisfied with high probability under a large class of practical scenarios [9]. The performance guarantee of EMaC presented in [9] is restated as follows.

Theorem 1. *Let* Ω *the random location set of size* L*. If the measurements are noiseless, then there exists a constant* $c > 0$ *such that*

- *under the strong incoherence condition, if*

$$L > c_1 \max\left(\mu_1, \mu_2, \mu_3\right) K \log^2\left(MN\right); \tag{12}$$

- *under the weak incoherence condition, if*

$$L > c_1 \mu_1^2 K^2 \log^2(MN); \tag{13}$$

then $\mathbf{Y}$ *is the unique solution of EMaC with probability at least* $1 - (MN)^{-2}$.

Theorem 1 indicates that as long as the number of samples per pulse is greater than $O(K\log^2(MN))$, and the locations of targets are not so close, it is with high probability that recovery is successful with the EMaC algorithm. This means that we can greatly reduce the power consumption in a MIMO radar while achieving a high resolution.

IV. Numerical Examples

We consider two sets of numerical examples. In the first example, we assume $M = 64$ and $N = 1$, and the goal is to estimate the DOA and the amplitudes of each target from a randomly sampled $L = 32$ samples. The pencil parameter is chosen as $k_1 = 32$. We generate 4 different targets with two that are closely located, and all of them are off the DFT grid, as shown in Fig. 2 (a). Fig. 2 (d) shows that the EMaC procedure perfectly estimates the true DOA in the noise-free scenario. The actual DOAs are estimated via applying ESPRIT on the recovered samples with an order estimate $P = 8$. We compare with CS approaches using BP assuming a DFT basis in Fig. 2 (d) and BP assuming a DFT frame with an oversampling factor of 4 in Fig. 2 (c), where both of them produce spurious targets around the truth targets. Fig. 3 shows the estimates when SNR is 20dB, where EMaC still obtains better performance than CS approaches. Note that no denoising is performed on the recovered $\mathbf{Y}$ before estimating the DOAs.

In the second example, we assume $M = 8$ and $N = 8$, and the goal is to jointly estimate the DOA and DOD of each target from $L = 32$ samples. The pencil parameters are chosen as $k_1 = 4$ and $k_2 = 4$. We randomly generate 6 different targets with the same amplitudes and random phases, and assume the SNR is 20dB. We then attempt recovery via BP with an oversampling factor of 4 in Fig. 4 (a), and via EMaC in Fig. 4 (b). We can see that EMaC produces less spurious targets around the truth targets without a grid assumption.

V. Conclusions

We explore a sub-Nyquist sampling strategy in a bistatic MIMO radar with M transmit and N receive antennas to reconstruct the sparse scattering scene with $K \ll MN$ targets. We develop a front-end that subsamples the output of the

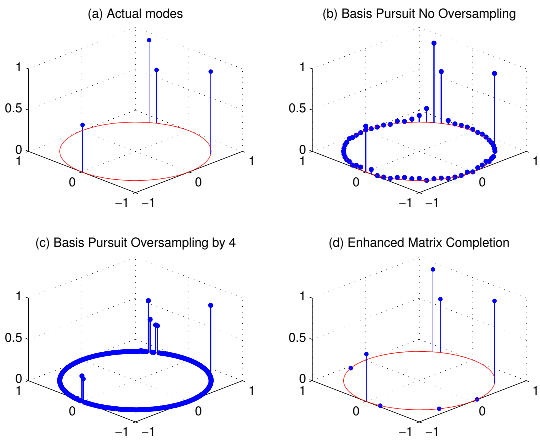

Fig. 2. The actual DOA arranged on a unit circle in (a), along with BP without oversampling in (b), with oversampling factor $c = 4$ in (c), with EMaC in (d). The EMaC perfect recovers the DOA without any grid assumptions.

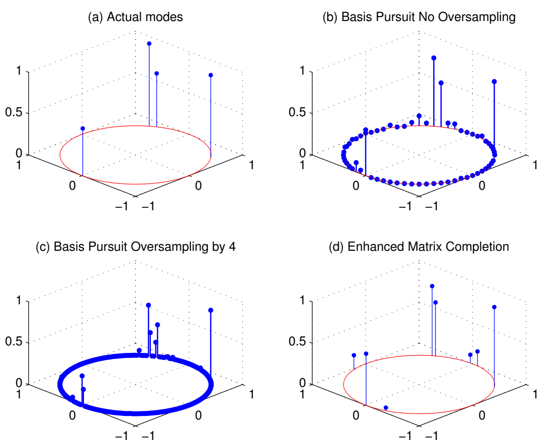

Fig. 3. The actual DOA arranged on a unit circle in (a), along with BP without oversampling in (b), with oversampling factor $c = 4$ in (c), with EMaC in (d) when the SNR is 20dB.

matched filter bank at each receive antenna uniformly at random with a total of L samples per pulse. Sparse recovery is then obtained via EMaC that makes no grid assumptions over the target scene. We demonstrate that as long as $L = O(K \log^2(MN))$, it is possible to recover the target scene under a mild condition, thus greatly reducing the amount of power consumption. We note that the same framework can also be applied to reduce the size of filter banks at the front-end by only implementing the sampled branches. Future work includes generalizing the current framework to process multiple pulses to further estimate Doppler shifts.

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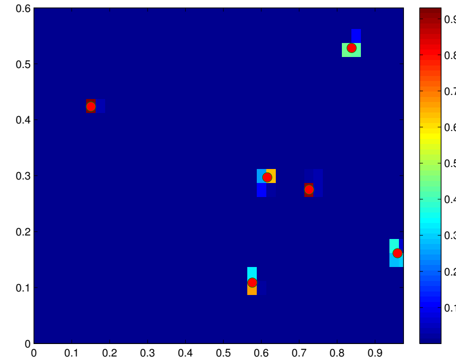
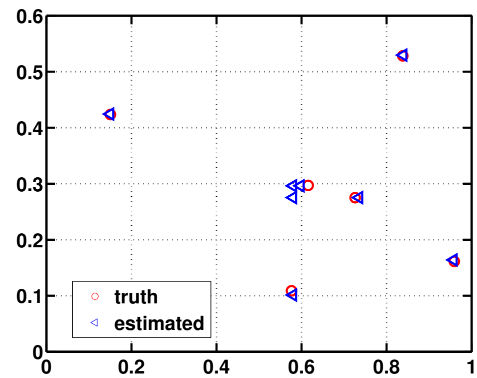


(a) Basis Pursuit with Oversampling factor 4



(b) EMaC

Fig. 4. The actual and recovered targets via BP with oversampling factor $c = 4$ in (a), with EMaC in (b).

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