AFFINE-PERMUTATION SYMMETRY: INVARIANCE AND SHAPE SPACE

David Sepiashvili, José M. F. Moura

Carnegie Mellon University
Electrical and Computer Engineering
5000 Forbes Avenue
Pittsburgh, PA
{davids2,moura}@ece.cmu.edu

Victor H. S. Ha

Samsung Electronics
Mobile Solution Lab
Digital Media R&D Center
Suwon, Korea
victor.ha@samsung.com

ABSTRACT

Studying similarity of objects by looking at their shapes arises naturally in many applications. However, under different viewpoints one and the same object appears to have different shapes. In addition, the correspondence between their feature points are unknown to the viewer.

In this paper, we introduce the concept of intrinsic shape of an object that is invariant to affine-permutation shape distortions.

We study geometry of the intrinsic shape space in the framework of differentiable manifolds with the emphasis on the computational aspects. We represent the intrinsic shape space as a folded Grassmann manifold. This allows us to easily analyze and compare different intrinsic shapes under the affine-permutation distortion without explicitly computing and recovering these intrinsic shapes.

We present the mathematical equations for connecting two intrinsic shapes by a geodesic, measuring their similarity, and morphing one intrinsic shape onto another.

1. INTRODUCTION

In many imaging environments, sensors and objects are arbitrarily oriented with respect to one another. As the relative 3D position of the object is unknown and often changes in time, the shape of the object captured by the sensor looks distorted in different images. The *affine shape distortion model* approximates these shape distortions closely and thus is used widely in the image processing community.

When the shape is distorted, for example, by an affine shape distortion, the feature points that comprise the shape move to different locations on the image plane. Since the image is scanned in a fixed order, such as the raster scan order, by the input device, the feature points of distorted shapes are not guaranteed to be read in the same order, thus adding another degree of ambiguity. We identify this type of shape irregularity as *permutation*. The permutation of feature points is commonly addressed in the well-known feature correspondence problem.

The combined affine-permutation shape distortion is a frequently observed source of difficulty in many image processing applications such as target detection, classification, pattern recognition, and registration.

In this paper, we explore effective ways to deal with affine-permutation distortions of shapes. In Section 2 we define a shape space and show that the above distortions are modeled as a group action on the shape space. We introduce the notion of an *intrinsic shape* as an orbit of all equivalent shapes. The intrinsic shape of an object is invariant to an affine class of motions and permutations of its feature points. Thus, the intrinsic shape uniquely represents the object in a broad range of imaging environments.

In Section 3 we breakdown an action of the affine-permutation group into a sequence of primitive group actions.

In Section 4, we derive an efficient algorithm referred to as BLAISER (the BLind Algorithm for Intrinsic Shape Recovery) for computing the intrinsic shape of an object.

In Sections 5, 6, and 7, we look at the space of intrinsic shapes in the framework of differentiable manifolds and study its geometry. We solve the two-point boundary value problem of given two shapes finding the geodesic connecting their corresponding intrinsic shapes as well as the distance between them. This is achieved directly from the coordinatization of the two shapes without computing the intrinsic shape of an object.

The geometry in the intrinsic shape space allows us to specify two shapes and then morph one onto the other obtaining the intermediate shapes. An example is presented in Section 8.

2. PROBLEM FORMULATION

We describe the shape of an object by a set of unlabeled n feature points in $\mathcal{R}^p$ (for 2D shapes $p = 2$, for 3D shapes $p = 3$, etc.). The shape is represented by an $n \times p$ tall-skinny matrix $\mathbf{X}$ of real entries that are the Cartesian coordinates of the n feature points in $\mathcal{R}^p$. The matrix $\mathbf{X} \in \mathcal{X}$ is referred to as the configuration matrix, while the set of all shape matrices $\mathcal{X}$ is defined as the configuration space, or shape space. We note that we consider only non-singular shapes, i.e. shapes with $\mathrm{rank}(\mathbf{X}) = p$.

We start by defining an equivalence relationship on $\mathcal{X}$. We say that two shapes $\mathbf{X}_1, \mathbf{X}_2 \in \mathcal{X}$ are equivalent iff they are related by:

$$\mathbf{X}_1 = P\mathbf{X}_2 A + \mathbf{1}\delta^T$$

where $A \in \mathcal{GL}_p$ is an affine distortion matrix acting on the right, and $P \in \mathcal{P}$ is the permutation matrix acting on the left and changing the order of rows in $\mathbf{X}$. δ is a $p \times 1$ translation vector that moves all feature points in $\mathcal{R}^p$uniformly in the same direction. $\mathbf{1}$ is $n \times 1$ vector of ones.

Rewriting in *vec* notation, the above relationship becomes:

$$vec(\mathbf{X}_1) = (A^T \otimes P)\cdot vec(\mathbf{X}_2) + (\delta \otimes \mathbf{1})$$

where the symbol $\otimes$ denotes the Kronecker product, see [1]. The above equation is, as shown in [2], an action of the affine-permutation group $\mathcal{G} = \{g : g = (A^T \otimes P, \delta \otimes \mathbf{1})\}$ on the configuration space.

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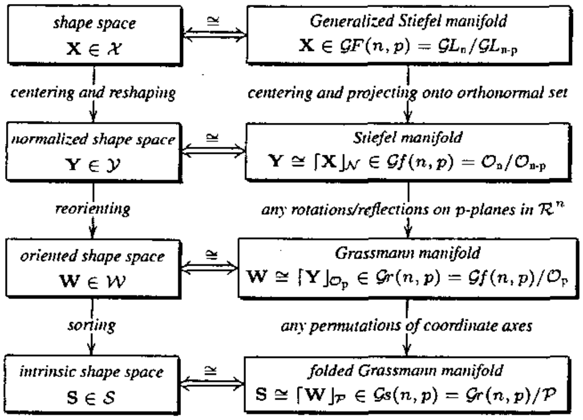

Fig. 1. Shape space reduction

The quotient space $\mathcal{Gs}(n,p) = \mathcal{X}/G$ is a set of equivalence classes under the action of group $\mathcal{G}$, also referred to as orbits. If we let semi-open brackets $\lceil \mathbf{X} \rfloor_{\mathcal{G}}$ to denote these equivalence classes, then we can write:

$$\lceil \mathbf{X} \rfloor_{\mathcal{G}} = \{\mathbf{X} \cdot g : g \in \mathcal{G}\} \in \mathcal{Gs}(n,p)$$

We now define the *intrinsic shape* to be the orbit $\lceil \mathbf{X} \rfloor_{\mathcal{G}} \in \mathcal{Gs}(n,p)$. It is a set of all shapes that are equivalent to $\mathbf{X}$. We also mark exactly one shape in each orbit $\mathbf{S} \in \lceil \mathbf{X} \rfloor_{\mathcal{G}}$ to be a unique canonical representative of the orbit. $\mathbf{S}$ fully identifies its orbit $\lceil \mathbf{X} \rfloor_{\mathcal{G}}$. For this reason we can write $\mathbf{S} = \lceil \mathbf{X} \rfloor_{\mathcal{G}}$ and refer to it as the intrinsic shape of an object. The space $\mathcal{S}$ of all such $\mathbf{S}$ is in bijective correspondence with $\mathcal{Gs}(n,p)$. Both $\mathcal{Gs}(n,p)$ and $\mathcal{S}$ are different representations of the intrinsic shape space.

In this paper we take two approaches to the problem of studying similarity of objects. The first approach is known as the canonical representative marking approach. The first approach uses the notion of intrinsic shape $\mathbf{S} \in \mathcal{S}$. We identify a unique representative $\mathbf{S}$ for each orbit $\lceil \mathbf{X} \rfloor_{\mathcal{G}}$ by a set of properties and we denote it as the intrinsic shape. We also give an algorithm for blindly reconstructing $\mathbf{S}$ given any shape $\mathbf{X}$.

The second approach studies the similarity of objects directly in the quotient space $\mathcal{Gs}(n,p)$. The intrinsic shape $\lceil \mathbf{X} \rfloor_{\mathcal{G}} \in \mathcal{Gs}(n,p)$ is the orbit of all equivalent shapes. We study the geometry of $\mathcal{Gs}(n,p)$directly from the coordinatization of any arbitrary representative shape from the orbit. In addition to studying similarity of objects we also look at other geometric problems, such as connecting two intrinsic shapes by a geodesic and morphing one shape onto the other.

3. FROM SHAPE SPACE TO INTRINSIC SHAPE SPACE

In section 2 we discussed that we can go from the shape space $\mathcal{X}$ to either the intrinsic shape space $\mathcal{Gs}(n,p)$ or its bijective intrinsic shape space $\mathcal{S}$. However, the action of group $\mathcal{G}$ cannot be easily studied. For this reason, we break down the action of group $\mathcal{G}$ into a set of consecutive actions of more primitive groups, as shown in Figure 1.

The left-hand side of Figure 1, going from top to bottom, shows the reduction process of the shape space $\mathcal{X}$ into the space of intrinsic shapes $\mathcal{S}$. Each reduced space is obtained by acting on the space above it with some group action, forming orbits, and then keeping in the space the unique canonical representatives of each orbit. Thus, $\mathcal{S} \subset \mathcal{W} \subset \mathcal{Y} \subset \mathcal{X}$. Each step for the left-hand side is described in Section 4.

Each of these spaces is in bijective correspondence with the spaces of orbits, formed under the same group action. They are shown on the right-hand side, and discussed in detail in Section 5.

4. BLIND RECONSTRUCTION OF $\mathbf{S} \in \lceil \mathbf{X} \rfloor_{\mathcal{G}}$ IN 2D

In this section we briefly summarize the BLind Algorithm for Intrinsic ShapE Recovery (BLAISER) that recovers from any shape $\mathbf{X} \in \mathcal{X}$ the corresponding intrinsic shape $\mathbf{S} \in \lceil \mathbf{X} \rfloor_{\mathcal{G}}$. For details on the algorithm we refer the reader to [2].

We note that this section is for 2D shapes (case $p = 2$). With some effort it can be extended to higher dimensions as well.

Definition 1 *The intrinsic shape* $\mathbf{S}$ *in each equivalence class* $\lceil \mathbf{X} \rfloor_{\mathcal{G}}$ *is defined uniquely by the following four properties:*

1. *The center of mass is located at the origin.*
2. *The inner product* $\mathbf{S}^T\mathbf{S}$ *is the 2D identity matrix* $\mathbf{I}$.
3. *The reorientation point, which we define uniquely for every 2D shape, is aligned with the* x*–coordinate axis.*
4. *The columns of* $\mathbf{S}$ *are ordered in ascending* y *coordinate values, then in ascending* x *coordinate values for the columns with the same* y *values.*

By satisfying the first two properties, we are reducing the shape space $\mathcal{X}$ into the normalized shape space $\mathcal{Y}$, as shown in Figure 1. This is achieved by first centering the shape with respect to its center of mass, and then performing a series of three geometric operations — a non-uniform scaling followed by a rotation and another non-uniform scaling. These geometric operations are easily determined by the shape's second order central moments, and are discussed in details in [2]. The normalized shape is invariant to translation, uniform scaling, non-uniform scaling, and shearing.

Next, we identify the shape's reorientation point and reorient the shape using this unique point. We rotate and, if necessary, reflect the normalized shape $\mathbf{Y} \in \mathcal{Y}$ to arrive at a unique 2D orientation. This satisfies the third property and reduces $\mathbf{Y} \in \mathcal{Y}$ to $\mathbf{W} \in \mathcal{W}$, as shown in Figure 1. We note that the algorithm for uniquely identifying the reorientation point is omitted here. We refer the readers again to [2] for details on this algorithm. The idea behind the algorithm is that we are dividing the shape into subshapes of equal radius from the center of mass, and then studying fold number for each subshape. Then, we get a list of fundamental angles, [2], from which we uniquely identify the reorientation point and determine if the shape needs to be reflected. The oriented shape becomes invariant to rotations and reflections.

Finally, we sort columns of $\mathbf{W}$ as described in the fourth property and obtain the intrinsic shape $\mathbf{S}$. This step reduces the oriented shape space $\mathcal{W}$ to the intrinsic shape space $\mathcal{S}$. The intrinsic shape is thus invariant to translation, uniform and non-uniform scaling, shearing, rotation and reflection, and permutation of its feature points.

For further details on BLAISER we refer the reader to [2].

5. QUOTIENT GEOMETRY OF SMOOTH MANIFOLDS

We recall that a shape is represented by an arbitrary $n \times p$ *rank-p* matrix $\mathbf{X} \in \mathcal{X}$. We can view this matrix as a linearly independent

p-frame in $\mathcal{R}^n$. We can realize $\mathcal{X}$ as a quotient of general linear groups, which are also Lie groups as $\mathcal{X} \cong \mathcal{GF}(n,p) = \mathcal{GL}_{\mathrm{n}}/\mathcal{GL}_{\mathrm{n\text{-}p}}$. $\mathcal{GF}(n,p)$ obtained this way is a smooth manifold. It is frequently referred to as the *Generalized Stiefel manifold.* For background information we refer the reader to [3]. Also, as mentioned in Section 2, we will abuse the notation and say that $\mathbf{X} \in \mathcal{GF}(n,p)$.

Now we go back to Figure 1 and start reviewing its right-hand side in the framework of smooth manifolds. We construct $\mathbf{Y} \cong \lceil \mathbf{X} \rfloor_{\mathcal{N}} \in \mathcal{G}f(n,p)$ from $\mathbf{X} \in \mathcal{GF}(n,p)$ by first removing the mean, and then replacing the diagonal matrix of the singular values in its compact SVD decomposition with the identity. Removing the mean makes column vectors of $\mathbf{Y}$ sum to zero, while replacing singular values with ones makes $\mathbf{Y}$ be orthonormal ($\mathbf{Y}^T\mathbf{Y} = \mathbf{I}$). The obtained space $\mathcal{G}f(n,p) = \mathcal{O}_{\mathrm{n}}/\mathcal{O}_{\mathrm{n\text{-}p}}$ is a space of orthonormal p-frames, and is called the *Stiefel manifold,* [3].

In order to eliminate the orientation ambiguity we take a Lie group $\mathcal{O}_{\mathrm{p}}$ and consider a quotient space $\mathcal{G}f(n,p)/\mathcal{O}_{\mathrm{p}}$. Each element in the quotient space is a set of all possible rotated and reflected versions of $\mathbf{Y}$ and is given by $\lceil \mathbf{Y} \rfloor_{\mathcal{O}_{\mathrm{p}}} = \{\mathbf{Y}V : V \in \mathcal{O}_{\mathrm{p}}\}$. This shows that the space of oriented shapes $\mathcal{W}$ is in bijective correspondence with the space $\mathcal{G}r(n,p) = \mathcal{G}f(n,p)/\mathcal{O}_{\mathrm{p}}$. $\mathcal{G}r(n,p)$ is the *Grassmann manifold,* [3]. It can also be defined as the space of all p-dim subspaces in $\mathcal{R}^n$. Under this definition if $\mathbf{W}$ is a point on the Grassmann manifold $\mathcal{G}r(n,p)$ then $\mathbf{W} = span(\mathbf{Y})$. Both definitions are equivalent, but we will work with the orbits $\lceil \mathbf{Y} \rfloor_{\mathcal{O}_{\mathrm{p}}}$. We will use a shorthand symbol $\lceil \mathbf{Y} \rfloor$, omitting the group $\mathcal{O}_{\mathrm{p}}$ as its subindex. The detailed treatment of the Grassmann manifold is presented in Section 6.

The final step allows arbitrary permutations of n feature points, which are row permutations of the shape matrices. This ambiguity is eliminated by taking a finite group of permutations $\mathcal{P}$ and folding the Grassmann manifold into $\mathcal{G}r(n,p)/\mathcal{P}$. This is presented in Section 7.

Notions of time and distance on a smooth manifold We refer the reader for the necessary background on differentiable manifolds to classical textbooks on Riemannian geometry [4, 3, 5]. Here we informally present the notions of time and distance on manifolds.

Smooth curves are smooth mappings $\gamma : [\tau_1, \tau_2] \to \mathcal{Y}$ from the time domain $\mathcal{R}$ to a smooth manifold $\mathcal{Y}$. They carry a way of natural differentiation in time $\gamma' = d\gamma/dt$. The derivative $\mathbf{H}_t = \gamma'(t)$ at point $\mathbf{Y}_t = \gamma(t)$ is called the *tangent vector* at $\mathbf{Y}_t$. *Vector fields* are mappings that assign to each point on $\mathcal{Y}$ one of its tangent vectors. A *connection* is a way of differentiating vector fields along curves in $\mathcal{Y}$. A vector field along some curve is said to be a *parallel vector field* if its derivative is zero along the curve. Given a curve $\gamma(t)$ and a tangent vector $\mathbf{D}_o$ at point $\gamma(0)$, there exists a unique parallel vector field along the curve that extends the tangent vector. We say we *parallel transported* the tangent vector $\mathbf{D}_o$ along the curve to $\mathbf{D}_t$ at point $\gamma(t)$. We define a straight line, called a geodesic, to be a curve γ of zero acceleration, i.e., a curve that parallel transports its derivative tangent vector $\mathbf{H}_o = \frac{d\gamma}{dt}(0)$, see Figure 2 for the illustration.

We don't define distances directly on the manifold. Rather, we equip tangent spaces with the inner products, making $\mathcal{Y}$ a Riemannian manifold. A *compatible* with the Riemannian metric connection, also called a *Levi-Civita* connection, preserves inner products during parallel transport. Then a notion of distance on the manifold is defined in the following way. We pick a point $\mathbf{Y}_o \in \mathcal{Y}$ and some tangent vector $\mathbf{H}_o$ at point $\mathbf{Y}_o$. This defines a geodesic curve $\gamma(t)$ s.t. $\gamma(0) = \mathbf{Y}_o$ and $\gamma'(0) = \mathbf{H}_o$. Then the distance

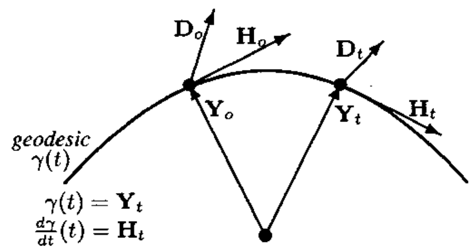

Fig. 2. Connecting points $\mathbf{Y}_o$ and $\mathbf{Y}_t$ by a geodesic $\gamma(t)$

between points $\mathbf{Y}_o = \gamma(0)$ and $\mathbf{Y}_\tau = \gamma(\tau)$ is the length of the $\gamma(t)$ curve:

$$d(\mathbf{Y}_o, \mathbf{Y}_\tau) = \oint_o^\tau \left\| \frac{d\gamma}{dt} \right\|_F dt = \tau \cdot \left\| \mathbf{H}_o \right\|_F \quad (1)$$

Since the curve $\gamma(t)$ is a geodesic, its *derivative vector field* $\gamma'(t)$ is the parallel vector field. At the same time a Levi-Civita connection preserves norms of parallel vector fields. Thus, the norm will be constant in the integration.

6. GRASSMANN MANIFOLD

We remind the reader that a point on the Grassmann manifold is the orbit $\lceil \mathbf{Y} \rfloor \in \mathcal{G}r(n,p)$ under the action of the Lie group $\mathcal{O}_{\mathrm{p}}$, where $\mathbf{Y}$ is an orthonormal $n \times p$ tall-skinny matrix representing a point on the Stiefel manifold $\mathcal{G}f(n,p)$. In this section we study in detail $\mathcal{G}r(n,p)$, with the emphasis on the computational aspects. We are working with any arbitrary $\mathbf{Y} \in \lceil \mathbf{Y} \rfloor$ matrix as the defining representative of the orbit $\lceil \mathbf{Y} \rfloor \in \mathcal{G}r(n,p)$.

We consider a natural embedding of $\mathcal{G}r(n,p)$ in $\mathcal{R}^{n \times n}$ and equip the manifold with the natural Levi-Civita connection. Then we study equations for the geodesic and its derivative vector field.

Solution for the Initial Value Problem Edelman, Arias and Smith [6] have proposed a computationally practical way of defining equations for the geodesic and its derivative vector field, given an initial point and a tangent direction at the point. This is known as an *initial value problem* in the theory of differential equations. Their approach does not require an explicit formula for the connection. For completeness we present their result below.

Let $\mathbf{Y}_o$ be a representative of the orbit $\lceil \mathbf{Y}_o \rfloor \in \mathcal{G}r(n,p)$ and let $\mathbf{H}_o$ be a valid tangent vector at $\mathbf{Y}_o$. We define the geodesic $\gamma(t)$ by the initial values $\gamma(0) = \lceil \mathbf{Y}_o \rfloor$ and $\gamma'(0) = \lceil \mathbf{H}_o \rfloor$. We perform an SVD decomposition of $\mathbf{H}_o = UEV^T$. We define matrices $C_t = cosm(t \cdot E)$ and $S_t = sinm(t \cdot E)$ to be the matrix-cosine and matrix-sine, which are defined well for symmetric matrices, [1]. Then the equations for the geodesic $\gamma(t) = \lceil \mathbf{Y}_t \rfloor$ and its derivative vector field along the geodesic $\gamma'(t) = \lceil \mathbf{H}_t \rfloor$ at time t are given by:

$$\begin{aligned} \mathbf{Y}_t &= \mathbf{Y}_o V C_t V^T + \mathbf{H}_o V S_t E^{-1} V^T \\ \mathbf{H}_t &= -\mathbf{Y}_o V S_t E V^T + \mathbf{H}_o V C_t V^T \end{aligned} \quad (2)$$

These equations tell us how the point $\lceil \mathbf{Y}_t \rfloor$ and its tangent $\lceil \mathbf{H}_t \rfloor$ evolve along the geodesic. This result is derived in [6].

Solution for the 2-point Boundary Value Problem The problem we would like to solve is quite different from the solution given by equation 2. Instead of defining equations for the geodesic and its derivative vector field, given an initial point and a tangent direction at the point, it is essential for us to define these equations, given an initial and a final points on the $\mathcal{G}r(n,p)$, which are to be connected by a geodesic. In the theory of differential equations this corresponds to a *two-point boundary value problem.* Such a problem is commonly solved numerically, using the shooting method [7]. We present our result for an explicit expression for the solution of the two-point boundary value problem, involving only standard linear algebra operations.

Let $\lceil \mathbf{Y}_o \rfloor, \lceil \mathbf{Y}_\tau \rfloor \in \mathcal{G}r(n,p)$ be two points on the $\mathcal{G}r(n,p)$, which are to be connected by a geodesic. We define the geodesic $\gamma(t)$ by the initial $\gamma(0) = \lceil \mathbf{Y}_o \rfloor$ and final $\gamma(\tau) = \lceil \mathbf{Y}_\tau \rfloor$ values. This is a two-point boundary value problem. Let $\mathbf{Y}_o \in \lceil \mathbf{Y}_o \rfloor$ and $\mathbf{Y}_\tau \in \lceil \mathbf{Y}_\tau \rfloor$ be some representatives for the initial and final values. We perform an SVD decomposition of $\mathbf{Y}_o^T \mathbf{Y}_\tau = V_o C_\tau V_\tau^T$, which gives two rotation matrices V_o, V_τ and a cosine matrix of the principal angles C_τ. Then we define the matrix $E_\tau = acosm(C_\tau)$, whose diagonal elements are the principal angles between the two subspaces spanned by $\lceil \mathbf{Y}_o \rfloor$ and $\lceil \mathbf{Y}_\tau \rfloor$, see [8]. The distance between the points is given by $\tau = \|E_\tau\|_F$. We define the matrix $E = E_\tau/\tau$ so that $\|E\|_F = 1$ Also we define $S_\tau = sinm(\tau \cdot E)$ as well as cosine and sine matrices of half angles $C_m = cosm(\frac{\tau}{2} \cdot E)$ and $S_m = sinm(\frac{\tau}{2} \cdot E)$.

In Euclidean space, we can take two points $\mathbf{y}_o, \mathbf{y}_\tau \in \mathcal{R}^n$, connect them by a line, and then find a point $\mathbf{y}_m$ in between them, which will lie at equal distance to both points. This mid-point will be given by $\mathbf{y}_m = (\mathbf{y}_o + \mathbf{y}_\tau)/2$. We present our result for a similar construct on the Grassmannian $\mathcal{G}r(n,p)$. It is of great importance for studying its properties. The equations of the $\gamma(t)$ geodesic's midpoint and of the tangent vector from its derivative vector field at the midpoint are given by:

$$\begin{aligned} \mathbf{Y}_m &= \tfrac{1}{2}\left(\mathbf{Y}_\tau V_\tau + \mathbf{Y}_o V_o\right) C_m^{-1} V_o^T \\ \mathbf{H}_m &= \tfrac{1}{2}\left(\mathbf{Y}_\tau V_\tau - \mathbf{Y}_o V_o\right) S_m^{-1} E V_o^T \end{aligned} \qquad (3)$$

In order to obtain a closed form solution for the two-point boundary value problem, all we need to do is to travel back from the point $\lceil \mathbf{Y}_m \rfloor$, given by equation (3), at time $t = 0$ to the point $\lceil \mathbf{Y}_o \rfloor$ at time $t = -\tau/2$ on the geodesic defined by equation (2). First, we present a closed-form solution for $\mathbf{H}_o$, such that $\|\mathbf{H}_o\|_F = 1$:

$$\mathbf{H}_o = -\mathbf{Y}_o V_o C_\tau {S_\tau}^{-1} E V_o^T + \mathbf{Y}_\tau V_\tau {S_\tau}^{-1} E V_o^T \qquad (4)$$

If we use brackets to denote stacking of matrices together to form a larger matrix, then an alternative solution can also be obtained from the equation:

$$\left[\mathbf{Y}_o | \mathbf{Y}_\tau\right]^T \cdot \left[\mathbf{H}_o | \mathbf{H}_\tau\right] = \begin{bmatrix} V_o & 0 \\ 0 & V_\tau \end{bmatrix} \cdot \begin{bmatrix} 0 & -ES_\tau \\ ES_\tau & 0 \end{bmatrix} \cdot \begin{bmatrix} V_o & 0 \\ 0 & V_\tau \end{bmatrix}^T \qquad (5)$$

From this equation we simultaneously solve for $\mathbf{H}_o$ and $\mathbf{H}_\tau$ using the pseudo-inverse of $[\mathbf{Y}_o|\mathbf{Y}_\tau]^T$.

We now present our result for the equations of the geodesic $\gamma(t) = \lceil \mathbf{Y}_t \rfloor$ and its derivative vector field along the geodesic $\gamma'(t) = \lceil \mathbf{H}_t \rfloor$ at time t. This is an explicit expression for the solution of the two-point boundary value problem:

$$\begin{aligned} \mathbf{Y}_t &= \mathbf{Y}_o V_o (C_t - S_t C_\tau {S_\tau}^{-1}) V_o^T + \mathbf{Y}_\tau V_\tau S_t S_\tau^{-1} V_o^T \\ \mathbf{H}_t &= -\mathbf{Y}_o V_o (S_t + C_t C_\tau S_\tau^{-1}) E V_o^T + \mathbf{Y}_\tau V_\tau C_t S_\tau^{-1} E V_o^T \end{aligned} \qquad (6)$$

The time-varying matrices in equation (6) are $C_t = cosm(tE)$ and $S_t = sinm(t \cdot E)$. All other matrices are obtained from an SVD decomposition of the inner product $\mathbf{Y}_o^T \mathbf{Y}_\tau$.

7. FOLDED GRASSMANN MANIFOLD $\mathcal{G}r(n,p)/\mathcal{P}$

As discussed in Section 5, the final step on the right-hand side of Figure 1 is folding the Grassmann manifold $\mathcal{G}r(n,p)$ with a group of permutations $\mathcal{P}$, which is not a Lie group. So, in order to preserve the geometry of $\mathcal{G}r(n,p)$ in $\mathcal{G}s(n,p) = \mathcal{G}r(n,p)/\mathcal{P}$, we need to require that the folded space $\mathcal{G}s(n,p)$ be *convex* in $\mathcal{G}r(n,p)$. We have preliminary results on such folding with proves for the $p = 1$ case. For $p > 1$ the folding is still under the investigation.

Currently, we perform folding by solving the optimization problem $\mathbf{P} = \arg\min_{\mathbf{P} \in \mathcal{P}} \|Y_o - PY_\tau\|_F$ which is a good approximation to the true problem $\mathbf{P} = \arg\min_{\mathbf{P} \in \mathcal{P}} d(Y_o, PY_\tau)$. The problem is linear, and is easily solved by the simplex algorithm.

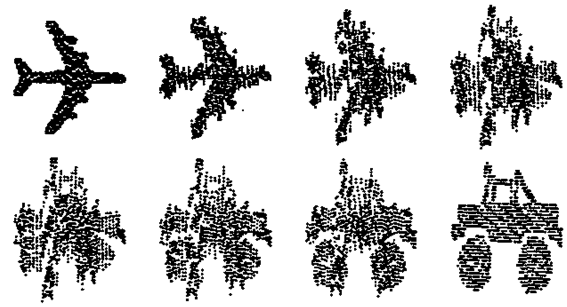

Fig. 3. Morphing an airplane onto a truck

8. MORPHING ONE SHAPE ONTO THE OTHER

In this section we apply the geometry in the intrinsic shape space $\mathcal{G}s(n,p)$ to the problem of morphing one intrinsic shape onto the other. In Figure 3 each of eight presented shapes represents a point in $\mathcal{G}s(n,p)$. We take the initial point in $\mathcal{G}s(n,p)$ to be an airplane, and the final point to be a truck. They are getting connected by a geodesic. We show equally spaced six intermediate shapes that we obtain by following this geodesic from the initial to the final point.

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