Manifold Gradient Descent Solves Multi-Channel Sparse Blind Deconvolution Provably and Efficiently

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Abstract
Multi-channel sparse blind deconvolution, or convolutional sparse coding, refers to the problem of learning an unknown filter by observing its circulant convolutions with multiple input signals that are sparse. This problem finds numerous applications in signal processing, computer vision, and inverse problems. However, it is challenging to learn the filter efficiently due to the bilinear structure of the observations with respect to the unknown filter and inputs, leading to global ambiguities of identification. In this paper, we propose a novel approach based on nonconvex optimization over the sphere manifold by minimizing a smooth surrogate of the sparsity-promoting loss function. It is demonstrated that the manifold gradient descent with random initializations will provably recover the filter, up to scaling and shift ambiguity, as soon as the number of observations is sufficiently large under an appropriate random data model. Numerical experiments are provided to illustrate the performance of the proposed method with comparisons to existing methods.

Keywords: nonconvex optimization, multi-channel sparse blind deconvolution, manifold gradient descent

1 Introduction
In various fields of signal processing, computer vision, and inverse problems, it is of interest to identify the location of sources from traces of responses collected from sensors. For example, neural or seismic recordings can be modeled as the convolution of a pulse shape (i.e. a filter), corresponding to characteristics of neuron or earth wave propagation, with a spike train modeling time of activations (i.e. a sparse input) [1, 2, 3]. When the filter is unknown, it leads to the so-called blind deconvolution problem. This problem is ill-posed without extra assumptions on the filter, since the number of unknowns is much larger than the number of observations [4, 5, 6, 7, 8]. Luckily, in many situations, one can make multiple observations sharing the same filter, but with diverse sparse inputs, either spatially or temporally, thanks to the advances of sensing technologies. The goal is thus to identify the filter as well as the sparse inputs leveraging multiple convolutional observations in an efficient manner, a problem termed as multi-channel sparse blind deconvolution (MSBD).

Mathematically, we model each observation \( y_i \in \mathbb{R}^n \) as a convolution, between a filter \( g \in \mathbb{R}^n \), and a sparse input, \( x_i \in \mathbb{R}^n \):

\[
y_i = g \circledast x_i = C(g)x_i, \quad i = 1, \ldots, p,
\]

where the total number of observations is given as \( p \). Here, we consider circulant convolution, denoted as \( \circledast \), whose operation is expressed equivalently via pre-multiplying a circulant matrix \( C(g) \) to the input, defined as

\[
C(g) = \begin{bmatrix}
g_1 & g_n & \cdots & g_2 \\
g_2 & g_1 & \cdots & g_3 \\
\vdots & \vdots & \ddots & \vdots \\
g_n & g_{n-1} & \cdots & g_1
\end{bmatrix}.
\]

Our goal is to recover both the filter \( g \) and sparse inputs \( \{x_i\}_{i=1}^p \) from the observations \( \{y_i\}_{i=1}^p \).

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1.1 Nonconvex Optimization on the Sphere

The problem is challenging due to the bilinear form of the observations with respect to the unknowns, as well as the sparsity constraint. A direct observation tells that the unknowns are not uniquely identifiable, since for any circulant shift matrix $S_k$ by $k$ entries and a non-zero scalar $\beta \neq 0$, we have

$$y_i = (\beta S_k g) \odot (\beta^{-1} S_{-k} x_i),$$  

(3)

for $k = 1, \ldots, n - 1$. Hence, we can only hope to recover $g$ and $\{x_i\}_{i=1}^p$ accurately up to certain circulant shift and scaling factor.

In this paper, we focus on the case that $\mathcal{C}(g)$ is invertible, which is equivalent to requiring all Fourier coefficients of $g$ are nonzero. This condition plays a critical role in guaranteeing the identifiability of the model as long as $p$ is large enough [9].

Under this assumption, there exists a unique inverse filter, $g_{\text{inv}} \in \mathbb{R}^n$ such that

$$\mathcal{C}(g_{\text{inv}}) \mathcal{C}(g) = \mathcal{C}(g_{\text{inv}}) \mathcal{C}(g_{\text{inv}}) = I.$$  

(4)

This allows us to convert the bilinear form (1) into a linear form, by multiplying $\mathcal{C}(g_{\text{inv}})$ on both sides:

$$\mathcal{C}(g_{\text{inv}}) y_i = \mathcal{C}(g_{\text{inv}}) \mathcal{C}(g) x_i = x_i, \quad i = 1, \ldots, p.$$  

Consequently, we can equivalently aim to recover $g_{\text{inv}}$ via exploiting the sparsity of the inputs $x_i$. An immediate thought is to seek a vector $h$ that minimizes the cardinality of $\mathcal{C}(h) y_i = \mathcal{C}(y_i) h$:

$$\min_{h \in \mathbb{R}^n} \frac{1}{p} \sum_{i=1}^p \|\mathcal{C}(y_i) h\|_0,$$

where $\| \cdot \|_0$ is the pseudo-$\ell_0$ norm that counts the cardinality of the nonzero entries of the input vector. However, this simple formulation is problematic for two obvious reasons:

1) first, due to scaling ambiguity, a trivial solution is $h = 0$;

2) second, the cardinality minimization is computationally intractable.

The first issue can be addressed by adding a spherical constraint $\|h\|_2 = 1$ to avoid scaling ambiguity. The second issue can be addressed by relaxing to a convex smooth surrogate that promotes sparsity. In this paper, we consider the function

$$\psi_\mu(z) = \mu \log \cosh(z/\mu),$$  

(5)

which serves as a convex surrogate of $\| \cdot \|_0$, where $\mu$ controls the smoothness of the surrogate. With slight abuse of notation, we assume $\psi_\mu(z) = \sum_{i=1}^n \psi_\mu(z_i)$ is applied in an entry-wise manner, where $z = [z_i]_{1 \leq i \leq n}$.

Putting them together, we arrive at the following optimization problem:

$$\min_{h \in \mathbb{R}^n} f_\mu(h) := \frac{1}{p} \sum_{i=1}^p \psi_\mu(\mathcal{C}(y_i) h) \quad \text{s.t.} \quad \|h\|_2 = 1,$$

(6)

which is a nonconvex optimization problem due to the sphere constraint. As we shall see later, while this approach works well when $\mathcal{C}(g)$ is an orthogonal matrix, further care needs to be taken when it is a general invertible matrix in order to guarantee a benign optimization geometry. Following [4, 10], we introduce the following pre-conditioned optimization problem:

$$\min_{h \in \mathbb{R}^n} f(h) = \frac{1}{p} \sum_{i=1}^p \psi_\mu(\mathcal{C}(y_i) R h) \quad \text{s.t.} \quad \|h\|_2 = 1,$$

(7)

where $R$ is a pre-conditioning matrix depending only on the observations $\{y_i\}_{i=1}^p$ that we will formally introduce in Section 2.

\(^{1}\)It is established in [9] that under the Bernoulli-Gaussian model (c.f. Definition 1) on sparse coefficients, the filter is identifiable with high probability, provided that $g$ is invertible, $\theta \in (1/n, 1/4)$ and $p > Cn \log n$ for some constant $C$. 


Figure 1: An illustration of the landscape of the empirical loss function \( f_o(h) \) or \( f(h) \) with or without the pre-conditioning matrix \( R \) in \( \mathbb{R}^3 \), where the sparse inputs are generated according to a Bernoulli-Gaussian model with \( p = 30 \) observations and activation probability \( \theta = 0.3 \). (a) orthogonal filter \( C(g) = I \), no pre-conditioning is applied; (b) a general filter, no pre-conditioning is applied; (c) the same general filter as (b) with pre-conditioning.

1.2 Optimization Geometry and Manifold Gradient Descent

Encouragingly, despite nonconvexity, under a suitable random model of the sparse inputs, the empirical loss functions exhibit benign geometric curvatures as long as the sample size \( p \) is sufficiently large. As an illustration, Fig. 1 shows the landscape of \( f_o(h) \) and \( f(h) \) when \( n = 3 \) and \( p = 30 \), and the sparse inputs \( \{x_i\}_{i=1}^p \) follow the standard Bernoulli-Gaussian model (with an activation probability \( \theta = 0.3 \), see Definition 1). When the filter is orthogonal, e.g. \( C(g) = I \), it can be seen from Fig. 1 (a) that the function \( f_o(h) \) in (6) has benign geometry without pre-conditioning, where the local minimizers are approximately all shift and sign-flipped variants of the ground truth (i.e., the basis vectors), and are symmetrically distributed across the sphere. On the other end, for filters that are not orthogonal, the geometry of \( f_o(h) \) in (6) is less well-posed without pre-conditioning, as illustrated in Fig. 1 (b). By introducing pre-conditioning, which intuitively stretches the loss surface to mirror the orthogonal case, the pre-conditioned loss function \( f(h) \) given in (7) for the same non-orthogonal filter used in Fig. 1 (b) is much easier to optimize over, as illustrated in Fig. 1 (c).

Motivated by this benign geometry, it is therefore natural to optimize \( h \) over the sphere. One simple and low-complexity approach is to minimize \( f(h) \) over the sphere via (projected) manifold gradient descent (MGD),

\[
h_{t+1} := h_t - \eta_t \frac{\partial f(h_t)}{\|\partial f(h_t)\|_2},
\]

where \( \eta_t \) is the stepsize, \( \partial f(h) \) is the Riemannian manifold gradient with respect to \( h \) (defined in Sec. 2.2). Surprisingly, this simple approach works remarkably well even with random initializations for appropriately chosen step sizes. As an illustration, Fig. 2 depicts that MGD converges within a few number of iterations for the problem instance in Fig. 1 (c). Based on such empirical success, our goal is to address the following question: can we establish theoretical guarantees of MGD to recover the filter for MSBD?

In this paper, we formally establish the benign geometry of the empirical loss function over the sphere, and prove that MGD, with a small number of random initializations, is guaranteed to recover the filter with high probability in polynomial time. Our result is stated informally below.

**Theorem 1 (Informal).** Assume the sparse inputs are generated using a Bernoulli-Gaussian model, where the activation probability \( \theta \in (0, 1/3) \). As long as the sample size is sufficiently large, i.e. \( p = O(\text{poly}(n)) \), manifold gradient descent, initialized from at most \( O(\log n) \) independent and uniform random points, recovers the filter accurately with high probability, for properly chosen \( \mu \), and step size \( \eta \).

Our theorem provides justifications to the empirical success of MGD with random initializations. This result is achieved through an integrated analysis of geometry and optimization. Namely, we identify a union of subsets, corresponding to neighborhoods of equivalent global minimizers, and show that this region has
large gradients pointing towards the direction of minimizers. Consequently, if the iterates of MGD lie in this region, and never jump out of this region during its execution, we can guarantee that MGD converges to the global minimizers. Luckily, this region is large enough, so that the probability of a random initialization selected uniformly over the sphere has at least a constant probability falling into the region. By independently initializing a few times, it is guaranteed with high probability at least one of the initializations successfully land into the region of interest and return a faithful estimate of the filter.

1.3 Paper Organization and Notations

The rest of this paper is organized as follows. Section 2 presents the problem formulation and main results. Section 3 outlines the analysis framework and sketches the proof. Section 4 provides numerical experiments on both synthetic and real data with comparisons to existing algorithms. Section 5 discusses the related literature and we conclude in Section 6 with some future directions.

Throughout the paper, we use boldface letters to represent vectors and matrices. Let $x^\top$, $x^\mathsf{H}$ denote the transpose and conjugate transpose of $x$, respectively. Let $[n]$ denote the index set $\{1, 2, \cdots, n\}$. For a vector $x \in \mathbb{R}^n$, let $x_j$ denote its $j$th element. Let $x_D$, $D \subseteq [n]$ denote the length-$|D|$ vector composed of the elements in the index set $D$ of $x$, and let $x_{\backslash D}$ denote the vector obtained by removing the elements of $x$ in the index set $D$. For example, $x_{1:j}$ denotes the length-$j$ vector composed of the first $j$ entries of $x$, i.e., the vector $[x_1, x_2, \cdots, x_j]^\top$, and $x_{\backslash \{i\}}$ denotes the length-$(n-1)$ vector composed of all entries of $x$ except the $i$-th one, i.e. the vector $x_{1:i-1,i+1:n}$. If an index $j \notin [n]$ for an $n$-dimensional vector, then the actual index is computed as in the modulo $n$ sense. $S_j$ denotes a circular shift by $j$ positions, i.e., $S_j(x)_k = x_{k-j}$ for $j, k \in [n]$. Let $\|\cdot\|_p$, $p \in [1, \infty]$ represent the $\ell_p$ norm of a vector, and $\|\cdot\|$ denote the operator norm of a matrix. Let $\sigma_i(A)$ be the $i$th largest eigenvalue of a matrix $A$. Let $\odot$ denote the Hadamard product for two vector $x, y \in \mathbb{R}^n$ of the same dimension. Let $I$ denote an identity matrix, and $e_i \in \mathbb{R}^n$, $i \in [n]$ be the $i$-th standard basis vector. If $A \preceq B$, then $B - A$ is positive semidefinite. Throughout the paper, we use $c_1, c_2, C, \ldots$ to denote universal constants whose values may change from line to line.

2 Main Results

To begin, we state a few key assumptions. In this paper, we assume that the sparse inputs are generated according to the well-known Bernoulli-Gaussian model, defined below.

**Definition 1** (Bernoulli-Gaussian model [11]). The inputs $x_i$, $i = 1, \cdots, p$, are said to satisfy the Bernoulli-Gaussian model with parameter $\theta \in (0, 1)$, i.e. $x_i \sim_{iid} \mathsf{BG}(\theta)$, if $x_i = \Omega_i \odot z_i$, where $\Omega_i$ is an i.i.d. Bernoulli
vector with parameter $\theta$, and $z_i$ is an independent random vector with i.i.d. random Gaussian variables drawn from $\mathcal{N}(0,1)$.

Furthermore, the geometry of the loss function $f(h)$ turns out to be highly related to the condition number of the matrix $C(g)$, which is defined below.

**Definition 2** (Condition number). We use $\kappa$ to denote the condition number of $C(g)$, i.e. $\kappa = \sigma_1(C(g))/\sigma_n(C(g))$.

When $C(g)$ is orthogonal, we have $\kappa = 1$. Let the Fourier transform of $g$ be $\hat{g} = Fg$, then $\kappa$ is equivalent to the ratio of the largest and the smallest absolute values of $\hat{g}$, i.e. $\kappa := \|\hat{g}\|_{\text{max}}/\|\hat{g}\|_{\text{min}}$. Therefore, $\kappa$ measures the flatness of the spectrum $\hat{g}$, which plays a similar role as the coherence introduced in early works of blind deconvolution with a single snapshot [5, 8].

Since $g_{\text{inv}}$ can only be identified up to scaling and shift ambiguities, without loss of generality, we assume $\|g_{\text{inv}}\|_2 = 1$. To measure the success of recovery, we use the following distance metric that takes into account the shift ambiguity:

$$\text{dist}(h, g_{\text{inv}}) = \min_{j \in [n]} \|g_{\text{inv}} \pm S_j(h)\|_2.$$  \hfill (9)

### 2.1 Geometry of the Empirical Loss

We start by describing the geometry of $f_\phi(h)$ when $C(g)$ is an orthonormal matrix, where pre-conditioning is not needed. Without loss of generality, we can assume $C(g) = I$, \footnote{Denote $\tilde{h} = C(g)h$, we have $\|\tilde{h}\|_2 = \|C(g)h\|_2 = 1$ due to the orthonormality of $C(g)$. Rewriting the loss function with respect to $\tilde{h}$ will confirm this assertion.} which corresponds to the ground truth $g_{\text{inv}} = e_1$ and $y_i = x_i$. Therefore, the loss function $f_\phi(h)$ in (6) can be equivalently reformulated as

$$\min_{h \in \mathbb{R}^n} f_\phi(h) = \frac{1}{p} \sum_{i=1}^{p} \psi_\mu(C(x_i)h) \quad \text{s.t.} \quad \|h\|_2 = 1.$$ \hfill (10)

Our geometric theorem characterizes benign properties of curvatures around the local neighborhood of $\{\pm e_i\}_{i=1}^n$, shifted and sign-flipped copies of the ground truth. Following [12, 13], we introduce $2n$ subsets,

$$S^{(i\pm)}_\xi = \left\{ h : h_i \geq 0, \frac{h_i^2}{\|h_{\setminus \{i\}}\|_\infty^2} \geq 1 + \xi \right\}, \quad i \in [n],$$ \hfill (11)

where $\xi \in [0,\infty)$. Clearly, $e_i \in S^{(+)}_\xi$ and $-e_i \in S^{(-)}_\xi$, for all $i \in [n]$. The quantity $\xi$ captures the size of the local neighborhood — the smaller $\xi$ is, the larger the size of $S^{(i\pm)}_\xi$.

Due to symmetry, we focus on describing the geometry of $f_\phi(h)$ in one of such subsets, say $S^{(+)}_\xi$. For convenience, we introduce a reparametrization trick [10]. Define $w = h_{1:n-1} \in \mathbb{B}^{n-1}$, corresponding to the first $(n - 1)$-entries of $h$, where $\mathbb{B}^{n-1} := \{w : \|w\|_2 \leq 1\}$ is the unit ball in $\mathbb{R}^{n-1}$. Given $w$, the vector $h$ can be written as

$$h(w) = \left( w, \sqrt{1 - \|w\|_2^2} \right), \quad w \in \mathbb{B}^{n-1}.$$ \hfill (12)

Therefore, $w = 0$ is equivalently to $h(0) = e_n$, which is the shifted ground truth within $S^{(+)}_\xi$. The loss function $f_\phi(h)$ can be rewritten with respect to $w$ as

$$\phi_\phi(w) = f_\phi(h) = \frac{1}{p} \sum_{i=1}^{p} \psi_\mu(C(x_i)h(w)).$$ \hfill (13)

A short calculation reveals that, \footnote{When $h(w) \in S^{(+)}_\xi$, we have $h_w^2 \geq (1 + \xi)\|h_{\setminus \{i\}}\|_\infty^2$, which leads to $1 = \|h\|_2^2 \leq h_w^2 + (n - 1)\|h_{\setminus \{i\}}\|_\infty^2 \leq \left(1 + \frac{\xi}{1 + \xi}\right) h_w^2 = \left(1 + \frac{n - 1}{n + \xi}\right) (1 - \|w\|_2^2)$.}

$$\|w\|_2^2 \leq \frac{n - 1}{n + \xi} \quad \text{whenever} \quad h(w) \in S^{(+)}_\xi.$$ \hfill (14)
Theorem 2 (Geometry in the orthogonal case). Without loss of generality, suppose \( C(g) = I \). For any \( \xi_0 \in (0,1) \), there exist constants \( c_1, c_2, c_3, c_4, c_5, C \) such that when \( \mu < c_1 \min \{ \xi_0^{\frac{1}{6}} n^{-\frac{3}{4}} \} \) and

\[
p \geq C n^4 \log \left( \frac{n^3 \log^{3/2} p \log n}{\mu \xi_0} \right),
\]

the following holds with probability at least \( 1 - c_3 p^{-7} - \exp(-c_4 n) \) for \( h(w) \in S^{(n)}_{\xi_0} \):

\[
\begin{align*}
&\text{(large gradient)} \quad \frac{w^\top \nabla \phi_0(w)}{\|w\|_2} \geq c_2 \xi_0 \theta, \quad \text{if } w \in Q_1, \\
&\text{(strong convexity)} \quad \nabla^2 \phi_0(w) \geq \frac{c_2 n \theta}{\mu} I, \quad \text{if } w \in Q_2.
\end{align*}
\]

Furthermore, the function \( \phi_0(w) \) has exactly one unique local minimizer \( w_0^* \) near \( 0 \), such that

\[
\|w_0^* - 0\|_2 \leq c_5 \mu \sqrt{\frac{\log^2 p}{p}}.
\]

Theorem 2 has the following implications when \( h(w) \in S^{(n)}_{\xi_0} \), as long as the sample size \( p \) is sufficiently large and satisfies (16):

- The function \( \phi_0(w) \) either has a large gradient when \( \|w\|_2 \) is large (c.f. (17a)), or is strongly convex when \( \|w\|_2 \) is small (c.f. (17b)), indicating the geometry is rather benign and suitable for optimization using first-order methods such as MGD;

- There are no spurious local minima, and the unique local optimizer is close to the ground truth according to (18) with an error decays on the rate of \( O \left( \frac{p}{\sqrt{\log p}} \right) \) as the sample size \( p \) increases.

Theorem 2 also suggests that a larger sample size is necessary to guarantee a benign geometry when the subset \( S^{(n)}_{\xi_0} \) gets larger, when \( \xi_0 \) is set smaller. By a simple union bound, we can ensure a similar geometry applies to all \( 2^n \) subsets \( S^{(n)}_{\xi_0} \) defined in (11).

**Extension to the general case.** To extend the geometry in Theorem 2 to the general case when \( C(g) \) is invertible, we adopt the trick in [4, 10] and introduce the pre-conditioning matrix \( R \):

\[
R = \left[ \frac{1}{\theta n p} \sum_{i=1}^{p} C(y_i)^\top C(y_i) \right]^{-1/2}.
\]

The main purpose of the pre-conditioning is to convert the loss function to one similar to the orthogonal case studied above. Recall the loss function after pre-conditioning in (7), we define \( U \) as

\[
U := C(g) \mathbb{E}[R] = C(g) \left( C(g)^\top C(g) \right)^{-1/2},
\]

where the expectation is taken with respect to the randomness of \( \{x_i\}_{i=1}^{p} \). It is easy to check that \( U \) is a circulant orthonormal matrix, and consequently, following similar arguments as (10), we can rewrite (7) as

\[
\min_{h \in \mathbb{R}^n} f(h) = \frac{1}{p} \sum_{i=1}^{p} \psi_i(C(x_i)C(g)RU^\top h) \quad \text{s.t. } \|h\|_2 = 1,
\]

where the shifted and sign-flipped ground truth has been rotated to \( \{\pm e_i\}_{i=1}^{n} \), which is the same as the orthogonal case. The theorem below suggests that under the same reparameterization \( h = h(w) \) in (12), a similar geometry as Theorem 2 can be guaranteed for \( \phi(w) = f(h(w)) \).
the function holds for 

In state the assumptions of Theorem 3. For the MGD algorithm in Alg. 1, if the initialization 

Theorem 4. Instate the assumptions of Theorem 3. For the MGD algorithm in Alg. 1, if the initialization satisfies 

The next theorem demonstrates that with an initialization in one of the 

Theorem 3 demonstrates that a similar benign geometry to Theorem 2 can be guaranteed for the general case, as long as a proper pre-conditioning is applied, and the sample size is sufficiently large. In particular, the sample size (22) increases with the increase of the condition number of $C(g)$.

2.2 Convergence Guarantees of MGD

Owing to the benign geometry in the subsets of interest \( \left\{ S_{\xi_0}^{(i)} \right\} \) (defined in (11)), a simple MGD algorithm is presented and summarized in Alg. 1, where \( \partial f(h) = (I - hh^T) \nabla f(h) \) is the Riemannian manifold gradient with respect to \( h \), and \( \nabla f(h) \) is the Euclidean gradient of \( f(h) \).

The proposed MGD algorithm, with proper step size, will converge to the local minimizer \( w^* \) in that subset in a polynomial time.

\[ p \geq C \frac{\kappa^8 \theta^2 \log n \log^2 n}{\theta^4 \mu^2 \eta^2} , \]

(22)

the geometry (17) holds for \( \phi(w) \) with probability at least \( 1 - c_3 p^{-7} - \exp(-c_4 n) \) for \( h(w) \in S_{\xi_0}^{(n^+)}. \) In addition, the function \( \phi(w) \) has exactly one unique local minimizer \( w^* \) near 0, such that

\[ \|w^* - 0\|_2 \leq \frac{c_2 \kappa^4}{\theta^2} \sqrt{\frac{n \log p \log^2 n}{p}} \]

\[ \text{for } k = 0 \text{ to } T \text{ do } \]

\[ h^{(k+1)} \leftarrow \frac{h^{(k)} - \eta \cdot \partial f(h^{(k)})}{\|h^{(k)} - \eta \cdot \partial f(h^{(k)})\|_2} \]

Output: Return \( h^{(T)} \)

**Algorithm 1:** Manifold Gradient Descent for MSBD

Input: Observation \( \left\{ y_i \right\}_{i=1}^{p} \), sparsity \( \theta \), step size \( \eta \), initialization \( h^{(0)} \) on the sphere;

Theorem 3 (Geometry in the general case). Suppose \( C(g) \) is invertible with condition number \( \kappa \). For any \( \xi_0 \in (0, 1) \), \( \theta \in (0, \frac{1}{2}) \), there exist constants \( c_1, c_2, c_3, c_4, C \) such that when \( \mu < c_1 \min\{\theta, \xi_0^{1/6} n^{-3/4}\} \) and

\[ p \geq C \frac{\kappa^8 \theta^2 \log n \log^2 n}{\theta^4 \mu^2 \xi_0^2} , \]

(23)

the sample size (22) increases with the increase of the condition number of $C(g)$.
with $O(\log n)$ random initializations selected uniformly over the sphere, it is guaranteed to obtain a vector $h^{(T)}$ that satisfies

$$\text{dist}(h^{(T)}, g_{\text{inv}}) \lesssim \frac{n^4}{\theta^2} \sqrt{\frac{n \log^3 p \log^2 n}{p}} + \epsilon$$

for any $\epsilon > 0$, in $T \lesssim \frac{n^2 \log p}{n\theta^2} + \frac{n \log p}{\theta^2} \log \left( \frac{\mu}{\epsilon} \right)$ iterations.

Corollary 1 provides theoretical footings to the success of MGD for solving the highly nonconvex MSBD problem. In particular, consider the interesting regime when $\theta = O(1)$ and $\kappa = O(1)$, it is sufficient to set $\mu = O((\log n)^{-1/6} n^{-3/4})$, which leads to a sample size $p = O(n^{4.5})$ up to logarithmic factors. This significantly improves over the prior work of Li and Bresler [4], which requires a sample complexity of $p = O(n^3)$ up to logarithmic factors. See further discussions in Section 5.

3 Overview of the Analysis

In this section, we outline the proof of the main results, while leaving the details to the appendix. We first deal with the simpler case when $\mathcal{C}(g)$ is an orthonormal matrix employing the objective function $\phi_o(w)$ (i.e., $f_o(h)$) without pre-conditioning in Section 3.1, and then extend the analysis to the general case where the objective function $\phi(w)$ (i.e., $f(h)$) is pre-conditioned in Section 3.2. Finally, we discuss the convergence guarantee of MGD in Section 3.3.

3.1 Proof Outline of Theorem 2

The proof of Theorem 2 is divided into several steps.

1. First, we characterize the landscape of the population loss function $E[\phi_o(w)]$;

2. Second, we prove the pointwise concentration of the directional gradient and the Hessian of the empirical loss $\phi_o(w)$ around those of the population one $E[\phi_o(w)]$ in the region of interest;

3. Third, we extend such concentrations to the uniform sense, thus the benign geometric properties of $E[\phi_o(w)]$ carry over to the empirical version $\phi_o(w)$.

To begin, the lemma below describes the geometry of $E[\phi_o(w)]$, whose proof is given in Appendix B.1.

Lemma 1 (Geometry of the population loss in the orthogonal case). Without loss of generality, suppose $\mathcal{C}(g) = I$. For any $\xi_0 \in (0, 1)$, $n \in (0, \frac{1}{3})$, there exists some constant $c_1$ such that when $\mu < c_1 \min\{\theta, \xi_0^{1/6} n^{-3/4}\}$, we have for $h(w) \in S_{c_0}^{(n+1)}$:

\[
\begin{align*}
\text{(large directional gradient)} \quad \frac{w^T \nabla \phi_o(w)}{\|w\|_2} & \geq \frac{\xi_0 \theta}{480 \sqrt{10 \pi}}, \quad w \in Q_1, \quad (24a) \\
\text{(strong convexity)} \quad \nabla^2 \phi_o(w) & \geq \frac{n \theta}{5 \sqrt{2\pi\mu}} I, \quad w \in Q_2. \quad (24b)
\end{align*}
\]

To extend the benign geometry to the empirical loss with a finite sample size $p$, we first need to prove the pointwise concentration of these quantities around their expectations for a fixed $w$, using the Bernstein’s inequality. The next two propositions demonstrate the pointwise concentration results, whose proofs are provided in Appendix B.2 and B.3.

Proposition 1. For any $w$ satisfies $\|w\|_2 \leq \sqrt{\frac{n-1}{n}}$, there exists a universal constant $C$ such that for any $t > 0$:

$$\mathbb{P}\left[ \left| \frac{w^T \nabla \phi_o(w)}{\|w\|_2} - \frac{w^T \nabla \mathbb{E} \phi_o(w)}{\|w\|} \right| \geq t \right] \leq 2 \exp\left( \frac{-pt^2}{2Cn^3 \log(n) + 2\sqrt{Cn^3 \log(n)}t} \right).$$
Proposition 2. For any \( w \) satisfies \( \|w\|_2 \leq \frac{\mu}{4\sqrt{2}} \), there exists a universal constant \( C \) such that for any \( t > 0 \),

\[
P \left[ \| \nabla^2 \phi_o(w) - \nabla^2 \mathbb{E} \phi_o(w) \| \geq t \right] \leq 4n \exp \left( \frac{-p \mu^2 t^2}{9C^2 n^2 \log^2 n + 3C \mu n \log(n) t} \right).
\]

The concentration of the Hessian and directional gradient between the empirical and population objective functions at a fixed point suggests that the empirical objective function may inherit the benign geometry of the population one outlined in Lemma 1. However, one needs to carefully extend the pointwise concentrations in Proposition 1 and 2 through a covering argument, which requires bounding the Lipschitz constants of the Hessian and directional gradients. The rest of the proof of Theorem 2 is provided in Appendix B.4.

3.2 Proof Outline of Theorem 3

To extend the benign geometry to the general case, we show that through pre-conditioning, the landscape of \( \phi(w) \) is not too far from that of \( \phi_o(w) \). Recall that the pre-conditioned loss function (21) is

\[
\phi(w) = \frac{1}{p} \sum_{i=1}^{p} \psi_{\mu} \left( \mathcal{C}(x_i) \mathcal{C}(g) R U^\top h(w) \right)
= \frac{1}{p} \sum_{i=1}^{p} \psi_{\mu} \left( \mathcal{C}(x_i) \left[ I + \left( \mathcal{C}(g) R U^{-1} - I \right) \right] h(w) \right), \tag{25}
\]

where \( \Delta = \mathcal{C}(g) R U^{-1} - I = (U' - U) U^{-1} \) and \( U' = \mathcal{C}(g) R \). It is easy to find that \( \mathbb{E}[U'] = U \), where \( U \) is defined in (20). As \( R \) converges to \( \left[ \mathcal{C}(g)^\top \mathcal{C}(g) \right]^{-1/2} \) when \( p \) increases, we have \( U' \) converges to \( U \). Therefore, by bounding the size of \( \Delta \), we can control the deviation between \( \phi_o(w) \) and \( \phi(w) \). To this end, the rest of the proof is divided into two steps.

First, we show that the spectral norm of \( \Delta \) is bounded when the sample size is sufficiently large in Lemma 2, whose proof is given in Appendix C.1.

Lemma 2 (Spectral norm of \( \Delta \)). There exist some constants \( C_1, C_f \), such that when \( p \geq \frac{C_1 \kappa^4 \log^2(n) \log(p)}{\mu^2} \), with probability at least \( 1 - 2np^{-8} \),

\[
\|\Delta\| \leq C_f \kappa^4 \sqrt{\log^2 n \log p \over \mu^2 p}. \tag{26}
\]

Second, we show that the deviation between the directional gradient and the Hessian of \( \phi(w) \) and \( \phi_o(w) \) can be bounded by the spectral norm of \( \Delta \), as shown in Lemma 3. The proof can be found in Appendix C.2.

Lemma 3 (Deviation between \( \phi_o(w) \) and \( \phi(w) \)). There exist some constants \( C_g, c_h, C_1 \), such that when \( p \geq \frac{C_1 \kappa^4 \log^2(n) \log(p)}{\mu^2} \), with probability at least \( 1 - (np)^{-8} \), we have

\[
\|\nabla \phi_o(w) - \nabla \phi(w)\| \leq c_g \frac{n^{3/2} \log(np)}{\mu} \|\Delta\|, \quad w \in \mathcal{Q}_1, \tag{27a}
\]

\[
\|\nabla^2 \phi_o(w) - \nabla^2 \phi(w)\| \leq c_h \frac{n^{2.5} \log^{3/2}(np)}{\mu^2} \|\Delta\|, \quad w \in \mathcal{Q}_2. \tag{27b}
\]

To complete the proof of Theorem 3, we need to show that the perturbations of the Hessian and the gradient between \( \phi_o(w) \) and \( \phi(w) \) are sufficiently small, which hold as long as the sample size is sufficiently large, in view of Lemma 2. Consequently, we can propagate the benign geometry of \( \phi_o(w) \) in Theorem 2 to \( \phi(w) \). The complete proof is provided in Appendix C.3.

3.3 Proof Outline of Theorem 4

To capitalize on the benign geometry established in Theorem 3, one of the key arguments is to ensure that the iterates stay in the 2n subsets \( \mathcal{S}^{(i \pm 1)}_{\mathcal{Q}_0}, i \in [n] \) implicitly. This requires bounding properties of the directional gradient of \( f(h) \), supplied in the following lemma whose proof can be found in Appendix D.1.
Lemma 4 (Uniform concentration of the directional gradient). Instate the assumptions of Theorem 3. For $h \in H_k = \{ h : h \in S_{0}^{(n)} , h_k \neq 0 , h_n^2 / h_k^2 < 4 \} , \text{there exist some constants } c_a , c_b , C_1 , \text{such that with probability at least } 1 - \theta (np)^{-8} - 2 \exp (-c_a n) ,$

$$
\begin{align*}
\partial f ( h ) ^{\top} \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) & \geq \frac{c_0 \xi_0 \theta}{2} , \\
\| \partial f ( h ) \|_2 & \leq \| \nabla f ( h ) \|_2 \leq C_1 n \sqrt{\log ( np )} .
\end{align*}
$$

(28a)

(28b)

The following lemma, proved in Appendix D.2, then shows that the iterates of MGD will always stay in one of the subsets $\{ S_{0}^{(i \pm)} , i \in [ n ] \} \text{that it initializes in, as long as the sample complexity } p \text{is large enough and the step size is properly chosen.}$

Lemma 5 (Implicitly staying in the subsets). Instate the assumptions of Theorem 3. For the MGD algorithm in Alg. 1, if the initialization satisfies in the $h^{(0)} \in S_{0}^{(i \pm)} \text{for any } i \in [ n ] , \text{then if the step size satisfies}$

$$
\eta \leq \frac{c}{n^{\theta / 2} \sqrt{\log ( np )}} 
$$

for some small enough constant $c$ , the iterates $h^{(k)} , k = 1 , 2 , \cdots$ will stay in $S_{0}^{(i \pm)} . \text{The proof of Theorem 4 then follows by analyzing the convergence in two stages, corresponding to when}$

$$
\text{the iterates lie in the region with large directional gradients, and the region with strong convexity respectively.} 
$$

The details are given in Appendix D.3.

Till this point, the only left ingredient is to make sure a valid initialization can be obtained efficiently. By setting $\xi_0$ sufficiently small, it is known from the following lemma [13, Lemma 3] that the union of $\{ S_{0}^{(i \pm)} , i \in [ n ] \}$ is large enough to ensure a random initialization will land in it with a constant probability.

Lemma 6. [13, Lemma 3] When $\xi_0 = \frac{1}{4 \log n} , \text{an initialization selected uniformly at random on the sphere}$

$$
\text{lies in one of these } 2n \text{ subsets } \{ S_{0}^{(i \pm)} , i \in [ n ] \} \text{with probability at least } 1/2. 
$$

Finally, combining Lemma 6 and Theorem 4, by setting $\xi_0 = 1/(4 \log n) , \text{we can guarantee to recover } g_{inv}$

$$
\text{accurately up to global ambiguity with high probability, as long as Alg. 1 is initialized uniformly at random over the sphere with } O ( \log n ) \text{ times.} \text{This leads precisely to Corollary 1.} 
$$

4 Numerical Experiments

In this section, we examine the performance of the proposed approach with comparison to [4] , which is also based on MGD using a different loss function $L ( h ) = - \frac{1}{4p} \sum_{i=1}^{p} \| C ( y_i ) R h \|_2^4 \text{over the sphere, on both}$

$$
synthetic and real data. 
$$

4.1 Blind Deconvolution with Synthetic Data

We first compare the success rate of the proposed approach and that in [4] , following a similar simulation setup as in [4] . In each experiment, the sparse inputs are generated following BG ($\theta$ ) , and $C ( g ) \text{with specific}$

$$
\kappa \text{is synthesized by generating the DFT } \hat{g} \text{of } g \text{which is random with the following rules: 1) The DFT } \hat{g} \text{is}$

$$
symmetric to ensure that } g \text{is real, i.e., } \hat{g}_j = \hat{g}^*_{n+2-j} , \text{where } * \text{denotes the conjugate operation. 2) The gains}$

$$
of } \hat{g} \text{follow a uniform distribution on } [ 1 , \kappa ) , \text{and the phases of } g \text{follow a uniform distribution on } [ 0 , 2\pi ) . 
$$

In all experiments, we run MGD for no more than $T = 200 \text{iterations with a fixed step size of } \eta = 0.1 \text{and}$

$$
\text{apply backtracking line search for both methods for computational efficiency. For our formulation, we set } \mu = \min ( 10n^{-5/4} , 0.05 ) . \text{For each parameter setting, we conduct 10 Monte Carlo simulations to compute}$

$$
\text{the success rate. Recall that the desired } h \text{is a signed shifted version of } g_{inv} \text{, i.e., } C ( g ) h = \pm e_j \text{,}$

$$
( j \in [ n ] ) . \text{Therefore, to evaluate the accuracy of the output } h^{(T)} , \text{we compute } C ( g ) R h^{(T)} \text{\text{with the ground truth } g , \text{and}$

$$
declare that the recovery is successful if } \| C ( g ) R h^{(T)} \|_{\infty} / \| C ( g ) R h^{(T)} \|_2 > 0.99 . 
$$

Fig. 3 (a) and (d) show the success rate of the proposed approach and that in [4] \text{with respect to } n \text{and } p , \text{where } \theta = 0.3 \text{and } \kappa = 8 \text{are fixed. It can be seen that the proposed approach succeeds at a much smaller}$

$$
sample size, where } p \text{is smaller than } n . \text{This indicates possible room for improvements of our theory. Fig. 3}$$
Figure 3: Success rates of the proposed approach (first row) and the approach in [4] (second row) under various parameter settings.

(b) and (e) shows the success rate of the proposed approach and that in [4] with respect to $\theta$ and $p$, where $n = 64$ and $\kappa = 8$ are fixed. The proposed approach continues to work well even at a relatively high value of $\theta$ up to around 0.5. Finally, Fig. 3 (c) and (f) shows the success rate of the proposed approach and that in [4] with respect to $\kappa$ and $p$, where $n = 64$ and $\theta = 0.3$ are fixed. Again, the performance of the proposed approach is insensitive to the condition number $\kappa$ as long as the sample size $p$ is large enough. On the other end, the approach in [4] performs significantly worse than the proposed approach under the examined parameter settings.

4.2 Experiments on 2D Image Deconvolution

To further evaluate our method, we perform the task of blind image reconstruction and compare with [4]. Suppose multiple circulant convolutions $\{y_i\}_{i=1}^p$ (illustrated in Fig. 4 (a) for the RGB image and Fig. 4 (c) for the R channel only) of an unknown 2D image (illustrated in Fig. 4 (d), the Hamerschlag hall on the campus of CMU) and multiple sparse inputs $\{x_i\}_{i=1}^P$ (illustrated in Fig. 4 (b)) are observed. Here, the size of the observations is $n = 128 \times 128$, $\theta = 0.1$, and the number of observations $p = 1000$, which is significantly smaller than $n$.

We apply the proposed reconstruction method to each channel of the image, i.e. R, G, B, respectively using the corresponding channel of the observations $\{y_i\}_{i=1}^p$, and obtain the final recovery by summing up the recovered channels. For each channel, the recovered image is computed as $\hat{g} = F^{-1} \left[ F \left( R \hat{h} \right) \odot^{-1} \right]$, where $\hat{h}$ denotes the output of the algorithm, $F$ is the 2D DFT operator, and $x \odot^{-1}$ is the entry-wise inverse of a vector $x$. The second row of Fig. 4 shows the true image, final recovered image by our method and [4] (after aligning the shift and sign) in (d), (e) and (f) respectively. It shows that the proposed approach again obtains much better recovery than that in [4].
5 Related Work

In this section, we discuss some existing literature most related to ours, focusing on algorithms with provable guarantees.

**Provable blind deconvolution.** The problem of blind deconvolution with a single snapshot (or equivalently, channel) has been studied recently under different geometric priors such as sparsity and subspace assumptions on both the filter and the input, using convex and nonconvex optimization formulations [5, 7, 8, 14, 15, 16, 17, 18, 19]. With the presence of multiple channels, one expects to identify the filter with fewer prior assumptions. For the same problem as ours, Wang and Chi [20] proposed a linear program which has stringent requirements on the conditioning number of the filter. Other algorithms for multi-channel blind deconvolution include sparse spectral methods [21] and nonconvex regularization [22]. A different model called “sparse-and-short” deconvolution is studied in [6, 23].

The work of Li and Bresler [4] is the most related to ours, which considered the same problem by running perturbed manifold gradient descent with a random initialization, over a spherically constrained loss function based on $\ell_4$ norm maximization. However, the required sample complexity is significantly worse. Specifically, to reach a similar accuracy as ours, it requires $O(n^9)$ samples, while we only require $O(n^{4.5})$ samples ignoring logarithmic factors, leading to an order-of-magnitude improvement. One key observation is that the large sample complexity required in [4] is partially due to bounding the uniform concentration of the gradient and the Hessian of the empirical loss function around their population counterparts, which is sufficient but in fact not necessary to obtain the benign geometry. Indeed, to optimize the sample complexity, we only require the uniform concentration of directional gradient over a large region near the global minimizer, which can be guaranteed at a significantly reduced sample complexity. Second, this region is large enough such as with a logarithmic number of random initializations we are guaranteed to land into this region with high probability and recover the signal of interest via MGD. Last, we observe the proposed loss function also empirically outperforms the $\ell_4$ norm used in [4]. At the time of finishing this paper, we became aware of another concurrent work [24], which obtains $O(n^4)$ sample complexities using a different loss function. In addition, a refinement procedure is proposed [24] to allow exact recovery of the filter.
Provable dictionary learning. Learning a specifying invertible transform from data has been extensively studied, e.g. in [25, 11, 12, 13, 26, 27, 28]. In addition, provable algorithms for learning overcomplete dictionaries are also proposed in [29, 30, 31, 32]. Our problem can be regarded as learning a convolutional invertible transform, where the proposed algorithm is inspired by the approach in [12] that characterizes a local region large enough for the success of gradient descent with random initializations. However, the approach in [12] is only applicable to an orthogonal dictionary, while we deal with a general invertible convolutional kernel. Compared to sample complexities required in learning complete dictionaries [25], our result demonstrates the benefit of exploiting convolutional structures in reducing the sample complexity.

Provable nonconvex statistical estimation. Our work belongs to the recent line of activities of designing provable nonconvex procedures for high-dimensional statistical estimation, see [33] for an overview. Our approach interpolates between two popular approaches, namely, global analyses of optimization landscape (e.g. [25, 10, 34, 35, 36, 37, 38, 39, 4]) that are independent of algorithmic choices, and local analyses with careful initializations and local updates (e.g. [40, 41, 42, 43, 44, 15, 45, 46]).

6 Discussions

This paper proposes a novel nonconvex approach for multi-channel sparse blind deconvolution based on manifold gradient descent with random initializations. Under a Bernoulli-Subgaussian model for sparse inputs, we demonstrate that the proposed approach succeeds as long as the sample complexity satisfies $p = O(n^{4.5} \text{polylog} p)$, a result significantly improving prior art in [4]. We conclude the paper by some discussions on future directions.

- **Improve sample complexity.** Our numerical experiments indicate that there is still room to further improve the sample complexity of the proposed algorithm, which may require a more careful analysis of the trajectory of the gradient descent iterates, as done in [47].

- **Efficient exploitation of negative curvature.** We remark that it is possible to characterize the global geometry over the sphere, where the remaining region contains saddle points with negative curvatures. However, a direct analysis leads to an increase of sample complexity which is undesirable and therefore not pursued in this paper. On the other end, it seems random initialization *without restarts* also works well in practice, which warrants further investigation.

- **Super-resolution blind deconvolution.** The model studied in this paper assumes the same temporal resolution of the input and the output, while in practice the sparse activations of the input can occur at a much higher resolution. This lead to the consideration of a refined model, where the observation is given as $y = F_{n \times D} \text{diag}(\hat{g}) F_{n \times D} x$, where $F_{n \times D}$ is the oversampled DFT matrix of size $n \times D$, $D \geq n$. The approach taken in this paper cannot be applied anymore, and new formulations are needed to address this problem.

- **Convolutional dictionary learning.** Our work can be regarded as a first step towards developing sample-efficient algorithms for convolutional dictionary learning [48] with performance guarantees. An interesting model for future investigation is when multiple filters are present, and the observation is modeled as $y = \sum_{\ell=1}^{L} C(g_\ell)x_\ell$, with $L$ the number of filters. The goal is thus to simultaneously learn multiple filters $\{g_\ell\}_{\ell=1}^{L}$ from a number of observations in the form of $y$.

References


A Prerequisites

Before beginning, we first introduce some additional notations and useful facts. For convenience, let $X \in \mathbb{R}^{n \times p}$ denote the inputs $X = [x_1, x_2, \ldots, x_p]$. Denote the first-order derivative of $\psi_\mu(x)$ as $\psi_\mu'(x) = \tanh(x/\mu)$ and the second-order derivative as $\psi_\mu''(x) = \left(1 - \tanh^2(x/\mu)\right)/\mu$. The gradient of $\psi_\mu(\mathcal{C}(x_i)\mathbf{h})$ with respect to $\mathbf{h}$ can be written as

$$\nabla_\mathbf{h} \psi_\mu(\mathcal{C}(x_i)\mathbf{h}) = \mathcal{C}(x_i)^\top \tanh\left(\frac{\mathcal{C}(x_i)\mathbf{h}}{\mu}\right),$$

where with slightly abuse of notation, we allow $\tanh(\cdot)$ to take a vector-value in an entry-wise manner.

Recall the reparameterization $\mathbf{h} = \mathbf{h}(\mathbf{w}) = \left(\mathbf{w}, \sqrt{1 - \|\mathbf{w}\|^2_2}\right)$, we obtain

$$\phi_o(\mathbf{w}) = f_o(\mathbf{h}(\mathbf{w})) = \frac{1}{p} \sum_{i=1}^{p} \psi_\mu(\mathcal{C}(x_i)\mathbf{h}(\mathbf{w})).$$

In addition, let $J_h(\mathbf{w})$ be the Jacobian matrix of $\mathbf{h}(\mathbf{w})$, i.e.

$$J_h(\mathbf{w}) = \left[ I_n - \frac{\mathbf{w}}{h_n(\mathbf{w})} \right] \in \mathbb{R}^{(n-1) \times n},$$

where $h_n(\mathbf{w}) = \sqrt{1 - \|\mathbf{w}\|^2_2}$. 

---


where \( h_n(w) = \sqrt{1 - \|w\|_2^2} \) is the last entry of \( h(w) \). By the chain rule, the gradient of \( \psi_\mu(C(x_i)h(w)) \) with respect to \( w \) is given as

\[
\nabla_w \psi_\mu(C(x_i)h(w)) = J_h(w)\nabla_h \psi_\mu(C(x_i)h) = J_h(w) \cdot C(x_i)^\top \tanh \left( \frac{C(x_i)h(w)}{\mu} \right).
\]

Moreover, the Hessian of \( \psi_\mu(C(x_i)h(w)) \) is given as

\[
\nabla_w^2 \psi_\mu(C(x_i)h(w)) = \frac{1}{\mu} J_h(w)C(x_i)^\top \left[ I - \text{diag} \left( \tanh^2 \left( \frac{C(x_i)h(w)}{\mu} \right) \right) \right] C(x_i)J_h(w)^\top
\]

\[\]

- \[\frac{1}{h_n} S_{n-1}(x_i)^\top \tanh \left( \frac{C(x_i)h(w)}{\mu} \right) J_h(w)J_h(w)^\top.\]

### A.1 Useful concentration inequalities

We first introduce some notations and properties about sub-Gaussian variables. A random variable \( X \) is called sub-Gaussian if its sub-Gaussian norm satisfies \( \|X\|_{\psi_2} < \infty \) [49]. Similarly, we have \( \|x\|_{\psi_2} < \infty \) for a sub-Gaussian random vector \( x \). For Bernoulli-Gaussian random variables / vectors, we have the following two facts, which imply that they are also sub-Gaussian.

**Fact 1.** [12, Lemma F.1] A Bernoulli-Gaussian random variable \( X \in \text{BG}(\theta) \) is sub-Gaussian, i.e. there exists some constant \( C_\theta \) such that \( \|X\|_{\psi_2} \leq C_\theta \). Similarly, for a Bernoulli-Gaussian random vector \( x \sim \text{BG}(\theta) \) and any deterministic vector \( v \in \mathbb{R}^n \), we have \( \|v^\top x\|_{\psi_2} \leq C_\theta \|v\|_2 \).

**Fact 2.** [10, Lemma 21] Assume \( x, y \in \mathbb{R}^n \) satisfy \( x \sim \text{BG}(\theta) \) and \( y \sim \text{BG}(\theta) \). Then for any deterministic vector \( v \in \mathbb{R}^n \), we have \( \mathbb{E}(\|v^\top x\|^m) \leq \mathbb{E}(\|v^\top y\|^m) \), and \( \mathbb{E}(\|x\|^m) \leq \mathbb{E}(\|y\|^m) \) for all integers \( m \geq 1 \).

The second fact allows us to bound the moments of a Bernoulli-Gaussian vector via the moments of a Gaussian vector, which are given below.

**Lemma 7.** [10, Lemma 35] Let \( y \in \mathbb{R}^n \) be \( y \sim \text{BG}(\theta) \), we have for any \( m \geq 1 \), \( \mathbb{E}(\|y\|^m) \leq m! \theta^m \).

**Lemma 8.** [10, Lemma 34] Let \( y \in \mathbb{R}^n \) be \( y \sim \text{BG}(\theta) \), we have for any \( m \geq 1 \), \( \mathbb{E}(\|y\|^m) \leq m! (2\theta)^m \).

In addition, let us list a few more useful facts about sub-Gaussian random variables.

**Fact 3.** [49, Lemma 2.6.8] If \( X \) is sub-Gaussian, then \( X - \mathbb{E}X \) is also sub-Gaussian with \( \|X - \mathbb{E}X\|_{\psi_2} \leq C \|X\|_{\psi_2} \) for some constant \( C \).

**Fact 4.** [49, Proposition 2.6.1] If \( X_1, X_2, \ldots, X_n \) are zero-mean independent sub-Gaussian random variables, then there exists some constant \( C \) such that \( \|\sum_{i=1}^n X_i\|_{\psi_2}^2 \leq C \sum_{i=1}^n \|X_i\|_{\psi_2}^2 \).

**Fact 5.** [49, Eq. (2.14-2.15)] If \( X \) is sub-Gaussian, it satisfies the following bounds:

\[
\mathbb{P}(\|X\| \geq t) \leq 2 \exp \left( -ct^2/\|X\|_{\psi_2}^2 \right)
\]

for all \( t \geq 0 \), \( \mathbb{E}(\|X\|^m) \leq C \|X\|_{\psi_2} \sqrt{m} \) for all \( m \geq 1 \).

Combining standard tail bounds with union bounds, we have the following facts.

**Fact 6.** For independent sub-Gaussian vectors \( \{x_i\}_{i=1}^p \in \mathbb{R}^n \) with \( \|x_i\|_{\psi_2} \leq B \), \( i = 1, \ldots, p \), for some constant \( B \), there exists constant \( C \) such that with probability at least \( 1 - p^{-8} \), we have

\[
\max_{i \in [p]} \|x_i\| \leq C \sqrt{n \log p}.
\]

**Fact 7.** For \( X \in \mathbb{R}^{n \times p} \) with \( X \sim \text{iid \, BG}(\theta) \), \( \theta \in (0, 1/2) \), with probability at least \( 1 - \theta(np)^{-7} \), we have

\[
\|X\|_{\infty} \leq 4 \sqrt{\log(np)}.
\]
Finally, let us record the useful Bernstein’s inequality for random vectors and matrices, which does not require the quantities of interest to be centered. This is a direct consequence of Fact 3 on centering and [50, Theorem 6.2].

**Lemma 9.** (Moment-controlled Bernstein’s inequality) Let \( \{X_k \in \mathbb{R}^{m \times n}\}_{k=1}^p \) be a set of independent random matrices. Assume there exist \( \sigma, R \) such that for all \( m \geq 2, \mathbb{E}(\|X_k\|^m) \leq \frac{m!}{2} \sigma^2 R^{m-2} \). Denote \( S = \frac{1}{p} \sum_{k=1}^p X_k \), then we have for any \( t > 0 \),

\[
\mathbb{P}(\|S - \mathbb{E}(S)\| > t) \leq 2n \exp\left( \frac{-pt^2}{2\sigma^2 + 2Rt} \right).
\]

Let \( \{x_k \in \mathbb{R}^n\}_{k=1}^p \) be a set of independent random vectors. Assume there exist \( \sigma, R \) such that \( \mathbb{E}(\|x_k\|^m) \leq \frac{m!}{2} \sigma^2 R^{m-2} \). Denote \( s = \frac{1}{p} \sum_{k=1}^p x_k \), then we have for any \( t > 0 \),

\[
\mathbb{P}(\|s - \mathbb{E}(s)\|_2 > t) \leq 2(n+1) \exp\left( \frac{-pt^2}{2\sigma^2 + 2Rt} \right).
\]

### A.2 Technical lemmas

In this section, we provide some technical lemmas that are used throughout the proof. We start with some useful properties about the \( \tanh(\cdot) \) function since it appears frequently in our derivation.

**Lemma 10.** Let \( X \sim \mathcal{N}(0, \sigma_x^2), Y \sim \mathcal{N}(0, \sigma_y^2) \), then we have

\[
\mathbb{E}[\tanh(aX)X] = a \sigma_x^2 \mathbb{E}[1 - \tanh^2(aX)]
\]

\[
\mathbb{E}[\tanh(aX + Y)X] = a \sigma_x^2 \mathbb{E}[1 - \tanh^2(a(X + Y))]
\]

**Proof.** Using integration by part, we have

\[
\mathbb{E}[\tanh(aX)X] = \frac{1}{\sqrt{2\pi} \sigma_x} \int_{-\infty}^{\infty} \tanh(aX)X \exp\left( \frac{-X^2}{2\sigma_x^2} \right) dX
\]

\[
= -\frac{1}{\sqrt{2\pi} \sigma_x} \cdot 2\sigma_x \tanh(aX) \exp\left( \frac{-X^2}{2\sigma_x^2} \right) \bigg|_0^{\infty} + \frac{1}{\sqrt{2\pi} \sigma_x} \int_{-\infty}^{\infty} \sigma_x^2 (1 - \tanh^2(aX)) \exp\left( \frac{-X^2}{2\sigma_x^2} \right) dX
\]

\[
= a \sigma_x^2 \mathbb{E}[1 - \tanh^2(aX)]
\]

and

\[
\mathbb{E}[\tanh(aX + Y)X] = \frac{1}{2\sigma_x \sigma_y} \int_{-\infty}^{\infty} X \exp\left( \frac{-X^2}{2\sigma_x^2} \right) \int_{-\infty}^{\infty} \tanh(aX + Y) \exp\left( \frac{-Y^2}{2\sigma_y^2} \right) dY dX
\]

\[
= -\frac{1}{2\sigma_x \sigma_y} \cdot \sigma_x^2 \left[ \int_{-\infty}^{\infty} \tanh(aX + Y) \int_{-\infty}^{\infty} \exp\left( \frac{-Y^2}{2\sigma_y^2} \right) dY \exp\left( \frac{-X^2}{2\sigma_x^2} \right) \right]_{-\infty}^{\infty}
\]

\[
+ \frac{1}{2\sigma_x \sigma_y} \int_{-\infty}^{\infty} \sigma_x^2 (1 - \tanh^2(a(X + Y))) \exp\left( \frac{-X^2}{2\sigma_x^2} \right) \exp\left( \frac{-Y^2}{2\sigma_y^2} \right) dY dX
\]

\[
= a \sigma_x^2 \mathbb{E}[1 - \tanh^2(a(X + Y))].
\]

**Lemma 11.** \( \psi'_\mu(x) = \tanh(x/\mu) \) and \( \psi''_\mu(x) = \psi'''_\mu(x) = (1 - \tanh^2(x/\mu)) / \mu \) are Lipschitz continuous with Lipschitz constants \( 1/\mu \) and \( 2/\mu^2 \), respectively.

**Proof.** Since \( \psi'_\mu(x) \) is continuous and third-order differentiable, we have for any \( x \) and \( x' \),

\[
|\psi'_\mu(x) - \psi'_\mu(x')| \leq \int_x^{x'} |\psi''_\mu(z)| dz \leq |x - x'| \max_z |\psi''_\mu(z)| \leq \frac{|x - x'|}{\mu},
\]

\[
|\psi''_\mu(x) - \psi''_\mu(x')| \leq \int_x^{x'} |\frac{d^3 \psi_\mu(z)}{dz^3}| dz \leq \int_x^{x'} \left| -\frac{2}{\mu^3} \tanh\left( \frac{z}{\mu} \right) \cdot \left( 1 - \tanh^2\left( \frac{z}{\mu} \right) \right) \right| dz \leq \frac{2|x - x'|}{\mu^2},
\]

where we use the fact that \( |\tanh(x)| \leq 1 \) and \( 1 - \tanh^2(x) \leq 1 \) for all \( x \in \mathbb{R} \).
Lemma 12. Let $x \sim_{iid} \text{BG} (\theta)$ for $\theta \in (0, 1]$. There exists some constants $c_1$ and $c_2$, such that
\[
P \left( \|C(x)\| \geq t \right) \leq 2n \exp \left( - \frac{t^2}{c_1 n} \right), \quad \text{and} \quad \mathbb{E} \|C(x)\|^{2m} \leq \frac{m!}{2} \left( c_2 n \log n \right)^m
\]
for all $m \geq 1$.

Proof. Since a circulant matrix is diagonalizable by the DFT matrix, the spectral norm of $C(x)$ is the maximum magnitude of the DFT coefficients of $x$, where the $i$th coefficient is given as $\hat{x}_i = f_i^\top x$, where $f_i = [1, e^{2\pi i/n}, \ldots, e^{2\pi in(n-1)/n}]^\top$ is the $i$th column of the DFT matrix. Since $x \sim_{iid} \text{BG}(\theta)$ is sub-Gaussian, by Fact 1, $\hat{x}_i$ is also sub-Gaussian with $\|\hat{x}_i\|_{\psi_2} \leq C\|f_i\|_2 = C\sqrt{n}$. Therefore, by the union bound, together with Fact 5, we have
\[
P \left( \|C(x)\| \geq t \right) = \mathbb{P} \left( \max_{i \in [n]} |\hat{x}_i| \geq t \right) \leq 2n \exp \left( - \frac{t^2}{c_1 n} \right),
\]
for some constant $c_1$. Equipped with the above bound, we can bound the moments of $\|C(x)\|^2$, where
\[
\mathbb{E} \|C(x)\|^{2m} = \int_0^\infty \mathbb{P}(\|C(x)\|^{2m} > u) du = \int_0^\infty \mathbb{P}(\|C(x)\| > t) \cdot 2mt^{2m-1} dt,
\]
where the second equality follows by a change of variable $t = u^{1/2m}$. To continue, we break the bound as
\[
\mathbb{E} \|C(x)\|^{2m} \leq \int_{0}^{\sqrt{c_1 n \log n}} 1 \cdot 2mt^{2m-1} dt + \int_{\sqrt{c_1 n \log n}}^{\infty} 2n \exp \left( - \frac{t^2}{2c_1 n} \right) 2mt^{2m-1} dt \\
\leq \left( 4c_1 n \log n \right)^m + \int_{0}^{\infty} \exp \left( - \frac{t^2}{2c_1 n} \right) 2mt^{2m-1} dt \\
= \left( 4c_1 n \log n \right)^m + (2c_1 n)^m m! \leq \frac{m!}{2} \left( c_2 n \log n \right)^m,
\]
where the second line used the fact $\exp \left( - \frac{t^2}{2c_1 n} \right) > 2n \exp \left( - \frac{t^2}{c_1 n} \right)$ when $t \geq \sqrt{2c_1 n \log n}$, and the third line used the definition of the Gamma function. The proof is completed.

Lemma 13. Let $\{x_i\}_{i=1}^p \in \mathbb{R}^n$ be drawn according to $x_i \sim_{iid} \text{BG} (\theta)$, $\theta \in (0, 1/2)$. There exists some constant $C$, such that
\[
\left\| \frac{1}{\theta np} \sum_{i=1}^p C(x_i)^\top C(x_i) - I \right\| \leq C \sqrt{\frac{\log^2(n) \log(p)}{\theta^2 p}}
\]
holds with probability at least $1 - 2np^{-8}$.

Proof. By assumption, it is easy to check
\[
\mathbb{E} \left[ \frac{1}{\theta np} \sum_{i=1}^p C(x_i)^\top C(x_i) \right] = \mathbb{E} \left[ \frac{1}{\theta n} C(x_1)^\top C(x_1) \right] = I.
\]
The remaining of the proof is to verify the quantities needed to apply Lemma 9. Specifically, we bound the $m$th-moment of $\frac{1}{\theta n} C(x_1)^\top C(x_1)$ as
\[
\mathbb{E} \left\| \frac{1}{\theta n} C(x_i)^\top C(x_i) \right\|^m = \frac{1}{\theta^m n^m} \mathbb{E} \|C(x_i)\|^{2m} \leq \frac{m!}{2} \left( \frac{c \log(n)}{\theta} \right)^m,
\]
where the last line comes from Lemma 12. Let $\sigma^2 = \frac{\log^2(n)}{\theta^2}$, $R = \frac{C \log(n)}{\theta}$ in Lemma 9, we have
\[
P \left( \left\| \frac{1}{\theta np} \sum_{i=1}^p C(x_i)^\top C(x_i) - I \right\| \geq t \right) \leq 2n \exp \left( \frac{-p\theta^2 t^2}{2C^2 \log^2(n) + 2C \log(n) \theta t} \right),
\]
Setting $t = c\sqrt{\frac{\log^2(n) \log(p)}{\theta^2 p}}$, we complete the proof.
B Proofs for Section 3.1

B.1 Proof of Lemma 1

Recall the two regions introduced in (15):

\[ Q_{1} := \left\{ w : \frac{\mu}{\sqrt{2}} \leq \|w\|_2 \leq \sqrt{\frac{n - 1}{n + \xi_0}} \right\}, \quad Q_{2} := \left\{ w : \|w\|_2 \leq \frac{\mu}{\sqrt{2}} \right\}. \]

We further divide \( Q_{1} \) into two subregions,

\[ R_{0} = \left\{ w : \frac{\mu}{\sqrt{2}} \leq \|w\|_2 \leq \frac{1}{20\sqrt{5}} \right\}, \quad R_{1} = \left\{ w : \frac{1}{20\sqrt{5}} \leq \|w\|_2 \leq \sqrt{\frac{n - 1}{n + \xi_0}} \right\}, \]

which we will prove the desired bound separately.

Note that

\[ \mathbb{E} (\phi_{\omega}(w)) = n \cdot \mathbb{E} (\psi_{\mu}(x^\top h(w))), \tag{37} \]

since every row of \( C(x) \) has the same distribution as \( x \sim_{iid} \text{BG}(\theta) \). Therefore, the strong convexity bound (24b) in \( Q_{2} \) follows directly from the following lemma from [10, Proposition 8] by a multiplication factor of \( n \).

Lemma 14. [10, Proposition 8] For any \( \theta \in (0, 1/2) \), if \( \mu \leq \frac{1}{20\sqrt{n}} \), it holds for all \( w \) with \( \|w\|_2 \leq \frac{\mu}{\sqrt{2}} \) that

\[ \nabla^2_{ww} \mathbb{E} [\psi_{\mu}(x^\top h(w))] \succeq \frac{\theta}{\sqrt{2\pi}} I. \]

Similarly, by the following lemma from [10, Proposition 7], we have the desired bound (24a) in \( R_{0} \).

Lemma 15. [10, Proposition 7] For any \( \theta \in (0, 1/3) \), if \( \mu \leq 9/50 \), it holds for all \( w \in R_{0} \) such that

\[ \frac{w^\top \nabla_w \mathbb{E} (\psi_{\mu}(x^\top h(w)))}{\|w\|_2} \geq \frac{\theta}{20\sqrt{2\pi}}. \]

Therefore, the remainder of the proof is to show that (24a) also applies to \( R_{1} \). To ease presentation, we introduce a few short-hand notations. For \( x = \Omega \odot z \sim_{iid} \text{BG}(\theta) \in \mathbb{R}^n \), we denote the first \( n - 1 \) dimension of \( x, z \) and \( \Omega \) as \( \bar{x}, \bar{z} \) and \( \bar{\Omega} \), respectively. Denote \( I \) as the support of \( \Omega \) and \( J \) as the support of \( \bar{\Omega} \).

Plugging in (32), we rewrite the directional gradient as following:

\[
\mathbb{E} \left[ \frac{w^\top \nabla_w \psi_{\mu}(x^\top h(w))}{\|w\|_2^2} \right] = \frac{1}{\|w\|_2^2} \mathbb{E} \left[ \frac{x^\top \cdot h(w)}{\mu} \cdot \left( \frac{w^\top \bar{x} - \frac{x_n \|w\|_2^2}{h_n}}{h_n} \right) \right]
\]

\[
= (1 - \theta) \frac{\mathbb{E}_\bar{x} \left[ \tanh \left( \frac{w^\top \bar{x}}{\mu} \right) \right]}{\|w\|_2^2} \mathbb{E}_\bar{x, z_n} \left[ \tanh \left( \frac{w^\top \bar{x} + h_n z_n}{\mu} \right) \right] + \theta \mathbb{E}_\bar{x, z_n} \left[ \tanh \left( \frac{w^\top \bar{x} - \frac{w^2}{h_n} z_n}{\mu} \right) \right], \tag{38}
\]

where the second line is expanded over the distribution of \( \Omega_n \sim \text{Bernoulli}(\theta) \). Conditioned on the support of \( \bar{\Omega} \), we have \( X = w^\top \bar{x} | \bar{\Omega} \sim \mathcal{N}(0, \|w\|_2^2) \). Moreover, denote \( Y = h_n z_n \sim \mathcal{N}(0, h_n^2) \). Therefore, invoking Lemma 10, we can express \( I_1 \) and \( I_2 \) respectively as

\[
I_1 = \mathbb{E}_\bar{\Omega} \left[ \mathbb{E}_X \left( \tanh \left( \frac{X}{\mu} \right) \right) \right] = \frac{1}{\mu} \mathbb{E}_\bar{\Omega} \left[ \|w\|_2^2 \mathbb{E}_X \left( 1 - \tanh^2 \left( \frac{X}{\mu} \right) \right) \right], \]

\[
I_2 = \mathbb{E}_\bar{\Omega} \left[ \mathbb{E}_{X,Y} \left( \tanh \left( \frac{X + Y}{\mu} \right) \right) \left( X - \frac{\|w\|_2^2}{h_n^2} Y \right) \right] = \frac{1}{\mu} \mathbb{E}_\bar{\Omega} \left[ \left( \|w\|_2^2 - \frac{\|w\|_2^2}{h_n^2} \right) \mathbb{E}_{X,Y} \left( 1 - \tanh^2 \left( \frac{X + Y}{\mu} \right) \right) \right].
\]

Plugging the above equalities back into (38), and using \( \|w\|_2^2 = \sum_{i=0}^{n-1} w_i^2 I\{\Omega_i = 1\} \), \( \|w\|_2^2 = \sum_{i=0}^{n-1} w_i^2 I\{\Omega_i = 0\} \), we arrive at

\[
\mathbb{E} \left[ \frac{w^\top \nabla_w \psi_{\mu}(x^\top h(w))}{\|w\|_2^2} \right] = \frac{(1 - \theta)}{\mu \|w\|_2^2} \mathbb{E}_\bar{\Omega} \left[ \sum_{i=1}^{n-1} w_i^2 \cdot I\{\Omega_i = 1\} \cdot \mathbb{E}_z \left( 1 - \tanh^2 \left( \frac{w^\top \bar{x}_{\{i\}} + w_i z_i}{\mu} \right) \right) \right]
\]

20
where $Q_i$ is written as

$$Q_i = (1 - \theta)E_\Omega \left[ 1 \cdot \mathbb{E}_z \left( 1 - \tanh^2 \left( \frac{w_{i}^T \hat{x}_{(i)} + w_i z_i}{\mu} \right) \right) \right] - \theta E_\Omega \left[ 1 \cdot \mathbb{E}_z \left( 1 - \tanh^2 \left( \frac{w_{i}^T \hat{x}_{(i)} + h_n z_n}{\mu} \right) \right) \right].$$

Evaluating $E_\Omega$ over $\Omega \setminus \{i\}$ and $\Omega_i$ sequentially, and combining terms, we can rewrite $Q_i$ as,

$$Q_i = (1 - \theta)\theta \cdot E_{\Omega \setminus \{i\}} \left[ \mathbb{E}_z \left( 1 - \tanh^2 \left( \frac{w_{i}^T \hat{x}_{(i)} + w_i z_i}{\mu} \right) \right) \right] - \mathbb{E}_z \left( 1 - \tanh^2 \left( \frac{w_{i}^T \hat{x}_{(i)} + h_n z_n}{\mu} \right) \right]$$

Our goal is to lower bound $E_{\Omega}[K]$ first. Let $X := w_{i}^T \hat{x}_{(i)} + w_i z_i | \bar{\Omega} \sim \mathcal{N}(0, \|w_{\mathcal{J}\setminus(i)}\|_2^2 + w_i^2) := \mathcal{N}(0, \sigma_X^2)$ and $Y := w_{i}^T \hat{x}_{(i)} + h_n z_n | \bar{\Omega} \sim \mathcal{N}(0, \|w_{\mathcal{J}\setminus(i)}\|_2^2 + h_n^2) := \mathcal{N}(0, \sigma_Y^2)$. By the fundamental theorem of calculus, we have

$$K = \tanh^2 \left( \frac{Y}{\mu} \right) - \tanh^2 \left( \frac{X}{\mu} \right)$$

$$= \frac{2}{\mu} \int_{|X|}^{|Y|} \tanh \left( \frac{x}{\mu} \right) \cdot \left( 1 - \tanh^2 \left( \frac{x}{\mu} \right) \right) dx$$

$$\geq \frac{2}{\mu} \int_{|X|}^{|Y|} \left[ 2 \exp \left( \frac{-2x}{\mu} \right) - \exp \left( \frac{-4x}{\mu} \right) \right] \left[ 1 - 2 \exp \left( \frac{-2x}{\mu} \right) \right] dx$$

$$\geq \frac{2}{\mu} \int_{|X|}^{|Y|} \left[ 2 \exp \left( \frac{-2x}{\mu} \right) - 5 \exp \left( \frac{-4x}{\mu} \right) \right] dx$$

$$= 2 \left[ \exp \left( \frac{-2|X|}{\mu} \right) - \exp \left( \frac{-2|Y|}{\mu} \right) \right] - \frac{5}{2} \left[ \exp \left( \frac{-4|X|}{\mu} \right) - \exp \left( \frac{-4|Y|}{\mu} \right) \right],$$

where the third line follows from the bounds $2 \exp(-2x/\mu) - \exp(-4x/\mu) \leq 1 - \tanh^2 (x/\mu)$ and $\tanh(x/\mu) \leq 1 - \exp(-2x/\mu)$ in [10, Lemma 29]. To continue, we record the lemma rephrased from [10, Lemma 32, 40] and obtain the following lemma by directly repeating integration by parts.

Lemma 16. [10, Lemma 32, 40] Let $X \sim \mathcal{N}(0, \sigma_X^2)$. For any $a > 0$, we have

$$\frac{1}{\sqrt{2\pi}} \left( \frac{1}{a \sigma_X} - \frac{1}{a^3 \sigma_X^3} + \frac{3}{a^5 \sigma_X^5} - \frac{15}{a^7 \sigma_X^7} \right) \leq \mathbb{E} \left[ \exp(-aX) \mathbb{I} \{X > 0\} \right] \leq \frac{1}{\sqrt{2\pi}} \left( \frac{1}{a \sigma_X} - \frac{1}{a^3 \sigma_X^3} + \frac{3}{a^5 \sigma_X^5} \right).$$

Therefore, $K_1$ can be bounded as

$$K_1 = 2\mathbb{E} \left[ \exp \left( \frac{-2|X|}{\mu} \right) - \exp \left( \frac{-2|Y|}{\mu} \right) \right].$$
where the second inequality uses the fact

\[ \sigma - 1 \]

Plugging the above bounds back into (41), we have

\[ \frac{2}{\sqrt{2\pi}} \left[ \left( \frac{\mu}{\sigma_X} - \frac{\mu}{\sigma_Y} \right) \left( \frac{\mu^3}{4\sigma_X^3} - \frac{\mu^3}{4\sigma_Y^3} \right) + \left( \frac{3\mu^5}{16\sigma_X^5} - \frac{3\mu^5}{16\sigma_Y^5} \right) - \frac{15\mu^7}{2^8\sigma_Y^7} \right]. \]

Similarly, we have

\[ K_2 \leq \frac{5}{4\sqrt{2\pi}} \left[ \left( \frac{\mu}{\sigma_X} - \frac{\mu}{\sigma_Y} \right) \left( \frac{\mu^3}{4\sigma_X^3} - \frac{\mu^3}{4\sigma_Y^3} \right) + \left( \frac{3\mu^5}{16\sigma_X^5} - \frac{3\mu^5}{4\sigma_Y^5} \right) - \frac{15\mu^7}{4^6\sigma_Y^7} \right]. \]

Plugging the above bounds back into (41), we have

\[ \mathbb{E}_\xi [K] \geq \mathbb{E}_{X,Y} [K_1 - K_2] \]

\[ \geq \frac{2}{\sqrt{2\pi}} \left[ \left( \frac{\mu}{\sigma_X} - \frac{\mu}{\sigma_Y} \right) \left( \frac{\mu^3}{4\sigma_X^3} - \frac{\mu^3}{4\sigma_Y^3} \right) + \left( \frac{3\mu^5}{16\sigma_X^5} - \frac{3\mu^5}{16\sigma_Y^5} \right) - \frac{15\mu^7}{2^8\sigma_Y^7} \right] \]

\[ - \frac{5}{4\sqrt{2\pi}} \left[ \left( \frac{\mu}{\sigma_X} - \frac{\mu}{\sigma_Y} \right) \left( \frac{\mu^3}{16\sigma_X^3} - \frac{\mu^3}{16\sigma_Y^3} \right) + \left( \frac{3\mu^5}{4\sigma_X^5} - \frac{3\mu^5}{4\sigma_Y^5} \right) - \frac{15\mu^7}{4^6\sigma_Y^7} \right] \]

\[ = \frac{1}{\sqrt{2\pi}} \left[ \frac{3\mu}{4} \left( \frac{1}{\sigma_X} - \frac{1}{\sigma_Y} \right) \left( \frac{27\mu^3}{64} \left( \frac{1}{\sigma_X^3} + \frac{1}{\sigma_Y^3} \right) + \frac{113\mu^5}{4^5} \left( \frac{1}{\sigma_X^5} + \frac{1}{\sigma_Y^5} \right) - \frac{15\mu^7}{2^8\sigma_Y^7} - \frac{75\mu^7}{4^7\sigma_Y^7} \right) \right] \]

\[ = \frac{1}{\sqrt{2\pi}} \left[ \frac{1}{\sigma_X} - \frac{1}{\sigma_Y} \right] \left( \frac{3\mu}{4} \left( \frac{27\mu^3}{64} \left( \frac{1}{\sigma_X^3} + \frac{1}{\sigma_Y^3} \right) + \frac{113\mu^5}{4^5} \left( \frac{1}{\sigma_X^5} + \frac{1}{\sigma_Y^5} \right) - \frac{15\mu^7}{2^8\sigma_Y^7} - \frac{75\mu^7}{4^7\sigma_Y^7} \right) \right] \]

\[ \geq \frac{1}{\sqrt{2\pi}} \left[ \frac{1}{\sigma_X} - \frac{1}{\sigma_Y} \right] \left( \frac{3\mu}{4} \left( \frac{27\mu^3}{64} \left( \frac{1}{\sigma_X^3} + \frac{1}{\sigma_Y^3} \right) + \frac{113\mu^5}{4^5} \left( \frac{1}{\sigma_X^5} + \frac{1}{\sigma_Y^5} \right) - \frac{15\mu^7}{2^8\sigma_Y^7} - \frac{75\mu^7}{4^7\sigma_Y^7} \right) \right] \]

where the last line follows from the fact \( \sigma_X < \sigma_Y \) and \( \frac{131\mu^5}{2^5} \left( \frac{1}{\sigma_X^3} - \frac{1}{\sigma_Y^3} \right) > 0. \)

To continue, since \( \sigma_X = \sqrt{\|w_{\mathcal{J}\setminus(i)}\|^2 + h_n^2} < 1 \) and \( \sigma_Y = \sqrt{\|w_{\mathcal{J}\setminus(i)}\|^2 + h_n^2} < 1, \)

\[ \frac{1}{\sigma_X} - \frac{1}{\sigma_Y} = \frac{\sigma_Y^2 - \sigma_X^2}{\sigma_X \sigma_Y (\sigma_X + \sigma_Y)} \geq \frac{\sigma_Y^2 - \sigma_X^2}{2} \geq \frac{1}{2} \left( h_n^2 - h_t^2 \right) \geq \frac{1}{2} \left( h_n^2 - \frac{1}{1 + \xi_0} h_n^2 \right) \geq \frac{\xi_0}{4n}, \]

where the second inequality uses the fact \( h_n^2 / h_t^2 \geq 1 + \xi_0, h_n \geq 1/\sqrt{n} \) and \( \xi_0 \in (0, 1). \) In addition, we have \( \frac{1}{\sigma_X} \leq \frac{1}{h_n} \leq \frac{1}{h_t} \leq 2/\sqrt{n}, \) such that

\[ \frac{1}{\sigma_X} + \frac{1}{\sigma_X \sigma_Y} \leq 10n \implies \frac{27\mu^3}{64} \left( \frac{1}{\sigma_X^3} + \frac{1}{\sigma_Y^3} \right) + \frac{113\mu^5}{4^5} \left( \frac{1}{\sigma_X^5} + \frac{1}{\sigma_Y^5} \right) - \frac{15\mu^7}{2^8\sigma_Y^7} - \frac{75\mu^7}{4^7\sigma_Y^7} \leq \frac{\mu}{4}. \]

provided \( \mu \leq cn^{-1/2} \) for a sufficiently small \( c > 0. \) Plugging (44) and (45) back into (43), we have

\[ \mathbb{E}_\xi [K] \geq \frac{\mu \xi_0}{8\sqrt{2\pi} n} - \frac{\mu^7}{2\sqrt{2\pi} \sigma_X^7} \geq \frac{\mu \xi_0}{16\sqrt{2\pi} n}, \]

conditioned on the support \( \Omega_{(i)}, \) provided that \( \frac{1}{\sigma_X^3} \leq 2^7 n^{7/2} \) and \( \mu \leq \frac{\xi_0^{1/6}}{4^{-3/4}}. \)

Plugging (46) back into (40) and then into (39), finally, by the assumption \( \|w\|_2 \geq \frac{1}{20\sqrt{\pi}}, \) we have

\[ \frac{w^T \nabla \mathbb{E}_{\phi_0}(w)}{\|w\|_2} = n \mathbb{E} \left[ w^T \nabla \psi \left( x^T h(w) \right) \right] \geq n \frac{\|w\|_2 \theta (1 - \theta)}{\|w\|_2} \frac{\mu \xi_0}{16\sqrt{2\pi} n} \geq \frac{\theta \xi_0}{480\sqrt{10\pi}}, \]

where the final bound follows from the constraint \( \theta \in (0, 1/3). \)
B.2 Proof of Proposition 1

The directional gradient can be written as the sum of \( p \) i.i.d. random variables as following:

\[
\frac{\nabla w \psi_o(w)}{\|w\|_2} := \frac{1}{p} \sum_{i=1}^{p} X_i, \quad \text{where} \quad X_i = \frac{\nabla w \psi_{\mu} (C(x_i)h(w))}{\|w\|_2}.
\]

In order to apply the Bernstein’s inequality in Lemma 9, we turn to bound the moments of \( X_i \). Plugging in (32), we have

\[
X_i = \frac{w^\top J_h(w) \cdot C(x_i)}{\|w\|_2} \tanh \left( \frac{C(x_i)h(w)}{\mu} \right)
\]

\[
= \left[ \frac{w}{\|w\|_2} - \frac{\|w\|_2}{\|w\|_2} \right] C(x_i) \tanh \left( \frac{C(x_i)h(w)}{\mu} \right) \leq \sqrt{2n} \|C(x_i)\|,
\]

where the last inequality follows from \(|\tanh(\cdot)| \leq 1\) and \(\left\| \left[ \frac{w}{\|w\|_2} - \frac{\|w\|_2}{\|w\|_2} \right] \right\|_2 = \sqrt{1 + \frac{\|w\|_2^2}{\|w\|_2^2}} \leq \sqrt{1 + \frac{1}{n}} \leq \sqrt{2n} \).

Invoking Lemma 12, we have for any \( m \geq 2 \),

\[
E|X_i|^m \leq \left( \sqrt{2n} \right)^m E\|C(x_i)\|^m \leq \frac{m!}{2} \cdot \left( Cn^3 \log(n) \right)^{m/2}
\]

for some constant \( C \). Finally, we complete the proof by setting \( \sigma^2 = Cn^3 \log(n) \), \( R = \sqrt{Cn^3 \log(n)} \) and applying the Bernstein’s inequality in Lemma 9.

B.3 Proof of Proposition 2

The Hessian of \( \phi_o(w) \) can be written as the sum of \( p \) i.i.d. random matrices as following:

\[
\nabla^2 w \phi_o(w) := \frac{1}{p} \sum_{i=1}^{p} Y_i, \quad \text{where} \quad Y_i = \nabla^2 w \psi_{\mu} (C(x_i)h(w)).
\]

Plugging in (33), we divide \( Y_i \) into two parts as:

\[
Y_i = \frac{1}{\mu} J_h(w)C(x_i)^\top \left[ I - \operatorname{diag}\left( \tanh^2 \left( \frac{C(x_i)h(w)}{\mu} \right) \right) \right] C(x_i)J_h(w)^\top
\]

\[
- \frac{1}{h_n} S_{n-1}(x_i)^\top \tanh \left( \frac{C(x_i)h(w)}{\mu} \right) J_h(w)J_h(w)^\top.
\]

Therefore, we bound the sums of \( D_i \) and \( E_i \) respectively, using the Bernstein’s inequality in Lemma 9.

**Bound the concentration of \( E_i \):** We start by bounding the moments of \( E_i \). Recall the Jacobian matrix \( J_h(w) \) in (31), we have

\[
J_h(w)J_h(w)^\top = I + \frac{ww^\top}{h_n^2},
\]

and therefore, \( \|J_h(w)J_h(w)^\top\| = 1 + \frac{\|w\|_2^2}{h_n^2} \leq 5 \) since for \( \|w\|_2 \leq \frac{n}{4\sqrt{2}} \leq 1 \), we have \( h_n(w) \geq 1/2 \). Consequently, by the triangle inequality,

\[
\|E_i\| \leq \frac{1}{h_n} \|x_i\|_2 \left\| \tanh \left( \frac{C(x_i)h(w)}{\mu} \right) \right\|_2 \|J_h(w)J_h(w)^\top\| \leq 10\sqrt{n} \|x_i\|_2.
\]

We can bound the moments of \( E_i \) as

\[
E\|E_i\|^m \leq 10^m m^{m/2} E\|x_i\|^m_2 \leq 10^m m^{m/2} \cdot m! m^{m/2} \leq \frac{m!}{2} (2n)^{2} (20n)^{m-2},
\]

23
where the second line follows from Fact 2 and Lemma 7 that bound the moments of \( \|x_i\|_2 \).

Setting \( \sigma^2 = 400n^2 \), \( R = 20n \), we apply the Bernstein’s inequality in Lemma 9 and obtain:

\[
\Pr \left( \left\| \frac{1}{p} \sum_{i=1}^{p} E_i - \mathbb{E} \left( \frac{1}{p} \sum_{i=1}^{p} E_i \right) \right\| \geq \frac{t}{2} \right) \leq 2n \exp \left( \frac{-pt^2}{c_1 n^2 + c_2 nt} \right) \tag{50}
\]

for some large enough constants \( c_1 \) and \( c_2 \).

**Bound the concentration of \( D_i \):** Using the fact that \( 1 - \tanh^2 (\cdot) \leq 1 \), the spectral norm of \( D_i \) can be bounded as

\[
\|D_i\| \leq \frac{1}{\mu} \| \mathcal{C}(x_i) \|_2^2 \| J_h(w) \|_2^2 \leq \frac{5}{\mu} \| \mathcal{C}(x_i) \|_2^2 ,
\]

where we have used again \( \| J_h(w) \|_2^2 = \| J_h(w)J_h(w)^\top \| \leq 5 \) derived above. Invoking Lemma 12, we obtain

\[
\mathbb{E} [\|D_i\|^m] \leq \left( \frac{5}{\mu} \right)^m \mathbb{E} [\| \mathcal{C}(x_i) \|_2^{2m}] \leq \frac{m!}{2} \left( \frac{C n \log(n)}{\mu} \right)^m ,
\]

for some constant \( C \). Let \( \sigma^2 = \frac{C^2 \mu^2 \log^2(n)}{\mu^2} \), \( R = \frac{C n \log(n)}{\mu} \), by the Bernstein’s inequality in Lemma 9, we have:

\[
\Pr \left( \left\| \frac{1}{p} \sum_{i=1}^{p} D_i - \mathbb{E} \left( \frac{1}{p} \sum_{i=1}^{p} D_i \right) \right\| \geq \frac{t}{2} \right) \leq 2n \exp \left( \frac{-pt^2}{c_1 n^2 \log^2(n) + c_4 \mu n \log(n)t} \right) . \tag{52}
\]

for some constants \( c_1, c_4 \).

Recall the Hessian of interest is written as:

\[
\nabla_w^2 \phi_o(w) = \frac{1}{p} \sum_{i=1}^{p} Y_i = \frac{1}{p} \sum_{i=1}^{p} D_i - \frac{1}{p} \sum_{i=1}^{p} E_i . \tag{53}
\]

Combing the bounds for \( D_i \) (52) and \( E_i \) (50), we obtain the final bound as advertised:

\[
\Pr \left( \| \nabla_w^2 \phi_o(w) - \nabla_w^2 \mathbb{E} \phi_o(w) \| \geq t \right) \leq 4n \exp \left( \frac{-pt^2}{9C^2 n^2 \log^2(n) + 3C \mu \log(n)t} \right) .
\]

**B.4 Proof of Theorem 2**

We start by introducing the event

\[
\mathcal{A}_0 := \left\{ \| X \|_\infty \leq 4 \sqrt{\log(np)} \right\} ,
\]

which holds with probability at least \( 1 - \theta(np)^{-7} \) by Fact 7.

**B.4.1 Proof of (17a)**

To show that \( \frac{w^\top \nabla_w \phi_o(w)}{\|w\|_2^2} \) is lower bounded uniformly in the region \( Q_1 \), we will apply a standard covering argument. Let \( \mathcal{N}_1 \) be an \( \epsilon \)-net of \( Q_1 \), such that for any \( w \in Q_1 \), there exists \( w_1 \in \mathcal{N}_1 \) with \( \| w - w_1 \|_2 \leq \epsilon \).

By standard results [51, Lemma 5.7], the size of \( \mathcal{N}_1 \) is at most \( [3/\epsilon]^n \), where the value of \( \epsilon \) will be determined later. We have

\[
\frac{w^\top \nabla_w \phi_o(w)}{\|w\|_2^2} = \left[ \frac{w^\top \nabla_w \phi_o(w)}{\|w\|_2^2} \right] - \left[ \frac{w^\top \nabla_w \phi_o(w_1)}{\|w_1\|_2^2} \right] + \left[ \frac{w^\top \nabla_w \phi_o(w_1)}{\|w_1\|_2^2} \right] - \left[ \frac{w^\top \nabla_w \mathbb{E} \phi_o(w_1)}{\|w_1\|_2^2} \right] + \left[ \frac{w^\top \nabla_w \mathbb{E} \phi_o(w_1)}{\|w_1\|_2^2} \right] .
\]

In the sequel, we derive bounds for the terms I, II, III respectively.
For term I, as \( w_1 \in \mathcal{N}_1 \subseteq Q_1 \), by Lemma 1, we have
\[
I = \frac{w_1^T \nabla_{w} \mathbb{E}(\phi_o(w_1))}{\|w_1\|_2} \geq \frac{\theta}{480\sqrt{10\pi}} \xi_0 := c_1 \theta \xi_0.
\]

To bound term II, by the additivity of Lipschitz constants and [10, Proposition 13], we have
\[
\frac{w^T \nabla_{w} \phi_o(w)}{\|w\|_2} \text{ is } L_1\text{-Lipschitz with}
\]
\[
L_1 \leq \left( \frac{8\sqrt{2}n^{3/2}}{\mu} + 8n^{5/2} \right) \|X\|_\infty + \frac{4n^3}{\mu} \|X\|_\infty^2.
\]

Therefore, under the event \( \mathcal{A}_0 \), we have
\[
L_1 \leq \frac{c_2 n^3 \log(np)}{\mu} \text{ for some constant } c_2. \text{ Setting } \epsilon = \frac{c_1 \theta \xi_0}{3}, \text{ we obtain that}
\]
\[
II = \left| \frac{w^T \nabla_{w} \phi_o(w)}{\|w\|_2} - \frac{w_1^T \nabla_{w} \phi_o(w_1)}{\|w_1\|_2} \right| \leq L_1 \|w - w_1\|_2 \leq L_1 \epsilon \leq \frac{c_1 \theta \xi_0}{3}.
\]

Along the way, we determine the size of \( \mathcal{N}_1 \) is upper bounded by
\[
|\mathcal{N}_1| \leq \lceil \frac{3}{\epsilon} \rceil^n \leq \exp \left\{ n \log \left( \frac{c_1 n^3 \log(np)}{\mu \theta \xi_0} \right) \right\}.
\]

For term III, by setting \( t = \frac{c_1 \theta \xi_0}{3} \) in Proposition 1 and the union bound, we have the event
\[
\mathcal{A}_1 := \left\{ \max_{w_1 \in \mathcal{N}_1} \left| \frac{w_1^T \nabla_{w} \phi_o(w_1)}{\|w_1\|_2} - \frac{w^T \nabla_{w} \phi_o(w)}{\|w\|_2} \right| \leq \frac{c_1 \theta \xi_0}{3} \right\}
\]
holds with probability at least
\[
1 - |\mathcal{N}_1| \cdot 2 \exp \left( -\frac{pt^2}{2cn^3 \log n + 2\sqrt{cn^3 \log(n)}t} \right) \geq 1 - 2 \exp \left( -\frac{c_1 \theta \xi_0^2}{n} \right) + n \log \left( \frac{c_3 n^3 \log(np)}{\mu \theta \xi_0} \log(n) \right)
\]
\[
\geq 1 - 2 \exp(-c_5 n),
\]
provided \( p \geq \frac{c_4 n^3 \log(n)}{\mu \theta \xi_0} \).

Combining terms, conditioned on \( \mathcal{A}_0 \cap \mathcal{A}_1 \), which holds with probability at least \( 1 - \theta(np)^{-7} - 2 \exp(-c_5 n) \), we have that for all \( w \in Q_1 \), (17a) holds since,
\[
\frac{w^T \nabla_{w} \phi_o(w)}{\|w\|_2} \geq I - II - III \geq -\frac{c_1 \theta \xi_0}{3} - \frac{c_1 \theta \xi_0}{3} + c_1 \theta \xi_0 = \frac{c_1 \theta \xi_0}{3}.
\]

### B.4.2 Proof of (17b)

The proof is similar to the above proof of (17a) in Appendix B.4.1. Let \( \mathcal{N}_2 \) be an \( \epsilon \)-net of \( Q_2 \), such that for any \( w \in Q_2 \), there exists \( w_2 \in \mathcal{N}_2 \) with \( \|w - w_2\|_2 \leq \epsilon \). By standard results [51, Lemma 5.7], the size of \( \mathcal{N}_2 \) is at most \( \lceil 3\mu/(4\sqrt{2}\epsilon) \rceil^n \), where the value of \( \epsilon \) will be determined later. By the triangle inequality, we have for all \( w \in Q_2 \),
\[
\nabla_{w}^2 \phi_o(w) \succeq \inf_{w_2 \in \mathcal{N}_2} \nabla_{w}^2 \mathbb{E}(\phi_o(w_2)) - \nabla_{w}^2 \phi_o(w_2) - \nabla_{w}^2 \phi_o(w) \|I - \nabla_{w}^2 \phi_o(w_2) - \nabla_{w}^2 \mathbb{E}(\phi_o(w_2))\|I.
\]

In the sequel, we derive bounds for the terms \( H_1, H_2, H_3 \) respectively.

- For \( H_1 \), by Theorem 1, we have
\[
H_1 = \inf_{w_2 \in \mathcal{N}_2} \nabla_{w}^2 \mathbb{E}(\phi_o(w_2)) \geq \frac{n \theta}{5\sqrt{2\pi \mu}} I := \frac{c_1 n \theta}{\mu} I.
\]

25
To bound $H_2$, by the additivity of Lipschitz constants and [10, Proposition 14], we have $\nabla^2_w \phi_0(w)$ is $L_2$-Lipschitz with

$$L_2 \leq \frac{4n^3}{\mu^2} \|X\|_\infty^3 + \left(\frac{4n^2}{\mu} + \frac{8\sqrt{2}n^{3/2}}{\mu}\right) \|X\|_\infty^2 + 8n \|X\|_\infty.$$ 

Under the event $A_0$, we have $L_2 \leq \frac{c_5 n^3}{\mu^2} \log^{3/2}(np)$ for some constant $c_5$. Setting $\epsilon = \frac{c_5 n^2}{\mu L_2}$, we obtain

$$\|\nabla^2_w \phi_0(w_2) - \nabla^2_w \phi_0(w)\| \leq \frac{c_5 n^2}{3\mu}, \quad \text{and} \quad H_2 \geq \frac{c_5 n^2}{3\mu} I.$$ 

Along the way, we determine the size of $N_2$ is upper bounded by

$$|N_2| \leq \left[\frac{3\mu}{(4\sqrt{2}e)}\right]^n \leq \exp\left[n \log\left(\frac{c_7 n^2 \log^{3/2}(np)}{\theta}ight)\right].$$ 

To bound $H_3$, by setting $t = \frac{c_8 n^2}{\mu^2}$ in Proposition 2 and the union bound, we have the event

$$A_2 := \left\{ \max_{w_2 \in N_2} \|\nabla^2_w \phi_0(w_2) - \nabla^2_w E(\phi_0(w_2))\| \leq \frac{c_5 n^2}{3\mu} \right\}$$ 

holds with probability at least

$$1 - |N_2| \cdot 4n \exp\left(-\frac{-p^2 t^2}{9C^2 n^2 \log^2 n + 3C \mu n \log(n) t}\right) \geq 1 - 4n \exp\left(-\frac{-c_8 \mu t^2}{\log^2 n} + n \log\left(\frac{c_7 n^2 \log^{3/2}(np)}{\theta}\right)\right) \geq 1 - \exp(-c_9 n),$$

provided $p \geq \frac{C_9}{\theta^2} \log^2 n \log\left(\frac{c_7 n^2 \log^{3/2}(p)}{\theta}\right)$.

Combining terms, conditioned on $A_0 \cap A_2$, which holds with probability at least $1 - \theta(np)^{-7} - \exp(-c_9 n)$, we have (17b) holds since,

$$\nabla^2_w \phi_0(w) \succeq H_1 - H_2 - H_3 \geq \frac{c_5 n^2}{3\mu}.$$ 

### B.4.3 Proof of (18)

The characterized geometry of $\phi_0(w)$ implies that it has at most one local minimum in $Q_2$ due to strong convexity, which is denoted as $w^*_o$. We are going to show that $w^*_o$ is close to 0 in $Q_2$. By the optimality of $w^*_o$ and the mean value theorem, we have for some $t \in (0, 1)$:

$$\phi_0(0) \geq \phi_0(w^*_o) \geq \phi_0(0) + \langle \nabla_w \phi_0(0), w^*_o \rangle + w^*_o^\top \nabla^2 \phi_0(t w^*_o) w^*_o$$

$$\geq \phi_0(0) - \|w^*_o\|_2 \|\nabla_w \phi_0(0)\|_2 + \frac{c_5 n^2}{2\mu} \|w^*_o\|_2^2,$$

where the second line follows from (17b) and the Cauchy-Schwartz inequality. Therefore, we have

$$\|w^*_o\|_2 \leq \frac{2\mu}{c_5 n^2} \|\nabla_w \phi_0(0)\|_2.$$ \hspace{1cm} (54)

It remains to bound $\|\nabla_w \phi_0(0)\|_2$, which we resort to the Bernstein’s inequality in Lemma 7. As $\nabla_w \phi_0(0) = \frac{1}{p} \sum_{i=1}^p \nabla_w \psi_{\mu}(C(x_i) h(0))$, where it is straightforward to check $E \nabla_w \psi_{\mu}(C(x_i) h(0)) = 0$ due to symmetry. We turn to bound the moments of $\|\nabla_w \psi_{\mu}(C(x_i) h(0))\|_2$ as follows,

$$\|\nabla_w \psi_{\mu}(C(x_i) h(0))\|_2 \leq \|J_h(0) \cdot C(x_i) \|^\top \tanh\left(\frac{C(x) h(0)}{\mu}\right)\|_2$$

26
\begin{align*}
&\leq \|J_h(0)\| \|\mathcal{C}(x_i)\| \left|\tanh \left( \frac{\mathcal{C}(x) h(0)}{\mu} \right) \right|_2 \\
&\leq \sqrt{n} \|\mathcal{C}(x_i)\|,
\end{align*}

where the last inequality follows from \(\|J_h(0)\| = \|[I_{n-1} \ 0]\| = 1\) and \(|\tanh (\cdot)| \leq 1\). Invoking Lemma 12, we have for all \(m \geq 2,\)
\[
E \left[ \|\nabla w \psi_{\mu} \left( \mathcal{C}(x) h(0) \right) \|_2^m \right] \leq E |X_i|^m \leq (\sqrt{n})^m E \|\mathcal{C}(x_i)\|^m \leq \frac{m!}{2} \left( Cn^2 \log(n) \right)^{m/2}
\]
for some constant \(C\). Setting \(\sigma^2 = Cn^2 \log(n), R = \sqrt{Cn^2 \log(n)}\) in the Bernstein’s inequality in Lemma 7, we have
\[
\mathbb{P} \left( \|\nabla w \phi_{\mu}(0)\|_2 \geq t \right) \leq 2(n+1) \exp \left( \frac{-pt^2}{2Cn^2 \log(n) + 2\sqrt{Cn^2 \log(n)}t} \right).
\]
Let \(t = c_9 \sqrt{\frac{n^2 \log(n) \log(p)}{p}}\), we have
\[
\|\nabla w \phi_{\mu}(0)\|_2 \leq c_9 \sqrt{\frac{n^2 \log n \log p}{p}}
\]
with probability at least \(1 - 4np^{-7}\) when \(p \geq c_{10} n \log(n)\). Under the sample size requirement on \(p\), we have
\[
\|w^*_o - 0\|_2 \leq \frac{c_6 \mu}{\theta} \sqrt{\frac{\log n \log p}{p}} \leq \frac{\mu}{10},
\]
for some constant \(c_6\), which ensures \(w^*_o \in Q_2\).

## C Proofs for Section 3.2

### C.1 Proof of Lemma 2

Recalling \(\Delta = (U' - U) \cdot U^{-1}\), we have
\[
\|\Delta\| = \|(U' - U) \cdot U^{-1}\| = \|U' - U\|,
\]
since \(U\) is an orthonormal matrix, i.e., \(\|U^{-1}\| = 1\). Therefore, it is sufficient to bound \(\|U' - U\|\) instead. Plugging in the definition of \(U'\) and \(U\), we have
\[
\|U' - U\| = \|\mathcal{C}(g) R - \mathcal{C}(g) \left( \mathcal{C}(g)^\top \mathcal{C}(g) \right)^{-1/2} \| \\
\leq \|\mathcal{C}(g)\| \left\| \mathcal{C}(g)^\top \mathcal{C}(g) \right\|^{-1/2} \\
\leq \|\mathcal{C}(g)\| \left\| R^2 - \left( \mathcal{C}(g)^\top \mathcal{C}(g) \right)^{-1} \right\|_{\sigma_{\min}} \left( \mathcal{C}(g)^\top \mathcal{C}(g) \right)^{-1/2} \\
\leq \|\mathcal{C}(g)\|^2 \left\| \mathcal{C}(g)^\top \mathcal{C}(g) \right\|^{-1} \|\mathcal{C}(g)^\top \mathcal{C}(g) R^2 - I\| = \kappa^2 \|\mathcal{C}(g)^\top \mathcal{C}(g) R^2 - I\|,
\]
where the second inequality follows from the fact [52, Theorem 6.2] that for two positive matrices \(U, V\), we have \(\|U^{-1/2} - V^{-1/2}\| \leq \frac{\|U^{-1} - V^{-1}\|}{\sigma_{\min}(V^{-1/2})}\). We continue by plugging in the definition of \(R\),
\[
\|\mathcal{C}(g)^\top \mathcal{C}(g) R^2 - I\| = \left\| \mathcal{C}(g)^\top \mathcal{C}(g) \cdot \left( \frac{1}{\theta n p} \sum_{i=1}^p \mathcal{C}(g)^\top \mathcal{C}(x_i)^\top \mathcal{C}(x_i) \mathcal{C}(g) \right)^{-1} - I \right\|
\]

27
where $A = \left( C(g)^\top \left[ \frac{1}{\eta_{np}} \sum_{i=1}^p C(x_i)^\top C(x_i) - I \right] C(g) \right) \cdot \left( C(g)^\top C(g) \right)^{-1}$.

By Lemma 13, we have when $p \geq C n \log(n)$, $\frac{1}{\eta_{np}} \sum_{i=1}^p C(x_i)^\top C(x_i) - I \leq C \sqrt{\log^2(n) \log(p)}$ with probability at least $1 - 2np^{-8}$, and $\|A\| \leq C\kappa^2 \sqrt{\log^2(n) \log(p)}$. Then as long as $\|A\| \leq 1/2$, which holds when $p \geq C_2 n \kappa^4 \log^2(n) \log p$ for some large enough constant $C_2$, we have

$$\| (I + A)^{-1} - I \| \leq \| (I + A)^{-1} \| A \| \leq \frac{\|A\|}{1 - \|A\|} \leq 2\|A\|.$$  

Plugging this back into (57), we have

$$\| U' - U \| \leq C_3 \kappa^4 \sqrt{\frac{\log^2 n \log p}{\theta^2}}.$$  

(59)

### C.2 Proof of Lemma 3

We first record some useful facts. For any $h \in S_0^{n+}$, we have the Jacobian matrix $J_h(w) = [I, -\frac{w}{h_n}] \in \mathbb{R}^{(n-1) \times n}$ satisfies

$$\| J_h(w) \| \leq \| J_h(w) \|_F \leq \sqrt{n - 1 + \frac{\|w\|^2}{h_n^2}} \leq \sqrt{2n},$$

(60)

since $\|w\|_2 \leq 1$ and $h_n \geq \frac{1}{\sqrt{n}}$. In addition, by the union bound and Lemma 12, we have with probability at least $1 - (np)^{-8}$,

$$\max_{i \in [p]} \| C(x_i) \| \leq C \sqrt{n \log(np)},$$

(61)

for some constant $C$.

#### C.2.1 Proof of (27a)

Similar to (32), we can write the gradient $\nabla_w \phi(w)$ as

$$\nabla_w \phi(w) = \frac{1}{p} \sum_{i=1}^p J_h(w) (I + \Delta)^\top C(x_i)^\top \tanh \left( \frac{C(x_i) (I + \Delta) h(w)}{\mu} \right).$$

Recalling the expression of $\nabla_w \phi_o(w)$ in (32), we write

$$\nabla_w \phi_o(w) - \nabla_w \phi(w)$$

$$= \frac{1}{p} \sum_{i=1}^p J_h(w) C(x_i)^\top \tanh \left( \frac{C(x_i) h(w)}{\mu} \right) - \frac{1}{p} \sum_{i=1}^p J_h(w) (I + \Delta)^\top C(x_i)^\top \tanh \left( \frac{C(x_i) (I + \Delta) h(w)}{\mu} \right)$$

$$= \frac{1}{p} \sum_{i=1}^p J_h(w) C(x_i)^\top \cdot \left[ \tanh \left( \frac{C(x_i) h(w)}{\mu} \right) - \tanh \left( \frac{C(x_i) (I + \Delta) h(w)}{\mu} \right) \right]$$

$$- \frac{1}{p} \sum_{i=1}^p J_h(w) \Delta^\top C(x_i)^\top \tanh \left( \frac{C(x_i) (I + \Delta) h(w)}{\mu} \right).$$

Therefore, we continue to bound $\|g_1\|_2$ and $\|g_2\|_2$. 

28
First, under the sample size for some constant $\mu$

\[
\|g_1\|_2 \leq \|J_h(w)\| \cdot \max_{i \in [p]} \|C(x_i)\| \cdot \max_{i \in [p]} \left\| \tanh \left( \frac{C(x_i)\cdot h(w)}{\mu} \right) - \tanh \left( \frac{C(x_i)(I + \Delta)\cdot h(w)}{\mu} \right) \right\|_2 \\
\leq \frac{1}{\mu} \|J_h(w)\| \cdot \max_{i \in [p]} \|C(x_i)\|^2 \cdot \|\Delta\|. \quad (62)
\]

Here, the second line follows from for any $i \in [p]$,

\[
\left\| \tanh \left( \frac{C(x_i)\cdot h(w)}{\mu} \right) - \tanh \left( \frac{C(x_i)(I + \Delta)\cdot h(w)}{\mu} \right) \right\|_2 \\
\leq \left\| \left( \frac{C(x_i)\cdot h}{\mu} \right) - \left( \frac{C(x_i)(I + \Delta)\cdot h}{\mu} \right) \right\|_2 \\
= \frac{1}{\mu} \|C(x_i)\| \cdot \|\Delta\| \cdot \|h\|_2 = \frac{1}{\mu} \|C(x_i)\| \cdot \|\Delta\|_2. \quad (63)
\]

where the second line follows from Lemma 11, and the last equality is due to $\|h\|_2 = 1$.

To bound $\|g_2\|_2$, we have

\[
\|g_2\|_2 \leq \|J_h(w)\| \cdot \max_{i \in [p]} \|C(x_i)\| \cdot \max_{i \in [p]} \left\| \tanh \left( \frac{C(x_i)(I + \Delta)\cdot h(w)}{\mu} \right) \right\|_2 \cdot \|\Delta\| \\
\leq \sqrt{n} \|J_h(w)\| \cdot \max_{i \in [p]} \|C(x_i)\| \cdot \|\Delta\|, \quad (64)
\]

where the second line uses $|\tanh(\cdot)| \leq 1$, and $\left\| \tanh \left( \frac{C(x_i)(I + \Delta)\cdot h(w)}{\mu} \right) \right\|_2 \leq \sqrt{n}$.

Combining (62) and (64), we have

\[
\|\nabla_w \phi_o(w) - \nabla_w \phi(w)\| \leq \|g_1\|_2 + \|g_2\|_2 \leq \|J_h(w)\| \cdot \max_{i \in [p]} \|C(x_i)\| \cdot \|\Delta\| \left( \sqrt{n} + \frac{1}{\mu} \max_{i \in [p]} \|C(x_i)\| \right) \\
\leq C \frac{n^{3/2} \log(np)}{\mu} \|\Delta\|, 
\]

for some constant $C$, where the last line follows from (60) and (61), which holds with probability at least $1 - (np)^{-8}$.

**C.2.2 Proof of (27b)**

First, under the sample size $p \geq \frac{C_2 \gamma^6 n \log^2(n) \log p}{\theta^2}$, from Lemma 2, we can ensure $\|\Delta\| \leq 1$. Note that

\[
\|\nabla^2_w \phi_o(w) - \nabla^2_w \phi(w)\| = \left\| \frac{1}{p} \sum_{i=1}^{p} \nabla^2_w \psi_{\mu}(C(x_i)\cdot h(w)) - \frac{1}{p} \sum_{i=1}^{p} \nabla^2_w \psi_{\mu}(C(x_i)(I + \Delta)\cdot h(w)) \right\| \\
\leq \frac{1}{p} \sum_{i=1}^{p} \left\| \nabla^2_w \psi_{\mu}(C(x_i)\cdot h(w)) - \nabla^2_w \psi_{\mu}(C(x_i)(I + \Delta)\cdot h(w)) \right\|. \quad (65)
\]

Similar to (33), we can write the Hessian $\nabla^2_w \psi_{\mu}(C(x_i)(I + \Delta)\cdot h(w))$ as

\[
\nabla^2_w \psi_{\mu}(C(x_i)(I + \Delta)\cdot h(w)) = \frac{1}{\mu} J_h(w)(I + \Delta)\cdot C(x_i)^\top \left[ I - \text{diag} \left( \tanh^2 \left( \frac{C(x)(I + \Delta)\cdot h(w)}{\mu} \right) \right) \right] C(x_i)(I + \Delta) J_h(w)^\top \\
- \frac{1}{n} \mathbf{S}_{n-1}(\cdot)(I + \Delta) \tanh \left( \frac{C(x)(I + \Delta)\cdot h(w)}{\mu} \right) J_h(w) J_h(w)^\top. \quad (66)
\]
Subtracting $\nabla_w^2 \psi_n(C(x_i)h(w))$ in (33) from the above equation, we have

$$\nabla_w^2 \psi_n(C(x_i)h(w)) - \nabla_w^2 \psi_n(C(x_i)(I + \Delta)h(w))$$

$$= \frac{1}{\mu} J_h(w)C(x_i)^\top \left[ \text{diag} \left( \tanh^2 \left( \frac{C(x_i)(I + \Delta)h(w)}{\mu} \right) \right) \right] C(x_i)J_h(w)^\top$$

$$- \frac{1}{\mu} J_h(w) \Delta C(x_i)^\top \left[ I - \text{diag} \left( \tanh^2 \left( \frac{C(x_i)(I + \Delta)h(w)}{\mu} \right) \right) \right] C(x_i)(I + \Delta)J_h(w)^\top$$

$$- \frac{1}{\mu} J_h(w)C(x_i)^\top \left[ I - \text{diag} \left( \tanh^2 \left( \frac{C(x_i)(I + \Delta)h(w)}{\mu} \right) \right) \right] C(x_i)\Delta J_h(w)^\top$$

$$+ \frac{1}{h_n} S_{n-1}(x_i)^\top \left[ \tanh \left( \frac{C(x_i)(I + \Delta)h(w)}{\mu} \right) \right] C(x_i)\Delta J_h(w)^\top$$

$$+ \frac{1}{h_n} S_{n-1}(x_i)^\top \Delta \tanh \left( \frac{C(x_i)(I + \Delta)h(w)}{\mu} \right) J_h(w)J_h(w)^\top,$$

where in the sequel we'll bound these terms respectively.

- $I_1$ can be bounded as
  $$\|I_1\| \leq \frac{1}{\mu} \|J_h(w)\|^2 \|C(x_i)\|^2 \left\| \tanh^2 \left( \frac{C(x_i)(I + \Delta)h(w)}{\mu} \right) \right\|_{\infty} - \tanh^2 \left( \frac{C(x_i)h(w)}{\mu} \right)_{\infty}$$
  $$\leq \frac{2}{\mu^2} \|J_h(w)\|^2 \|C(x_i)\|^2 \|C(x_i)\Delta h(w)\|_{\infty}$$
  $$\leq \frac{2}{\mu^2} \|J_h(w)\|^2 \|C(x_i)\|^2 \|x_i\|_2 \|\Delta\|,$$

where the second line follows from Lemma 11, where the last line uses $\|h\|_2 = 1$.

- $I_2$ can be bounded as
  $$\|I_2\| \leq \frac{1}{\mu} \|J_h(w)\|^2 \|C(x_i)\|^2 \|\Delta\| \|1 + \|\Delta\|\| \leq \frac{2}{\mu} \|J_h(w)\|^2 \|C(x_i)\|^2 \|\Delta\|,$$

where we have used $1 - \tanh^2(\cdot) \leq 1$, $\|\Delta\| \leq 1$ respectively.

- Similar to $I_2$, $I_3$ can be bounded as
  $$\|I_3\| \leq \frac{1}{\mu} \|J_h(w)\|^2 \|C(x_i)\|^2 \|\Delta\|.$$

- $I_4$ can be bounded as
  $$\|I_4\| \leq \frac{1}{h_n} \|x_i\|_2 \|J_h(w)\|^2 \left\| \tanh \left( \frac{C(x_i)(I + \Delta)h(w)}{\mu} \right) \right\|_2 - \tanh \left( \frac{C(x_i)h(w)}{\mu} \right)$$
  $$\leq \frac{\sqrt{n}}{\mu} \|J_h(w)\|^2 \|C(x_i)\| \|x_i\|_2 \|\Delta\|,$$

where the second line follows from (63) and $h_n \geq 1/\sqrt{n}$.  

30
• $I_5$ can be bounded as

$$
\|I_5\| \leq \frac{1}{h_n} \|x_i\|_2 \|J_h(w)\|^2 \|\Delta\| \left\| \tanh \left( \frac{C(x_i)(I + \Delta)h(w)}{\mu} \right) \right\|_2 \\
\leq n \|x_i\|_2 \|J_h(w)\|^2 \|\Delta\|,
$$

where the second line uses $|\tanh(\cdot)| \leq 1$ and $h_n \geq 1/\sqrt{n}$.

Combining the above bounds back into (65), we have

$$
\left\| \nabla^2 \phi_o(w) - \nabla^2 \phi(w) \right\| \leq \|J_h(w)\| \|\Delta\| \max_{i \in [p]} \left( \frac{2}{\mu^2} \|C(x_i)\|^2 \|x_i\|_2 + \frac{3}{\mu} \|C(x_i)\|^2 + \frac{\sqrt{n}}{\mu} \|C(x_i)\| \|x_i\|_2 + n \|x_i\|_2 \right).
$$

Plugging in (60), (61), and Fact 6, where with probability at least $1 - (np)^{-8}$,

$$
\max_{i \in [p]} \|C(x_i)\| \leq C \sqrt{n \log(np)}, \quad \max_{i \in [p]} \|x_i\|_2 \leq C \sqrt{n \log p},
$$

we have

$$
\left\| \nabla^2 \phi_o(w) - \nabla^2 \phi(w) \right\| \leq C_9 \frac{n^{2.5}}{\mu^2} \log^{3/2}(np) \|\Delta\|.
$$

### C.3 Proof of Theorem 3

To begin, by Lemma 3, we have

$$
\left| \frac{w^T \nabla w \phi_o(w)}{\|w\|} - \frac{w^T \nabla w \phi(w)}{\|w\|} \right| \leq \left\| \nabla w \phi_o(w) - \nabla w \phi(w) \right\|_2 \leq c_9 \frac{n^{3/2} \log(np)}{\mu} \|\Delta\| \leq \frac{c_2 \xi_0 \theta}{2}, \quad (67a)
$$

$$
\left\| \nabla^2 \phi_o(w) - \nabla^2 \phi(w) \right\| \leq c_9 \frac{n^{2.5}}{\mu^2} \log^{3/2}(np) \|\Delta\| \leq \frac{c_2 \xi_0 \theta}{2\mu}, \quad (67b)
$$

as long as the sample size satisfies

$$
\|\Delta\| \leq c f \kappa^4 \sqrt{\frac{\log^2(n) \log(p)}{\theta^2 p}} \leq \frac{\xi_0 \theta \mu}{C n^{3/2} \log^{3/2}(np)}.
$$

for some constant $C$ in view of Lemma 2, where we have used the assumption that $\mu < cn^{-1/2}$ for some sufficiently small $c$. Translating this into the sample size requirement, it means

$$
p \geq C \frac{n^3 \log^4 p \log^2 n}{\theta^4 \mu^2 \xi_0^2}.
$$

Under the assumption of Theorem 2, and in view of (17a) and (17b), we have

$$
\nabla^2 \phi(w) \succeq \nabla^2 \phi_o(w) - \left\| \nabla^2 \phi_o(w) - \nabla^2 \phi(w) \right\|_2 I \succeq \frac{c_2 \xi_0 \theta}{2},
$$

$$
\nabla x \phi(w) \succeq \nabla x \phi_o(w) - \left\| \nabla x \phi_o(w) - \nabla x \phi(w) \right\|_2 I \succeq \frac{c_2 \xi_0 \theta}{2}.
$$

Now let $w^*$ be the local minimizer of $\phi(w)$ in the region of interest. Similar to the proof in Appendix B.4.3, we have

$$
\|w^*\|_2 \leq \frac{4\mu}{c_2 n \theta} \|\nabla w \phi(0)\|_2 \\
\leq \frac{4\mu}{c_2 n \theta} \|\nabla w \phi(0)\|_2 + \frac{4\mu}{c_2 n \theta} \|\nabla w \phi(0) - \nabla w \phi_o(0)\|_2
$$
We first prove the upper bound of \( \|w^* - 0\|_2 \leq \frac{c_9 \mu}{\sqrt{p}} \left( \sqrt{\frac{n^2 \log(n) \log(p)}{\mu}} + \frac{n^{3/2} \log(np)}{\mu^4} \sqrt{\frac{\log^2(n) \log(p)}{\theta^2 p}} \right) \).

where the first term is bounded by (55) and the second term is bounded by Lemma 3. Under the sample size requirement, the latter term dominates and therefore we have

\[
\|w^* - 0\|_2 \leq \frac{c\kappa^4}{\theta^2} \sqrt{\frac{n \log^3 p \log^2 n}{p}}.
\]

## D Proofs for Section 3.3

We start by stating a useful observation. Notice that \( \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) \) is on the tangent space of \( h \), i.e.,

\[
(I - hh^\top) \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) = \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right),
\]

we have the relation

\[
\partial f(h)^\top \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) = [ (I - hh^\top) \nabla f(h)]^\top \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) = \nabla f(h)^\top \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right),
\]

holds for both \( \partial f(h) \) and \( \partial f_o(h) \).

### D.1 Proof of Lemma 4

We first prove the upper bound of \( \|\partial f(h)\|_2 \) in (28a), which is simpler. Plugging the bound for \( \max_{i \in [p]} \|C(x_i)\| \) in (61) and \( \|\Delta\| \leq 1 \) ensured by the sample size requirement and Lemma 2, for any \( h \) on the unit sphere, with probability at least \( 1 - (np)^{-8} \), the manifold gradient satisfies

\[
\|\partial f(h)\|_2 \leq \|\nabla f(h)\|_2 = \left\| \frac{1}{\mu} \sum_{i=1}^{p} (I + \Delta)^\top C(x_i)^\top \cdot \tanh \left( \frac{C(x_i) (I + \Delta) h(w)}{\mu} \right) \right\|_2 \\
\leq \sqrt{n} \|I + \Delta\| \max_{i \in [p]} \|C(x_i)\| \\
\leq 2C \sqrt{n} \log(np).
\]

We now move to prove the lower bound of the directional gradient in (28a). To prove Lemma 4, first, we consider the directional gradient of \( f_o(h) \) for the orthogonal case, following the proof procedure of the Theorem 2 to obtain the empirical geometry of \( f_o(h) \) in the region of interest (shown in Lemma 17), which is proved in Appendix D.4.

**Lemma 17** (Uniform concentration for orthogonal case). Instate the assumptions of Theorem 2. There exist some constants \( c_a, c_b \), such that for \( h \in \mathcal{H}_k = \{h : h \in S^{(n+1)}_{\xi_0}, h_k \neq 0, h^2_k/h_k^2 < 4\} \), with probability at least

\[
1 - \theta(np)^{-8} - 2 \exp \left( -c_a n \right),
\]

\[
\partial f_o(h)^\top \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) \geq c_b \xi_0 \theta.
\]

Based on the result, we derive the bound for the directional gradient of \( f(h) \) in the general case by bounding the deviation between the directional gradient of \( f_o(h) \) and \( f(h) \). Using (68), we can relate the directional gradient of \( f(h) \) to that of \( f_o(h) \) as

\[
(\partial f(h) - \partial f_o(h))^\top \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) = (\nabla f(h) - \nabla f_o(h))^\top \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right).
\]
We have

\[ \left( \langle \partial f(h) - \partial f_o(h) \rangle \cdot \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) \right) \leq \|\nabla f(h) - \nabla f_o(h)\|_2 \|\frac{e_k}{h_k} - \frac{e_n}{h_n}\|_2 \leq \sqrt{5n} \|\nabla f(h) - \nabla f_o(h)\|_2, \]

where the last line follows from

\[ \left\| \frac{e_k}{h_k} - \frac{e_n}{h_n} \right\|_2 = \sqrt{\frac{1}{h_k^2} + \frac{1}{h_n^2}} \leq \sqrt{\frac{5}{h_n^2}} \leq \sqrt{5n}, \tag{71} \]

due to the assumption \( h_n^2/h_k^2 \leq 4 \) and \( h_n \geq 1/\sqrt{n} \). Therefore, it is sufficient to bound \( \|\nabla f(h) - \nabla f_o(h)\|_2 \).

By (29), we have

\[ \|\nabla f_o(h) - \nabla f(h)\|_2 = \left\| \frac{1}{p} \sum_{i=1}^{p} C(x_i)^\top \tanh \left( \frac{C(x_i)h}{\mu} \right) - \frac{1}{p} \sum_{i=1}^{p} (I + \Delta)^\top C(x_i)^\top \tanh \left( \frac{C(x_i)(I + \Delta)h}{\mu} \right) \right\|_2 \]

\[ \leq \left\| \frac{1}{p} \sum_{i=1}^{p} \Delta^\top C(x_i)^\top \tanh \left( \frac{C(x_i)(I + \Delta)h}{\mu} \right) \right\|_2 + \left\| \frac{1}{p} \sum_{i=1}^{p} C(x_i)^\top \left[ \tanh \left( \frac{C(x_i)h}{\mu} \right) - \tanh \left( \frac{C(x_i)(I + \Delta)h}{\mu} \right) \right] \right\|_2 \]

\[ \leq \max_{i \in [p]} \|C(x_i)\| \cdot \max_{i \in [p]} \left\| \tanh \left( \frac{C(x_i)(I + \Delta)h}{\mu} \right) \right\|_2 \|\Delta\| \]

\[ + \max_{i \in [p]} \|C(x_i)\| \cdot \max_{i \in [p]} \left\| \tanh \left( \frac{C(x_i)h}{\mu} \right) - \tanh \left( \frac{C(x_i)(I + \Delta)h}{\mu} \right) \right\|_2 \]

\[ \leq \max_{i \in [p]} \|C(x_i)\| \cdot \left( \sqrt{n} \|\Delta\| + \frac{1}{\mu} \max_{i \in [p]} \|C(x_i)\| \|\Delta\| \right) \]

\[ \leq C_1 n \log(np) \|\Delta\|. \]

with probability at least \( 1 - (np)^{-8} \), where the penultimate inequality follows from (63), and the last inequality follows from (61). By Lemma 2, there exists some constant \( C \), such that under the sample complexity requirement, we have

\[ \left| \langle \partial f(h) - \partial f_o(h) \rangle \cdot \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) \right| \leq C_1 n^{3/2} \log(np) \frac{k^4 \sqrt{\log^2(n) \log(p)}}{\theta^2 p} \leq \frac{c_0 \xi_0 \theta}{2}. \tag{72} \]

In addition, Theorem 17 guarantees that \( \partial f_o(h)^\top \cdot \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) \geq c_0 \xi_0 \theta \). Putting together, we have

\[ \partial f(h)^\top \cdot \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) \geq \partial f_o(h)^\top \cdot \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) - \left| \langle \partial f(h) - \partial f_o(h) \rangle \cdot \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) \right| \]

\[ \geq c_0 \xi_0 \theta - \frac{c_0 \xi_0 \theta}{2} = \frac{c_0 \xi_0 \theta}{2} \]

with probability at least \( 1 - (np)^{-8} \).

### D.2 Proof of Lemma 5

Owing to symmetry, without loss of generality, we will show that if the current iterate \( h \in S_{\xi_0}^{(n+)} \) with \( \xi_0 \in (0, 1) \), the next iterate

\[ h^+ = \frac{h - \eta \partial f(h)}{\|h - \eta \partial f(h)\|_2} \]

33
stays in $S_{\xi_0}^{(n+)}$ for sufficiently small step size $\eta$. For any $i \in [n-1]$, we have
\[
\left(\frac{h_n^+}{h_i^+}\right)^2 = \frac{(h_n - \eta[\partial f(h)]_n)^2}{(h_i - \eta[\partial f(h)]_i)^2} = \frac{(1 - \eta[\partial f(h)]_n/h_n)^2}{(h_i/h_n - \eta[\partial f(h)]_i/h_n)^2}.
\] (73)

By (28b) in Lemma 4, which bounds $\|\partial f(h)\|_{\infty} \leq \|\partial f(h)\|_2 \leq Cn\sqrt{\log(np)}$ for some constant $C$, and $h_n \geq 1/\sqrt{n}$, by setting $\eta \leq \frac{1}{10Cn^{3/2}\sqrt{\log(np)}}$, we can lower bound the numerator of (73) as
\[
\|\eta\partial f(h)\|_{\infty}/h_n \leq \frac{1}{10} \quad \text{and} \quad (1 - \eta[\partial f(h)]_n/h_n)^2 \geq \frac{2}{3}.
\] (74)

To continue, we take a similar approach to [12, Lemma D.1], and divide our discussions of the denominator of (73) for different coordinates in three subsets:
\[
\mathcal{J}_0 := \{i \in [n-1] : h_i = 0\},
\]
(75a)
\[
\mathcal{J}_1 := \left\{ i \in [n-1] : \frac{h_i^2}{h_n^2} \geq 4, h_i \neq 0 \right\},
\] (75b)
\[
\mathcal{J}_2 := \left\{ i \in [n-1] : \frac{h_i^2}{h_n^2} \leq 4 \right\}.
\] (75c)

- For any index $i \in \mathcal{J}_0$, we have $h_i = 0$, and then by (73) and (74),
\[
\left(\frac{h_n^+}{h_i^+}\right)^2 = \frac{(1 - \eta[\partial f(h)]_n/h_n)^2}{(h_i/h_n - \eta[\partial f(h)]_i/h_n)^2} \geq \frac{2/3}{(1/10)^2} \geq 2.
\]

- For any index $i \in \mathcal{J}_1$, we have
\[
\left(\frac{h_n^+}{h_i^+}\right)^2 = \frac{(1 - \eta[\partial f(h)]_n/h_n)^2}{(h_i/h_n - \eta[\partial f(h)]_i/h_n)^2} \geq \frac{2/3}{(1/4 + 1/10^2)} \geq 2.
\]

- For any index $i \in \mathcal{J}_2$, we have
\[
\left(\frac{h_n^+}{h_i^+}\right)^2 = \frac{h_n^2}{h_i^2} \left(1 + \eta \frac{\partial f(h)\top(e_i/h_i - e_n/h_n)}{1 - \eta[\partial f(h)]_i/h_i}\right)^2.
\]

Using (28a) in Lemma 4, we have $\partial f(h)\top(e_i/h_i - e_n/h_n) \geq \frac{\theta}{2}\xi_0 > 0$, and consequently,
\[
\left(\frac{h_n^+}{h_i^+}\right)^2 \geq \frac{h_n^2}{h_i^2} \left(1 + \eta \frac{\partial f(h)\top(e_i/h_i - e_n/h_n)}{1 - \eta[\partial f(h)]_i/h_i}\right)^2 \geq \frac{h_n^2}{h_i^2} \geq 1 + \xi_0,
\]

where the last inequality is due to $h \in S_{\xi_0}^{(n+)}$.

Combing the above, we have that for all $i \in [n-1]$, $\left(\frac{h_n^+}{h_i^+}\right)^2 \geq 1 + \xi_0$, i.e, $h^+ \in S_{\xi_0}^{(n+)}$.

### D.3 Proof of Theorem 4

First, as the step size requirement satisfies that in Lemma 5, the iterates never jumps out of $S_{\xi_0}^{(n+)}$, if initialized in it. Denote $h_u^+$ as the unnormalized update of $h$ with step size $\eta$ on the tangent space of $h$, i.e,
\[
h_u^+ = h - \eta \partial f(h) = h - \eta (I - hh\top) \nabla_h f(h).
\]

and $w_u^+$ the first $(n-1)$-entries of $h_u^+$, whose update can be written with respect to $\phi(w)$ as
\[
w_u^+ = w - \eta [I \ 0] \cdot (I - hh\top) \nabla_h f(h) = w - \eta (I - wu\top) J_h(w) \nabla_h f(h) = w - \eta (I - wu\top) \nabla_w \phi(w).
\] (76)

The normalized updates are respectively $h^+ = h_u^+ / \|h_u^+\|_2$ and $w^+ = w_u^+ / \|w_u^+\|_2$. By the property that $h \perp (I - hh\top) \nabla_h f(h)$, we have $\|h^+\|_2 \geq \|h\|_2 \geq 1$. 

34
Convergence in the region of $Q_1 \cap \{w : h(w) \in S^{(n+)}_{\xi_0}\}$. By (76), we have
\[
\|w^+_n\|^2 - \eta h_n^2 w^T \nabla_w \phi(w) + \eta^2 \left\|\left(I - w w^T\right) \nabla_w \phi(w)\right\|^2.
\] (77)

- First, $I_1$ can be bounded as
\[
I_1 = h_n^2 w^T \nabla_w \phi(w) \geq c_1 h_n^2 \|w\|_2 \xi_0 \theta.
\]
for some constant $c_1$, where the last inequality owes to (24a).

- Second, $I_2$ can be bounded as
\[
I_2 \leq \left\|\left[I \ 0\right]\left[I - h h^T\right] \left\|\left(I - h h^T\right) \nabla_h f(h)\right\|_2 \leq 1 \cdot (1 + \|h\|_2^2) \left\|\nabla_h f(h)\right\|_2 \leq c_2 n \sqrt{\log(np)}
\]
for some constant $c_2$, where the last inequality follows from (28b) in Lemma 4, which holds with probability at least $1 - (np)^{-8}$.

Sum up the above results for $I_1$ and $I_2$, we have with probability at least $1 - (np)^{-8}$,
\[
\|w^+\|^2_2 \leq \|w^+_0\|^2 \leq \|w\|^2_2 - \eta c_1 h_n^2 \|w\|_2 \xi_0 \theta + c_2 n \sqrt{\log(np)},
\]
where the first inequality follows from $\|h^+_n\|^2_2 \geq 1$. Setting $\eta \leq \frac{c_1 h_n^2 \|w\|_2 \xi_0 \theta}{n \sqrt{\log(np)}} \leq \frac{c_1 h_n^2 \|w\|_2 \xi_0 \theta}{2c_2 n \sqrt{\log(np)}}$ for some sufficiently small $c$, we have
\[
\|w^+\|^2_2 \leq \|w\|^2_2 - \eta c_1 h_n^2 \|w\|_2 \xi_0 \theta \leq \|w\|^2_2 - \eta \frac{c_1}{2n} \|w\|^2_2 \xi_0 \theta.
\] (78)

Denote the $k$-th iteration as $h^{(k)}$, we have for all $k = 0, 1, \cdots, T - 1$, $w^{(k)} \in Q_1$, and
\[
\left\|w^{(k+1)}\right\|^2_2 \leq \left\|w^{(k)}\right\|^2 - \eta \frac{c_1}{2n} \left\|w^{(k)}\right\|_2 \xi_0 \theta \leq \left\|w^{(k)}\right\|^2 - \eta \frac{c_1 \mu}{8 \sqrt{2n}} \xi_0 \theta.
\]
Telescoping these $T_1$ inequalities, we have
\[
\frac{c_1 \mu}{8 \sqrt{2n}} \xi_0 \theta \leq \left\|w^{(0)}\right\|^2_2 - \left\|w^{(T_1)}\right\|^2_2 \leq 1, \quad \Rightarrow \quad T_1 \leq \frac{Cn}{\mu \eta \xi_0 \theta},
\]
which means it takes at most $T_1$ iterations to enter $Q_2 \cap \{w : h(w) \in S^{(n+)}_{\xi_0}\}$.

Convergence in the region of $Q_2 \cap \{w : h(w) \in S^{(n+)}_{\xi_0}\}$: Denoting the unique local minima in $Q_2 \cap \{w : h(w) \in S^{(n+)}_{\xi_0}\}$ as $w^*$, whose norm is bounded in Theorem 3. By setting $p$ sufficiently large, we can ensure that the iterates stay in $Q_2$ following a similar argument as (77). To begin, we note that $\nabla_w \phi(w)$ is $L$-Lipschitz with
\[
L \leq \frac{C n^{3/2} \log(np)}{\mu},
\] (79)
which is proved in Appendix D.3.1, and $c_1 n \theta / \mu$-strongly convex in $Q_2 \cap \{w : h(w) \in S^{(n+)}_{\xi_0}\}$. For $w \in Q_2$, we have
\[
\frac{1}{2} \leq 1 - \frac{\mu^2}{32} \leq \left\|\left[I - w w^T\right] \right\| \leq 1 + \frac{\mu^2}{32}.
\] (80)

Without loss of generality, consider the case when $\|w\|_2 \geq \|w^*\|_2$, otherwise we already achieve $\|w^+ - 0\|_2 \leq \epsilon \|w^*\|_2 + \epsilon \leq \frac{c_1}{2n} \sqrt{n \log(p) \mu} + \epsilon$. Recall the next iterate with respect to $w$ is given in (76). By the fundamental theorem of calculus, we have
\[
\left\|w^+_n - w^*\right\|_2 = \left\|w - w^* - \eta \left(I - w w^T\right) \nabla_w \phi(w)\right\|_2 \leq \epsilon \|w^*\|_2 + \epsilon \leq \frac{c_1}{2n} \sqrt{n \log(p) \mu} + \epsilon.
\]
= \left\| I - \eta (I - w w^\top ) \int_0^1 \nabla w^2 \phi(w(t)) dt \right\|_2 (w - w^\star ) \\
\leq \left\| I - \eta (I - w w^\top ) \int_0^1 \nabla w^2 \phi(w(t)) dt \right\|_2 \| w - w^\star \|_2 \\
\leq \left( 1 - \frac{c_1 \eta \theta}{2\mu} \right) \| w - w^\star \|_2 . \quad (81)

where \( w(t) := w + t(w^\star - w) \), \( t \in [0, 1] \), and the step size \( \eta \leq \frac{c_\mu \xi_0 \theta}{n^2 \log(np)} \leq \frac{1}{2\mu} \). Moreover, since
\[ w^+ = w^+ / \| h^+ \|_2 = w^+ / (1 + K) \] for some \( K > 0 \), we have
\[
\| w^+ - w^\star \|_2^2 = \| (1 + K) w^+ - w^\star \|_2^2 \\
= \| w^+ - w^\star \|_2^2 + (2K + K^2) \| w^+ \|_2^2 - 2K w^+ \top w^\star \\
\geq \| w^+ - w^\star \|_2^2 + (2K + K^2) \| w^+ \|_2^2 - 2K \| w^+ \|_2 \| w^\star \|_2 \| w^\star \|_2 \geq \| w^+ - w^\star \|_2^2 \quad (82)
\]
where the last inequality owes to \( \| w^+ \|_2 \geq \| w^\star \|_2 \) (otherwise the algorithm can stop). Combining (81) and (82), we have the update \( w^+ \) satisfies
\[
\| w^+ - w^\star \|_2 \leq \left( 1 - \frac{c_1 \eta \theta}{2\mu} \right) \| w - w^\star \|_2 . \quad (83)
\]
Therefore, to ensure \( \| w^+ - w^\star \|_2 \leq \epsilon \), it takes no more than
\[
T_2 \leq \frac{c_1 \mu}{n \theta} \log \left( \frac{3\mu}{2\epsilon} \right) \quad (84)
\]
iterations. Plugging into the bound for \( \eta \), the total number of iterates is bounded by
\[
T_1 + T_2 \lesssim \frac{n}{\mu \xi_0 \theta} + \frac{\mu}{n \theta} \log \left( \frac{\mu}{\epsilon} \right) = \frac{n^3 \sqrt{\log(np)}}{\mu^2 \xi_0 \theta^2} + \frac{n \sqrt{\log(np)}}{\theta^2 \xi_0} \log \left( \frac{\mu}{\epsilon} \right) .
\]

D.3.1 Proof of (79)
Recalling \( \psi'_\mu (x) = \tanh(x/\mu) \), for any \( w_1, w_2 \in \mathbb{Q}_2 \), we have
\[
\nabla w \phi(w_1) - \nabla w \phi(w_2) \\
= \frac{1}{p} \sum_{i=1}^p \left[ J_h(w_1) - J_h(w_2) \right] (I + \Delta)^\top C(x_i)^\top \psi'_\mu (C(x_i) (I + \Delta) h(w_1)) \\
\quad + \frac{1}{p} \sum_{i=1}^p J_h(w_2) (I + \Delta)^\top C(x_i)^\top [ \psi'_\mu (C(x_i) (I + \Delta) h(w_1)) - \psi'_\mu (C(x_i) (I + \Delta) h(w_2))] .
\]

We bound \( \| g_1 \|_2 \) and \( \| g_2 \|_2 \) respectively. Under the sample size requirement, from Lemma 2, we can ensure \( \| \Delta \| \leq 1 \).
To bound \( \| g_1 \|_2 \), we have
\[
\| g_1 \|_2 \leq \left\| J_h(w_1) - J_h(w_2) \right\| \cdot \| I + \Delta \| \cdot \max_{i \in [p]} \left\| C(x_i) \right\| \cdot \left\| \tanh \left( \frac{C(x_i) (I + \Delta) h(w_1)}{\mu} \right) \right\|_2 \\
\leq C \sqrt{n^2 \log(np)} \left\| \left[ 0 \ (\frac{w_2}{n_i(w_2)} - \frac{w_1}{n_i(w_1)}) \right] \right\|
\]
where the second line follows from \( |\tanh(\cdot)| \leq 1 \) and (61), which holds with probability at least \( 1 - (np)^{-8} \).
It is seen from the fundamental theorem of calculus that
\[
\begin{align*}
\| [0 \left( w_2 \frac{w_2}{h_n(w_2)} - \frac{w_1}{h_n(w_1)} \right)] & = \left\| \frac{w_2}{h_n(w_2)} - \frac{w_1}{h_n(w_1)} \right\|_2 \\
\leq & \left\| \frac{w_2}{h_n(w_2)} - \frac{w_1}{h_n(w_1)} \right\|_2 + \left\| \frac{w_1}{h_n(w_2)} - \frac{w_1}{h_n(w_1)} \right\|_2 \\
\leq & 2 \| w_2 - w_1 \|_2 + \| w_1 \|_2 \left( \frac{1}{\sqrt{1 - \| w_2 \|_2^2}} - \frac{1}{\sqrt{1 - \| w_1 \|_2^2}} \right) \\
\leq & 2 \| w_2 - w_1 \|_2 + 8 \max \left\{ \| w_1 \|_2^2, \| w_2 \|_2^2 \right\} \cdot \| w_2 - w_1 \|_2 \\
\leq & C \| w_2 - w_1 \|_2
\end{align*}
\]
where we used the fact that \( \| w \|_2 \leq \mu / (4 \sqrt{2}) \) in \( Q_2 \). Therefore, we have
\[
\| g_1 \|_2 \leq C \sqrt{n^2 \log(np)} \| w_2 - w_1 \|_2, \tag{85}
\]
for some constant \( C \).

To bound \( \| g_2 \|_2 \), we have
\[
\begin{align*}
\| g_2 \|_2 & \leq \left\| J_h(w_2) \right\| \cdot \left\| I + \Delta \right\| \cdot \max_{i \in [p]} \| C(x_i) \| \cdot \max_{i \in [p]} \left\| \tanh \left( \frac{C(x_i) (I + \Delta) h(w_1)}{\mu} \right) - \tanh \left( \frac{C(x_i) (I + \Delta) h(w_2)}{\mu} \right) \right\|_2 \\
& \leq \frac{1}{\mu} \left\| J_h(w_2) \right\| \cdot \left\| I + \Delta \right\|^2 \cdot \max_{i \in [p]} \| C(x_i) \|^2 \cdot \left\| h(w_1) - h(w_2) \right\|_2 \\
& \leq \frac{C \sqrt{n^2 / \log(np)}}{\mu} \| w_1 - w_2 \|_2,
\end{align*}
\]
where the second line follows from (63), the last line follows from (60) and (61), which holds with probability at least \( 1 - (np)^{-8} \), and \( \| h(w_1) - h(w_2) \|_2 \leq \sqrt{2} \| w_1 - w_2 \|_2 \) since \( \| w_1 \|_2 \leq \mu / (4 \sqrt{2}) \), \( \| w_2 \|_2 \leq \mu / (4 \sqrt{2}) \). Combining the bounds on \( \| g_1 \|_2 \) and \( \| g_2 \|_2 \) achieve the desired result.

### D.4 Proof of Lemma 17

The proof follows a standard covering argument similar to the proof of Theorem B.4. To begin, we need the following propositions, proved in Appendix D.4.1, D.4.2, and D.4.3, respectively.

**Proposition 3.** For any \( \xi_0 \in (0, 1) \), \( \theta \in (0, \frac{1}{2}) \), \( k \in [n-1] \), there exists some constant \( c_1 \) such that when \( \mu < c_1 \min\{\theta, \frac{1}{2} \} n^{-3/4} \}, \text{for any } h \in \mathcal{H}_k \), we have
\[
\mathbb{E}(\partial f_\xi(h))^\top \left( e_k - \frac{e_n}{h_n} \right) \geq \frac{\theta \xi_0}{2 \sqrt{2} \pi}.
\]

**Proposition 4.** For any \( \xi_0 \in (0, 1) \), \( \theta \in (0, \frac{1}{2}) \), \( k \in [n-1] \), there exists some constant \( c_1 \) such that when \( \mu < c_1 \min\{\theta, \frac{1}{2} \} n^{-3/4} \}, \text{for any fixed } h \in \mathcal{H}_k \), there exists some constant \( C \) such that for any \( t > 0 \)
\[
\mathbb{P} \left( \left| \partial f_\xi(h)^\top \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) - \mathbb{E} \partial f_\xi(h)^\top \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) \right| \geq t \right) \leq 2 \exp \left( \frac{-pt^2}{2Cn^3 \log(n) + 2 \sqrt{2Cn^3 \log(n)t}} \right).
\]

**Proposition 5.** For any \( \xi_0 \in (0, 1) \), \( \theta \in (0, \frac{1}{2}) \), \( k \in [n-1] \), \( \partial f_\xi(h)^\top \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) \) is \( L_3 \)-Lipschitz in the domain \( \mathcal{H}_k \) with
\[
L_3 \leq \max_{i \in [p]} \left( \frac{\sqrt{5n}}{\mu} \| C(x_i) \|^2 + 4n^{3/2} \| C(x_i) \| \right).
\]
We now continue to the proof of Lemma 17. In the subset $\mathcal{H}_k$, for any $0 \leq \epsilon \leq 2\sqrt{\frac{n}{n+5}}$, we have an $\epsilon$-net $\mathcal{N}_3$ of size at most $(3/\epsilon)^n$, where $\epsilon$ will be determined later. Under the event (61) and the assumption of $\mu$, we have

$$L_3 \leq \frac{c_{10}n^{3/2}}{\mu} \log(np).$$

For all $h \in \mathcal{H}_k$, there exists $h' \in \mathcal{N}_3$ such that $\|h' - h\|_2 \leq \epsilon$. By Proposition 5, we have

$$\left| \partial f_0(h) \cdot \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) - \partial f_0(h') \cdot \left( \frac{e_k}{h'_k} - \frac{e_n}{h'_n} \right) \right| \leq L_3 \|h' - h\|_2 \leq \frac{c_{10}n^{3/2}}{\mu} \log(np) \epsilon \leq \frac{c_\theta \xi_0}{3},$$

which holds when $\epsilon \leq \frac{c_\theta \xi_0}{n^{3/2} \log(np)}$ for some sufficiently small $c$. The covering number of $\mathcal{N}_3$ satisfies

$$|\mathcal{N}_3| \leq \exp \left( n \log \left( \frac{c_\theta n^{3/2} \log(np)}{\mu \theta \xi_0} \right) \right).$$

Let $\mathcal{A}_3$ denote the event

$$\mathcal{A}_3 := \left\{ \max_{h \in \mathcal{H}_k} \left| \partial f_0(h) \cdot \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) - \mathbb{E} \left( \partial f_0(h) \right) \cdot \left( \frac{e_k}{h'_k} - \frac{e_n}{h'_n} \right) \right| \leq \frac{c_\theta \xi_0}{3} \right\},$$

which holds with probability at least

$$1 - |\mathcal{N}_3| \cdot 2 \exp \left( -\frac{pt^2}{2Cn^3 \log(n) + 2\sqrt{Cn^3 \log(n)t}} \right) \geq 1 - \exp \left( -\frac{c_{11} p \theta^2 \xi_0^2}{n^3} + n \log \left( \frac{c_{10} n^{3/2} \log(np)}{\mu \theta \xi_0} \right) \right) \geq 1 - \exp \left( -c_{12} n \right),$$

provided $p \geq C \frac{n^{3/2} \log(np)}{\mu \theta \xi_0}$, by the union bound and setting $t = \frac{c_\theta \xi_0}{3}$ in Proposition 4. Finally, we have for all $h \in \mathcal{H}_k$,

$$\partial f_0(h)^\top \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) = \left[ \partial f_0(h)^\top \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) - \partial f_0(h')^\top \left( \frac{e_k}{h'_k} - \frac{e_n}{h'_n} \right) \right] + \left[ \partial f_0(h')^\top \left( \frac{e_k}{h'_k} - \frac{e_n}{h'_n} \right) - \mathbb{E} \left( \partial f_0(h') \right) \cdot \left( \frac{e_k}{h'_k} - \frac{e_n}{h'_n} \right) \right] + \mathbb{E} \left( \partial f_0(h') \right) \cdot \left( \frac{e_k}{h'_k} - \frac{e_n}{h'_n} \right) \geq -\frac{c_\theta \xi_0}{3} - \frac{c_\theta \xi_0}{3} + c_\theta \xi_0 = \frac{c_\theta \xi_0}{3}.$$

### D.4.1 Proof of Proposition 3

First, recall a few notations introduced in Appendix B.1. For $x = \Omega \odot z \sim_{\text{iid}} \text{BG}(\theta) \in \mathbb{R}^n$, we denote the first $n - 1$ dimension of $x$, $z$ and $\Omega$ as $\bar{x}$, $\bar{z}$ and $\bar{\Omega}$, respectively. Denote $\mathcal{I}$ as the support of $\Omega$ and $\mathcal{J}$ as the support of $\bar{\Omega}$. For any $k \in [n - 1]$ with $h_k \neq 0$, by (68) and (29), we have

$$\mathbb{E} \left( \partial f_0(h) \right)^\top \cdot \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) = \mathbb{E} \left( \nabla f_0(h) \right)^\top \cdot \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) = n \cdot \mathbb{E} \left( \nabla \psi_\mu(x^\top h) \right)^\top \cdot \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right),$$

since the rows of $C(x)$ has the same distribution as $x \sim_{\text{iid}} \text{BG}(\theta)$. Further plugging in (32), we rewrite it as:

$$\mathbb{E} \left( \nabla \psi_\mu(x^\top h) \right)^\top \cdot \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) = \mathbb{E} \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right)^\top \tanh \left( \frac{x^\top h}{\mu} \right) x = \mathbb{E} \left[ \tanh \left( \frac{x^\top h}{\mu} \right) x \right] - \mathbb{E} \left[ \tanh \left( \frac{x^\top h}{\mu} \right) x \right] = \mathbb{E}_\Omega \mathbb{E}_z \left[ \tanh \left( \frac{x^\top h}{\mu} \right) x \right] - \mathbb{E}_\Omega \mathbb{E}_z \left[ \tanh \left( \frac{x^\top h}{\mu} \right) x \right].$$

38
Evaluating $\mathbb{E}_{\Omega}$ over $\Omega_k$, $\Omega_{(k)}$, and $\Omega_{\{k\}}$ sequentially, we can express $I_1$, $I_2$ respectively as:

\[
I_1 = \theta \mathbb{E}_{\Omega_{\{k\}}} \left[ \tanh \left( \frac{\mathbf{x}_i^\top \mathbf{w}_{\{k\}} + h_k z_k}{\mu} \right) \right] \frac{z_k}{h_k}.
\]

\[
I_2 = (1 - \theta) \mathbb{E}_{\Omega_{\{k\}}} \left[ \tanh \left( \frac{\mathbf{x}_i^\top \mathbf{w}_{\{k\}} + h_k z_k}{\mu} \right) \right] \frac{z_k}{h_k} + \theta \mathbb{E}_{\Omega_{\{k\}}} \left[ \tanh \left( \frac{\mathbf{x}_i^\top \mathbf{w}_{\{k\}} + h_k z_k + h_n z_n}{\mu} \right) \right] \frac{z_k}{h_k}.
\]

Introducing the short-hand notations $X_1 = h_k z_k \sim \mathcal{N}(0, h_k^2)$, $Y_1 = \mathbf{x}_i^\top \mathbf{w}_{\{k\}} + h_n z_n$, $X_2 = h_n z_n \sim \mathcal{N}(0, h_n^2)$, $Y_2 = \mathbf{x}_i^\top \mathbf{w}_{\{k\}} + h_k z_k$. Invoking Lemma 10, the sum of the second terms of $I_1$ and $I_2$ is

\[
\mathbb{E}_{\Omega_{\{k\}}} \left[ \mathbb{E}_{X_1,Y_1} \left( 1 - \tanh^2 \left( \frac{X_1 + Y_1}{\mu} \right) \right) \right] - \mathbb{E}_{\Omega_{\{k\}}} \left[ \mathbb{E}_{X_2,Y_2} \left( 1 - \tanh^2 \left( \frac{X_2 + Y_2}{\mu} \right) \right) \right] = 0.
\]

Consequently, we have

\[
\mathbb{E} \left( \nabla \psi_{\mu}^\top (\mathbf{x}^\top \mathbf{h}) \right) \cdot \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) = \frac{\theta(1 - \theta)}{\mu} \mathbb{E}_{\Omega_{\{k\}}} \left[ \left( 1 - \tanh^2 \left( \frac{\mathbf{x}_i^\top \mathbf{w}_{\{k\}} + h_k z_k}{\mu} \right) \right) \right] - \left( 1 - \tanh^2 \left( \frac{\mathbf{x}_i^\top \mathbf{w}_{\{k\}} + h_k z_k}{\mu} \right) \right)
\]

\[
\geq \frac{\theta(1 - \theta)}{\mu} \frac{\xi_0}{16 \sqrt{\pi}} = \frac{\theta \xi_0}{24 \sqrt{\pi}},
\]

where the second line follows from Lemma 10, and the last line follows from (46) and $\theta \in (0, 1/3)$. Finally, we have

\[
\mathbb{E} (\partial f_o(h))^\top \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) = n \cdot \mathbb{E} \left( \nabla \psi_{\mu}^\top (\mathbf{x}^\top \mathbf{h}) \right) \cdot \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) \geq \frac{\theta \xi_0}{24 \sqrt{\pi}}.
\]

### D.4.2 Proof of Proposition 4

We start by writing the directional gradient as the sum of p i.i.d. random variables:

\[
\partial f_o(h)^\top \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) = \nabla f_o(h)^\top \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) = \frac{1}{p} \sum_{i=1}^{p} \tanh \left( \frac{C(x_i) h}{\mu} \right)^\top C(x_i) \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right),
\]

where the first equality is due to (68) and the second equality is due to (29). Moreover,

\[
|Z_i| \leq \left\| \tanh \left( \frac{C(x_i) h}{\mu} \right) \right\|_2 \cdot \left\| \frac{e_k}{h_k} - \frac{e_n}{h_n} \right\|_2 \cdot \left\| C(x_i) \right\| \leq \sqrt{n} \left\| \frac{e_k}{h_k} - \frac{e_n}{h_n} \right\|_2 \cdot \left\| C(x_i) \right\| \leq \sqrt{5n} \left\| C(x_i) \right\|,
\]

where the second inequality follows from $|\tanh(|)\leq 1$ and the third inequality follows from (71). Therefore, for any $m \geq 2$, the moments of $|Z_i|$ can be controlled by Lemma 12 as

\[
\mathbb{E} |Z_i|^m \leq \left( \sqrt{5n} \right)^m \mathbb{E} \left\| C(x) \right\|^m \leq \frac{m!}{2} (C n^3 \log(n))^{m/2}.
\]

The proof is then completed by setting $\sigma^2 = Cn^3 \log(n)$, $R = \sqrt{C n^3 \log(n)}$ and applying the Bernstein’s inequality in Lemma 9.
D.4.3 Proof of Proposition 5

Using (88), we have for any \( h, h' \),

\[
\left| \partial f_o(h)^\top \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) - \partial f_o(h')^\top \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) \right|
\]

\[
= \frac{1}{p} \sum_{i=1}^{p} \left[ \tanh \left( \frac{C(x_i)h}{\mu} \right) C(x_i) \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) - \tanh \left( \frac{C(x_i)h'}{\mu} \right) C(x_i) \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) \right]
\]

\[
\leq \frac{1}{p} \sum_{i=1}^{p} \left[ \tanh \left( \frac{C(x_i)}{\mu} \right) - \tanh \left( \frac{C(x_i)h'}{\mu} \right) \right]^\top C(x_i) \left[ \frac{e_k}{h_k} - \frac{e_n}{h_n} \right] - \left[ \frac{e_k}{h_k} - \frac{e_n}{h_n} \right]
\]

where the second line follows by the triangle inequality. In the sequel, we’ll bound \( A_i \) and \( B_i \) respectively.

- To bound \( A_i \), we have

\[
A_i \leq \left\| \tanh \left( \frac{C(x_i)h}{\mu} \right) - \tanh \left( \frac{C(x_i)h'}{\mu} \right) \right\|_2 \cdot \| C(x_i) \| \cdot \left\| \frac{e_k}{h_k} - \frac{e_n}{h_n} \right\|_2
\]

\[
\leq \frac{\sqrt{5n}}{\mu} \| C(x_i) \|^2 \| h - h' \|_2,
\]

where the second line follows from (63) and (71).

- To bound \( B_i \), we have

\[
B_i \leq \left\| \tanh \left( \frac{C(x_i)h'}{\mu} \right) \right\|_2 \cdot \| C(x_i) \| \cdot \left\| \frac{e_k}{h_k} - \frac{e_n}{h_n} \right\|_2
\]

\[
\leq \sqrt{n} \| C(x_i) \| \sqrt{\left( \frac{1}{h_k'} - \frac{1}{h_k} \right)^2 + \left( \frac{1}{h_n'} - \frac{1}{h_n} \right)^2}
\]

\[
\leq 4n^{3/2} \| C(x_i) \| \sqrt{(h_k - h_k')^2 + (h_n - h_n')^2} \leq 4n^{3/2} \| C(x_i) \| \| h - h' \|_2
\]

where the first line follows from \( | \tanh(\cdot) | \leq 1 \), and the second line follows from \( h_n, h'_n \geq 1/\sqrt{n} \) and \( h_n^2/h_k^2 < 4, (h'_n)^2/(h'_k)^2 < 4 \).

Combining terms, we have

\[
\left| \partial f_o(h)^\top \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) - \partial f_o(h')^\top \left( \frac{e_k}{h_k} - \frac{e_n}{h_n} \right) \right| \leq \max_{i \in [p]} \left( \frac{\sqrt{5n}}{\mu} \| C(x_i) \|^2 + 4n^{3/2} \| C(x_i) \| \right) \| h - h' \|_2
\]