Analytical Convergence Regions of Accelerated First-Order Methods in Nonconvex Optimization under Regularity Condition

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October 2018

Abstract

Gradient descent (GD) converges linearly to the global optimum for even nonconvex problems when the loss function satisfies certain benign geometric properties that are strictly weaker than strong convexity. One important property studied in the literature is the so-called Regularity Condition (RC). The RC property has been proven valid for many problems such as deep linear neural networks, shallow neural networks with nonlinear activations, phase retrieval, to name a few. Moreover, accelerated first-order methods (e.g. Nesterov’s accelerated gradient and Heavy-ball) achieve great empirical success when the parameters are tuned properly but lack theoretical understandings in the nonconvex setting. In this paper, we use tools from robust control to analytically characterize the region of hyperparameters that ensure linear convergence of accelerated first-order methods under RC. Our results apply to all functions satisfying RC and therefore are more general than results derived for specific problem instances. We derive a set of Linear Matrix Inequalities (LMIs), based on which analytical regions of convergence are obtained by exploiting the Kalman-Yakubovich-Popov (KYP) lemma. Our work provides deeper understandings on the convergence behavior of accelerated first-order methods in nonconvex optimization.

1 Introduction

In many machine learning problems, we are interested in estimating an unknown set of parameters by minimizing certain loss function, e.g. the empirical risk, given as

$$\min_{z \in \mathbb{R}^n} f(z),$$

where $f(z)$ is nonconvex in general, and sometimes is even nonsmooth. Denote $x^*$ as the global minimizer of $f(z)$. In practice, it is very popular to use first-order methods, e.g. GD and its variants, to solve (1) due to its scalability to large-scale problems.

1.1 Convergence of GD under RC

While the convergence of GD in the convex setting is well-understood (Bubeck, 2015), its behavior is much less clear in the nonconvex setting with possibly nonsmooth loss functions. It turns out that, much weaker geometric properties are sufficient to guarantee the linear convergence of GD even when $f(z)$ is nonconvex and nonsmooth (Necoara et al., 2015). One popular property utilized in the literature is the Regularity Condition (RC) (Candès et al., 2015), defined as follows.

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Definition 1 (Regularity Condition). A function \( f(\cdot) \) is said to satisfy the Regularity Condition \( RC(\mu, \lambda) \) with positive constants \( \mu, \lambda > 0 \), if for all \( z \in \mathbb{R}^n \) we have

\[
\langle \nabla f(z), z - x^* \rangle \geq \frac{\mu}{2} \| \nabla f(z) \|^2 + \frac{\lambda}{2} \| z - x^* \|^2.
\] (2)

It is straightforward to check that one must have \( \mu \lambda \leq 1 \) by Cauchy-Schwartz inequality. The RC can be regarded as a combination of one-point strong convexity and smoothness, and does not require the function \( f(z) \) to be convex or smooth. In particular, it is straightforward that GD,\(^1\) which follows the update rule

\[
z_{k+1} = z_k - \alpha \nabla f(z_k), \quad k = 0, 1, \ldots
\] (3)

with \( \alpha \) being the step size and \( z_0 \) some initial guess, converges linearly:

\[
\| z_{k+1} - x^* \|^2 \leq (1 - \alpha \lambda) \| z_k - x^* \|^2,
\] (4)

as long as the step size satisfies \( \alpha \leq \mu \). This simple observation leads to the study of identifying nonconvex problems that satisfy RC, at least locally in a neighborhood of \( x^* \), as it implies that problems satisfying RC can be solved efficiently via GD, possibly with proper initializations. A partial list of relevant problems include phase retrieval (Candès et al., 2015; Chen and Candès, 2017; Zhang et al., 2017, 2016; Wang et al., 2017), deep linear neural networks (Zhou and Liang, 2017), shallow nonlinear neural networks (Li and Yuan, 2017), matrix sensing (Tu et al., 2016), to name a few.

1.2 Accelerated First-order Methods

In practice, accelerated gradient descent (AGD) methods are widely adopted to speed up the convergence of vanilla GD. Two widely-used acceleration schemes include Nesterov’s accelerated gradient (NAG) method (Nesterov, 2013), given as

\[
y_k = (1 + \beta)z_k - \beta z_{k-1}, \\
z_{k+1} = y_k - \alpha \nabla f(y_k), \quad k = 0, 1, \ldots
\] (5)

where \( \alpha > 0 \) is the step size, \( 0 \leq \beta < 1 \) is the momentum parameter; and Heavy-Ball (HB) method (Polyak, 1964), given as

\[
y_k = (1 + \beta)z_k - \beta z_{k-1}, \\
z_{k+1} = y_k - \alpha \nabla f(z_k), \quad k = 0, 1, \ldots
\] (6)

where \( \alpha > 0 \) is the step size, \( 0 \leq \beta < 1 \) is the momentum parameter.

Despite the empirical success, the convergence of AGD in the nonconvex setting remains unclear to a large extent. For example, it is not known whether AGD converges under RC, whether it converges linearly if it does and how to set the step size and the momentum parameter to guarantee its convergence.

1.3 Main Contribution

The main contribution of this paper is an analytical characterization of hyperparameter choices that ensure linear convergence of the AGD methods for all functions that satisfy RC. Specifically, for a given RC(\( \mu, \lambda \)), we characterize the analytical region for \( (\alpha, \beta) \) where AGD is guaranteed to converge linearly. Due to the complicated expressions of the region, we first provide an example to illustrate the flavor of our result in Figure 1. For \( \mu = 0.5 \) and \( \lambda = 0.5 \), Figure 1 provides regions for \( (\alpha, \beta) \) that guarantee HB and NAG to converge. Such characterizations apply to all functions as long as they satisfy RC. When the momentum parameter \( \beta = 0 \), it coincides with the range of step size allowable by GD, while providing much richer information when \( \beta > 0 \). A 3D visualization which further explains the relationship between convergence regions and the RC parameters can be found in Figure 2 of Section 2. To the best of our knowledge, our work provides the first analytical characterization of convergence regions of AGD under the RC.

\(^1\)For nonsmooth loss functions, \( \nabla f(x) \) should be understood as the generalized gradient or the subgradient (Clarke, 1975).
Figure 1: Visualization of the convergence regions of two AGD methods taking RC parameters as $\mu = 0.5, \lambda = 0.5$.

As will be explained in detail later, our main technical tools are borrowed from robust control, which provides further insights into the connections between optimization and control in the nonconvex setting. Inspired by Lessard et al. (2016), we view first-order optimization algorithms as linear dynamical systems subject to nonlinear feedback, by which the convergence of an algorithm becomes equivalent to the stability of the associated dynamical system. Such viewpoint allows us to use Linear Matrix Inequalities (LMIs) to characterize convergence regions. Different from Lessard et al. (2016), we analytically derive a region of $(\alpha, \beta)$ that guarantees the feasibility of the proposed LMIs by exploiting their connection to frequency domain inequalities via the so-called KYP lemma (Rantzer, 1996). We also extend our analytical results to a general case where RC only holds locally, and thus cover a wider range of applications.

1.4 Related Works

Convergence analysis has always been a key part in optimization theory. Typically this task is done in a case-by-case manner and the analysis techniques highly depend on the structure of algorithms and assumptions of objective functions. However, by translating iterative algorithms and prior information of objective functions as feedback dynamical systems, we can incorporate tools from control theory to carry out the convergence analysis in a more systematic way. Such framework is pioneered by Lessard et al. (2016), where the authors proposed small semidefinite programming to analyze the convergence of a class of optimization algorithms including GD, HB and NAG, by assuming the gradient of the loss function satisfies some integral quadratic constraints (IQCs). Notice the standard smoothness and convexity assumptions can be well decoded as IQCs. Afterwards, a series of work have appeared to unify the convergence rate analysis of more algorithms such as the ADMM (Nishihara et al., 2015), distributed methods (Sundararajan et al., 2017), proximal algorithms (Fazlyab et al., 2018), and stochastic finite-sum methods (Hu et al., 2017b, 2018) by generalizing the connections between control and optimization. Similar ideas are also used to give alternative insights of the momentum methods (Hu and Lessard, 2017; Wilson et al., 2016) and design new algorithms (Van Scoy et al., 2018; Cyrus et al., 2018; Dhingra et al., 2018; Kolarjani et al., 2018). Moreover, control tools are also useful in analyzing the robustness of algorithms against computation inexactness (Lessard et al., 2016; Cherukuri et al., 2017; Hu et al., 2017a).

None of these previous papers gives an analytical characterization of the behaviors of accelerated first-order methods under RC. In this paper, we aim to bridge the gap between the unified control analysis and non-convex optimization under RC.

1.5 Paper Organization

The rest of the paper is organized as follows. In Section 2, we state our main results and show some insights obtained from the analytical convergence regions. Section 3 presents some backgrounds and key concepts in control theory that are crucial for deriving the main results. Section 4 presents the main steps and ideas for
proving the main results. In Section 5, we briefly discuss how to extend the results to the case where RC only holds locally.

2 Main Results

We first present the analytical convergence region of HB under RC.

**Theorem 1.** Let \( x^* \in \mathbb{R}^n \) be a minimizer of the loss function \( f(\cdot) \) which satisfies RC(\( \mu, \lambda \)). For any step size \( \alpha > 0 \) and momentum parameter \( \beta \in (0, 1) \) lying in the region:

\[
\{(\alpha, \beta) : H_1(\beta) \leq \alpha \leq \frac{2(\beta + 1)(1 - \sqrt{1 - \mu \lambda})}{\lambda(1 + 2\beta)} \} \cup \{(\alpha, \beta) : 0 < \alpha \leq \min\{N_1(\beta), N_2(\beta)\}\},
\]

where

\[
H_1(\beta) = \frac{\mu \beta^2 + 6\mu \beta + \mu}{\beta + 1}, H_2(\beta) = \frac{P_2(\beta) - \sqrt{P_2(\beta)^2 - 4P_1(\beta)P_3(\beta)}}{2P_1(\beta)},
\]

with \( P_1(\beta) = 4\mu \lambda \beta - \beta^2 - 1 - 2\beta, P_2(\beta) = 2\mu + 2\mu^2 - 2\mu^3 - 2\mu, \) and \( P_3(\beta) = 4\mu^2 \beta^3 + 4\mu^2 \beta - 6\mu^2 \beta^2 - \mu^2 \beta^4 - \mu^2 \), the iterates \( z_k \) generated by the Heavy-Ball method (6) converge linearly to \( x^* \) as \( k \to \infty \).

The next theorem provides the analytical region of convergence for NAG under RC.

**Theorem 2.** Let \( x^* \in \mathbb{R}^n \) be a minimizer of the loss function \( f(\cdot) \) which satisfies RC(\( \mu, \lambda \)). For any step size \( \alpha > 0 \) and momentum parameter \( \beta \in (0, 1) \) lying in the region:

\[
\{(\alpha, \beta) : N_1(\beta) \leq \alpha \leq \frac{2(\beta + 1)(1 - \sqrt{1 - \mu \lambda})}{\lambda(1 + 2\beta)} \} \cup \{(\alpha, \beta) : 0 < \alpha \leq \min\{N_1(\beta), N_2(\beta)\}\},
\]

where

\[
N_1(\beta) = \frac{Q_1(\beta) - \sqrt{Q_1(\beta)^2 - (1 + 6\beta + \beta^2)Q_2(\beta)}}{2\lambda \beta(\beta + 1)},
\]

\[
N_2(\beta) = \left\{ \beta : \frac{Q_3(\beta) - \sqrt{Q_3(\beta)^2 - (1 - \beta)^2Q_2(\beta)}}{2\lambda \beta(\beta + 1)} \leq \alpha, g \left( \frac{(\mu - \alpha)(1 + \beta)^2 + (\lambda \alpha^2 - \alpha)(\beta + \beta^2)}{4\mu \beta - 4\alpha \beta} \right) = 0 \right\},
\]

with \( Q_1(\beta) = 1 + 7\beta + 2\beta^2, Q_2(\beta) = 4\mu \lambda \beta(1 + \beta), Q_3(\beta) = 1 - \beta + 2\beta^2, \) the iterates \( z_k \) generated by the Nesterov’s accelerated gradient method (5) converge linearly to \( x^* \) as \( k \to \infty \).

**Remark 1.** The bound \( N_2(\beta) \) is an implicit function of \( \beta \). It is hard to derive an explicit expression since \( g(\cdot) = 0 \) is a 4th order equation of \( \beta \). The function \( g(\eta) \) is:

\[
g(\eta) := 4\mu \beta \eta^2 - 2(2\mu + \beta^2 - \alpha \beta + \mu - \alpha) \eta + 2\mu + 2\mu^2 - 2\alpha - 2\alpha \beta + \lambda \alpha^2.
\]

The analytical results stated in the above theorems can provide rich insights on the convergence behaviors of AGD methods. Take the convergence region of HB as an example. In Figure 2 (a), we fix the RC parameter \( \lambda = 0.5 \) and vary \( \mu \) within [0.01, 1.9], while in Figure 2 (b), we fix \( \mu = 0.5 \) and vary the value of \( \lambda \) within [0.01, 1.9]. Observe that when we fix one of the RC parameter and increase the other, the stability region of \((\alpha, \beta)\) gets larger.

Notice that \( \mu \) plays a role similar to the inverse of the smoothness parameter, and therefore, it dominantly determines the step size, which is clearly demonstrated in Figure 2 (a). In addition, when we fix the values of a pair of \((\mu, \lambda)\) (e.g. Figure 1), the convergence region coincides with the bound of GD (see Subsection 3.3) when \( \beta = 0 \), which is as expected. Moreover, we can also see in Figure 1 that even when \( \alpha \) exceeds the value of the bound of GD, the Heavy-ball method can still ensure convergence when we choose \( \beta \) properly. This property has not been discussed in the literature.

We have seen that analytical convergence regions of AGD under RC can provide interesting insights. In the following, we will show how to derive such analytical result of a general first-order method for a class of nonconvex problems whose loss functions satisfy RC.
Fixing $\lambda = 0.5$ and varying $\mu$.

(b) Fixing $\mu = 0.5$ and varying $\lambda$.

Figure 2: Visualization of the convergence regions of HB when perturbing the RC parameters.

3 A Control View on Convergence Analysis under RC

Robust control theory has been tailored for convergence analysis of optimization methods (Lessard et al., 2016; Nishihara et al., 2015; Hu and Lessard, 2017; Hu et al., 2017b). The proofs of our main theorems also rely on such techniques. In this section, we will discuss how to transform convergence analysis under RC to robust stability analysis problems and derive a set of LMI conditions to guarantee convergence under RC. We will use GD as an example to illustrate how control tools can be used to show algorithm convergence under RC. The discussions in this section provide necessary backgrounds for understanding the proofs of the main theorems in Section 4.

3.1 A Review of Feedback Dynamical System Perspective on First-order Methods

As discussed in Lessard et al. (2016), GD (3), NAG (5), and HB (6) can all be viewed as linear dynamical systems subject to nonlinear feedback:

$$
\begin{align*}
\dot{z}_k^{(1)} &= (1 + \beta_1)z_k^{(1)} - \beta_1 z_k^{(2)} - \alpha u_k, \\
\dot{z}_k^{(2)} &= z_k^{(1)}, \\
y_k &= (1 + \beta_2)z_k^{(1)} - \beta_2 z_k^{(2)}, \\
u_k &= \nabla f(y_k).
\end{align*}
$$

To see this, let $z_k^{(1)} = z_k$, $z_k^{(2)} = z_{k-1}$. Then it can be easily verified that (7) represents GD when $(\beta_1, \beta_2) = (0, 0)$, HB when $(\beta_1, \beta_2) = (\beta_0)$, and NAG when $(\beta_1, \beta_2) = (\beta, \beta)$.

Let $\otimes$ denote the Kronecker product. Then we use the notation $G(A, B, C, D)$ to denote a dynamical system $G$ governed by the following iterative state-space model:

$$
\begin{align*}
\phi_{k+1} &= (A \otimes I_n)\phi_k + (B \otimes I_n)u_k, \\
y_k &= (C \otimes I_n)\phi_k + (D \otimes I_n)u_k.
\end{align*}
$$

If we define $\phi_k = \begin{bmatrix} z_k^{(1)} \\ z_k^{(2)} \end{bmatrix}$ as the state, $u_k$ as the input and $y_k$ as the output, the algorithms (7) can be represented as a dynamical system shown in Figure 3, where the feedback $\nabla f(y_k)$ is a static nonlinearity that depends on the gradient of the loss function, and $G(A, B, C, D)$ is a linear system specified by

$$
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} =
\begin{bmatrix}
1 + \beta_1 & -\beta_1 & -\alpha \\
1 & 0 & 0 \\
1 + \beta_2 & -\beta_2 & 0
\end{bmatrix}.
$$

(8)
Next, we will further explain how to transfer the convergence analysis of an algorithm under RC into the stability analysis of the feedback dynamical system.

### 3.2 Convergence Analysis under RC via Stability Analysis of a Feedback System

Convergence analysis of optimization algorithms typically consists of two steps. First, one must show that the algorithm has a fixed point. Second, one needs to prove that the algorithm converges to this optimal solution at a specified rate from some reasonable starting point. For dynamical systems, such analysis is called stability analysis and a fixed point is usually named as an equilibrium.

To illustrate, we define $\phi^* = \begin{bmatrix} x^* \\ x^* \end{bmatrix}$ as the equilibrium of the dynamical system (7). If the system is (globally) asymptotically stable, then $\phi_k \xrightarrow{k \to \infty} \phi^*$. It further implies $z_k \xrightarrow{k \to \infty} x^*$. In other words, the asymptotic stability of the dynamical system can indicate the convergence of the iterates to the fixed point. From now on, we can focus on the stability analysis of the feedback control system (7).

The stability analysis of the feedback control system (7) can be carried out using robust control theory. The main challenge is on the nonlinear feedback term $u_k = \nabla f(y_k)$. Our key observation is that RC is equivalent to a quadratic constraint imposed on the feedback block, i.e.,

$$
\begin{bmatrix}
    y_k - y^*
    u_k - u^*
\end{bmatrix}^T
\begin{bmatrix}
    -\lambda I_n & I_n \\
    I_n & -\mu I_n
\end{bmatrix}
\begin{bmatrix}
    y_k - y^*
    u_k - u^*
\end{bmatrix} \geq 0.
$$

Applying the quadratic constraint framework in Lessard et al. (2016), we can derive an LMI as a sufficient stability condition as stated in the following proposition. A formal proof of this result can be found in the appendix.

**Proposition 1.** Let $x^* \in \mathbb{R}^n$ be a minimizer of the loss function $f(\cdot)$ which satisfies $RC(\mu, \lambda)$. For a given first-order method characterized by $G(A, B, C, D)$, if there exists a matrix $P \succ 0$ and $\rho \in (0, 1)$ such that the following LMI (10) holds,

$$
\begin{bmatrix}
    A^T P A - \rho^2 P & A^T P B \\
    B^T P A & B^T P B
\end{bmatrix} + \begin{bmatrix}
    C & D \\
    0_{1 \times 2} & 1
\end{bmatrix}^T
\begin{bmatrix}
    -\lambda I_n & I_n \\
    I_n & -\mu I_n
\end{bmatrix}
\begin{bmatrix}
    C & D \\
    0_{1 \times 2} & 1
\end{bmatrix} \preceq 0,
$$

then the state $\phi_k$ generated by the first-order algorithm $G(A, B, C, D)$ converges to the fixed point $\phi^*$ exponentially, i.e.,

$$
\|\phi_k - \phi^*\| \leq \sqrt{\text{cond}(P)} \rho^k \|\phi_0 - \phi^*\| \text{ for all } k.
$$

**Remark 2.** For fixed $(A, B, C, D)$ and $\rho$, the condition (10) is linear in $P$ and hence a linear matrix inequality (LMI). The size of this LMI is $3 \times 3$, and the decision variable $P$ is a $2 \times 2$ matrix. The size of the LMI (10) is independent of the state dimension $n$.

The above LMI stability condition is similar to the one derived in Lessard et al. (2016) under the so-called sector bound nonlinearity. Different from Lessard et al. (2016), we focus on deriving analytical solutions (see Section 4), which offers deeper insight than numerical solutions regarding the convergence behavior of the AGD methods. In addition, we also extend the results to the case where RC holds only locally around the fixed point (see Section 5).
3.3 A Simple Example: Convergence of GD under RC

Before proving the general analytical results, we first use GD as a simple example to illustrate how to use the abovementioned framework to derive analytical convergence conditions.

Recall that (7) represents GD when $(\beta_1, \beta_2) = (0, 0)$. In this simple case, the state dimension can be reduced by half by letting $\phi_k = z_k$, $\phi_x = x^*$, $y_k = z_k$ and $u_k = \nabla f(z_k)$. Then we aim to prove

$$\|z_{k+1} - x^*\|_2^2 \leq \rho^2 \|z_k - x^*\|_2^2.$$  \hspace{1cm} (11)

Alternatively, by plugged in $z_{k+1} = z_k - \alpha \nabla f(z_k)$, (11) is equivalent to

$$\begin{bmatrix} z_k - x^* \\ \nabla f(z_k) \end{bmatrix}^T \begin{bmatrix} (1 - \rho^2)I_n & -\alpha I_n \\ -\alpha I_n & \alpha^2 I_n \end{bmatrix} \begin{bmatrix} z_k - x^* \\ \nabla f(z_k) \end{bmatrix} \leq 0.$$  \hspace{1cm} (12)

Similarly to (9), the RC condition is equivalent to the following inequality:

$$\begin{bmatrix} z_k - x^* \\ \nabla f(z_k) \end{bmatrix}^T \begin{bmatrix} -\lambda I_n & I_n \\ I_n & -\mu I_n \end{bmatrix} \begin{bmatrix} z_k - x^* \\ \nabla f(z_k) \end{bmatrix} \geq 0.$$  \hspace{1cm} (13)

Clearly, (13) is a quadratic constraint of the feedback $u_k = \nabla f(z_k)$ ($y_k = z_k$ for GD) in the system (7). To show the asymptotic stability of the system, if suffices to show that (12) holds whenever given (13). If we set $P = \frac{1}{s} > 0$, the condition (10) can be equivalently solved by looking for a positive $s$, such that

$$\begin{bmatrix} 1 - \rho^2 & -\alpha \\ -\alpha & \alpha^2 \end{bmatrix} + s \begin{bmatrix} -\lambda & 1 \\ 1 & -\mu \end{bmatrix} \preceq 0.$$  \hspace{1cm} (14)

(14) is a small $2 \times 2$ linear matrix inequality. By checking the principal minors of the left hand side of (14), we can obtain the convergence region of GD under RC.

**Theorem 3.** Let $x^* \in \mathbb{R}^n$ be a minimizer of the loss function $f(\cdot)$ which satisfies RC($\mu$, $\lambda$). If the step size obeys that $\alpha < \frac{2 - 2\sqrt{1 - \mu \lambda}}{\lambda}$, then GD can guarantee the linear convergence as

$$\|z_k - x^*\|^2 \leq \rho^{2k} \|z_0 - x^*\|^2,$$  \hspace{1cm} (15)

with a rate of

$$\rho^2 = \frac{2|\alpha - \mu|\alpha}{\mu^2} \sqrt{1 - \mu \lambda} + \frac{(\alpha - \mu)^2 + \alpha^2 (1 - \mu \lambda)}{\mu^2}.$$  \hspace{1cm}

As a comparison, recall that current result shown in (4) states that the RC leads to the linear convergence of GD with a rate of $1 - \alpha \lambda$ by choosing the step size $\alpha \leq \mu$. By simple calculation, we find that $\frac{2 - 2\sqrt{1 - \mu \lambda}}{\lambda} \geq \mu$ for all $\mu, \lambda$ satisfying $\mu \lambda \in (0, 1)$, which means our bound can cover the existing one (Candès et al., 2015). We also observe that when $\alpha \leq \mu$, the newly derived rate $\rho^2 \leq 1 - \alpha \lambda$, which means we also improve the existing rate. Further, one can check when taking $\alpha = \mu$, GD can achieve the optimal rate $\rho^2_{\text{min}} = 1 - \mu \lambda$.

4 Analytical Convergence Conditions of AGD

To be noted, (14) is a special case of (10). Similarly, by solving (10) we can obtain analytical convergence conditions of AGD under RC. In this section, we focus on how to derive the main results (Theorem 1 and 2).

The LMI (10) corresponding to an AGD is $3 \times 3$ instead of a $2 \times 2$ one (14) for GD, and more challenging to solve. This is due to that the matrix variable $P$ for the AGD methods is of higher dimension, and thus introduces more unknown decision variables. Our main idea is to transform the LMI (10) to equivalent frequency domain inequalities (FDIs) which can reduce unknown parameters using the classical KYP lemma (Rantzer, 1996). Then we can derive the main convergence results by solving the FDIs analytically.
4.1 The Kalman-Yakubovich-Popov (KYP) Lemma

We first introduce the KYP lemma and the reader is referred to Rantzer (1996) for an elegant proof.

**Lemma 1.** (Theorem 2 in Rantzer (1996)) Given \( A, B, M \), with \( \det(e^{j\omega}I - A) \neq 0 \) for \( \omega \in \mathbb{R} \), the following two statements are equivalent:

1. \( \forall \omega \in \mathbb{R}, \)
   \[
   \begin{bmatrix}
   (e^{j\omega}I - A)^{-1}B \\
   I
   \end{bmatrix}^* M 
   \begin{bmatrix}
   (e^{j\omega}I - A)^{-1}B \\
   I
   \end{bmatrix} \prec 0. \tag{16}
   \]

2. There exists a matrix \( P \in \mathbb{R}^{n \times n} \) such that \( P = P^T \) and
   \[
   M + \begin{bmatrix}
   A^T PA - P & A^T PB \\
   B^T PA & B^T PB
   \end{bmatrix} \prec 0. \tag{17}
   \]

The general KYP lemma only asks \( P \) to be symmetric instead of being positive definite (PD) in our problem. To ensure that the KYP lemma can be applied to solve (10), some adjustments of the lemma are necessary. In fact, we observe that if \( A \) of the dynamical system is Schur stable (i.e., the magnitudes of all eigenvalues of \( A \) are less than 1) and the upper left corner of \( M \)-denoted as \( M_{11} \) is positive semidefinite (PSD), then by checking the principal minor \( M_{11} + A^T PA - P < 0 \), we know \( P \) satisfying (17) must be PD. We define these conditions on \( A \) and \( M \) as KYP Conditions.

**Definition 2.** KYP Conditions \( \text{KYPC}(A, M) \) are restrictions to make sure that all solutions of symmetric \( P \) for (17) are PD. The \( \text{KYPC}(A, M) \) can be listed as:

1. \( \det(e^{j\omega}I - A) \neq 0 \) for \( \omega \in \mathbb{R} \);
2. \( A \) is Schur stable;
3. The left upper corner of \( M \) in (17) is PSD.

Thus we can conclude the following corollary.

**Corollary 1.** Under \( \text{KYPC}(A, M) \), the following two statements are equivalent:

1. \( \forall \omega \in \mathbb{R}, \)
   \[
   \begin{bmatrix}
   (e^{j\omega}I - A)^{-1}B \\
   I
   \end{bmatrix}^* M 
   \begin{bmatrix}
   (e^{j\omega}I - A)^{-1}B \\
   I
   \end{bmatrix} \prec 0. \tag{18}
   \]

2. There exists a matrix \( P \in \mathbb{R}^{n \times n} \) such that \( P > 0 \) and
   \[
   M + \begin{bmatrix}
   A^T PA - P & A^T PB \\
   B^T PA & B^T PB
   \end{bmatrix} \prec 0. \tag{19}
   \]

One can easily check, however, that \( A \) and \( M \) of a general AGD in (10) do not satisfy the \( \text{KYPC}(A, M) \). Therefore, we need to rewrite the dynamical system (7) in a different way to satisfy the \( \text{KYPC}(A, M) \), so that its stability analysis can be done by combining Proposition 1 and Corollary 1. In the following, we first introduce a way to rewrite the dynamical system to satisfy the \( \text{KYPC}(A, M) \).

4.2 How to Satisfy the KYP Conditions?

Recall that a general AGD can be written as:

\[
\begin{align*}
    z_{k+1} &= (1 + \beta_1)z_k - \beta_1 z_k - \alpha \nabla f(y_k), \\
    y_k &= (1 + \beta_2)z_k - \beta_2 z_{k-1}.
\end{align*}
\tag{20}
\]

Here we introduce an uncertain parameter to rewrite the algorithm:

\[
\begin{align*}
    z_{k+1} &= (1 + \delta + \beta_1 + \delta \beta_2)z_k - (\beta_1 + \delta \beta_2) z_{k-1} - \alpha \nabla f(y_k) - \delta y_k, \\
    y_k &= (1 + \beta_2)z_k - \beta_2 z_{k-1}.
\end{align*}
\tag{21}
\]

Observe that for any value of the uncertain parameter $\delta$, algorithm (21) remains the same update rule as (20). In fact, the only reason to make such adjustment is that (21) can generalize the representation of the dynamical systems corresponding to the targeted AGD. Similar to (7), we rewrite (21) as a dynamical system $G(\mathbf{A}', \mathbf{B}', \mathbf{C}', \mathbf{D}')$.

$$\begin{align*}
z_{k+1}^{(1)} &= (1 + \beta_1 + \delta + \delta \beta_2) z_k^{(1)} - (\beta_1 + \delta \beta_2) z_k^{(2)} + u_k, \\
z_{k+1}^{(2)} &= z_k^{(1)}, \\
y_k &= (1 + \beta_2) z_k^{(1)} - \beta_2 z_k^{(2)}, \\
u_k &= -\alpha \nabla f(y_k) - \delta y_k.
\end{align*}$$

(22)

Correspondingly,

$$\begin{bmatrix}
\mathbf{A}' & \mathbf{B}' \\
\mathbf{C}' & \mathbf{D}'
\end{bmatrix} = \begin{bmatrix}
1 + \beta_1 + \delta + \delta \beta_2 & -(\beta_1 + \delta \beta_2) \\
1 & 0 \\
1 + \beta_2 & -\beta_2
\end{bmatrix}. $$

In addition to the adjustment of the dynamics, the feedback of the $G(\mathbf{A}', \mathbf{B}', \mathbf{C}', \mathbf{D}')$ also varies from that in (7). As a consequence, the quadratic bound for the new feedback $u_k = -\alpha \nabla f(y_k) - \delta y_k$ is shifted as stated in the following lemma.

**Lemma 2.** Let $f$ be a loss function which satisfies $RC(\mu, \lambda)$ and $y_* = x^*$ be a minimizer. If $u_k = -\alpha \nabla f(y_k) - \delta y_k$, then $y_k$ and $u_k$ can be quadratically bounded as

$$\begin{bmatrix}
y_k - y_* \\
u_k - u_*
\end{bmatrix}^T M' \begin{bmatrix}
y_k - y_* \\
u_k - u_*
\end{bmatrix} \geq 0. $$

(23)

where $M' = \begin{bmatrix}
-(2\alpha \delta + \lambda \alpha^2 + \mu \delta^2) & -\alpha - \mu \delta \\
-\alpha - \mu \delta & -\mu
\end{bmatrix}$.

Now we have general representations of $\mathbf{A}', \mathbf{M}'$ with one unknown parameter $\delta$. We need to certify the region of $(\alpha, \beta_1, \beta_2)$ such that its corresponding $(\mathbf{A}', \mathbf{M}')$ has at least one $\delta$ satisfying the KYPC($\mathbf{A}', \mathbf{M}'$).

**Lemma 3.** Let $f$ be a loss function which satisfies $RC(\mu, \lambda)$. To ensure that at least one representation of the dynamical system (22) can satisfy $KYPC(\mathbf{A}', \mathbf{M}')$, the step size $\alpha$ and the momentum parameters $\beta_1, \beta_2$ should obey the following restriction:

$$0 < \alpha < \frac{2(1 + \beta_1)(1 + \sqrt{1 - \mu \lambda})}{\lambda (1 + 2 \beta_2)}. $$

(24)

By Lemma 3, if the parameters of a fixed AGD with $(\alpha, \beta_1, \beta_2)$ satisfy (24), then all feasible symmetric $\mathbf{P}'$’s for (19) can be guaranteed to be PD. Now we are ready to use the KYP lemma to complete the convergence analysis of an accelerated algorithm.

### 4.3 Stability Region of AGD under RC

By Proposition 1, we can solve the stability of the new system (22) by finding some feasible $\mathbf{P} \succ 0$ to the following time domain matrix inequality for some rate $0 < \rho^2 < 1$:

$$\begin{bmatrix}
\mathbf{A}'^T \mathbf{P} \mathbf{A}' - \rho^2 \mathbf{P} & \mathbf{A}'^T \mathbf{P} \mathbf{B}' \\
\mathbf{B}'^T \mathbf{P} \mathbf{A}' & \mathbf{B}'^T \mathbf{P} \mathbf{B}'
\end{bmatrix} + \begin{bmatrix}
\mathbf{C}' & \mathbf{D}' \\
0_{1 \times 2} & 1
\end{bmatrix}^T \mathbf{M}' \begin{bmatrix}
\mathbf{C}' & \mathbf{D}' \\
0_{1 \times 2} & 1
\end{bmatrix} \preceq 0. $$

(25)

We are interested in obtaining the analytical region of $(\alpha, \beta_1, \beta_2)$ that guarantees the linear convergence of AGD under RC. So we use the following strict matrix inequality

$$\begin{bmatrix}
\mathbf{A}'^T \mathbf{P} \mathbf{A}' - \mathbf{P} & \mathbf{A}'^T \mathbf{P} \mathbf{B}' \\
\mathbf{B}'^T \mathbf{P} \mathbf{A}' & \mathbf{B}'^T \mathbf{P} \mathbf{B}'
\end{bmatrix} + \begin{bmatrix}
\mathbf{C}' & 0 \\
0_{1 \times 2} & 1
\end{bmatrix}^T \mathbf{M}' \begin{bmatrix}
\mathbf{C}' & 0 \\
0_{1 \times 2} & 1
\end{bmatrix} < 0. $$

(26)
Notice if (26) holds strictly, then there exists a sufficiently small \( \epsilon \) such that
\[
\begin{bmatrix}
A'^T P A' - P & A'^T P B' \\
B'^T P A' & B'^T P B'
\end{bmatrix} + \begin{bmatrix} C' & 0 \end{bmatrix}^T M' \begin{bmatrix} C' & 0 \end{bmatrix} \leq -\epsilon \begin{bmatrix} P & 0_{2 \times 1} \\
0_{1 \times 2} & 0
\end{bmatrix}.
\]  
(27)

Then (25) holds with \( \rho^2 = 1 - \epsilon < 1 \). We emphasize that solving (26) can still guarantee linear convergence and consider all possible rates \( \rho^2 < 1 \), while solving (25) can only provide convergence region under some specific rate \( \rho^2 \). Conventionally, we call the region obtained by (26) the "stability region" of the targeted algorithm.

Observe that now (26) is of the same form as (19). Together with Lemma 3, we are ready to use the KYP lemma (Corollary 1), and thus (26) can be equivalently solved by studying the following FDI:
\[
\begin{bmatrix}
(e^{j\omega} I - A')^{-1} B' \\
I
\end{bmatrix}^T \begin{bmatrix} C' & 0 \end{bmatrix}^T M' \begin{bmatrix} C' & 0 \end{bmatrix} \begin{bmatrix}
(e^{j\omega} I - A')^{-1} B' \\
I
\end{bmatrix} < 0, \forall \omega \in \mathbb{R}.
\]  
(28)

By simplifying (28) we observe that all uncertain terms can be canceled out and then conclude the following lemma.

**Lemma 4.** To find the stability region of a general AGD under RC(\( \mu, \lambda \)), or equivalently, to find the region of \((\alpha, \beta_1, \beta_2)\) such that there exists a feasible \( P \succ 0 \) satisfying (26), it is equivalent to find \((\alpha, \beta_1, \beta_2)\) which simultaneously obeys (24) and guarantees the following FDI:
\[
4(\alpha \beta_2 - \mu \beta_1) \cos^2 \omega + 2 \left[ \mu(1 + \beta_1)^2 + \lambda \alpha^2 \beta_2(1 + \beta_2) - \alpha(1 + \beta_1)(1 + 2 \beta_2) \right] \cos \omega \\
+ 2 \alpha(1 + \beta_1 + 2 \beta_1 \beta_2) - 2 \mu(1 + \beta_1 \beta_2) - \alpha \lambda \alpha^2 [\beta_2^2 + (1 + \beta_2)^2] < 0, \forall \omega \in \mathbb{R}.
\]  
(29)

We omit the proof of Lemma 4 since it follows easily from Corollary 1 and some simple calculation to simplify (28). The stability results of HB and NAG are of particular interest. Hence we take these two popular accelerated algorithms as examples to show the analytical results. The stability regions of HB and NAG can be obtained by letting \( \beta_2 = 0 \) and \( \beta_1 = \beta_2 = \beta \) in (29), respectively. Therefore we can conclude Theorem 1 and Theorem 2 in Section 2.

We have also verified our analytical formulas in Theorem 1 and Theorem 2 using the numerical solutions of (10) by grid search for feasible \((\alpha, \beta_1, \beta_2)\). Our theoretical characterization is consistent with the numerical simulations of (10). Since it is straightforward to implement (10) using existing semidefinite program solvers, we omit the details of this numerical implementation here.

## 5 Local Regularity Condition

So far, all the above derivations assume the RC holds globally. In addition, the existing control framework for optimization methods (Lessard et al., 2016; Fazlyab et al., 2018; Hu et al., 2017b; Hu and Lessard, 2017; Van Scoy et al., 2018; Cyrus et al., 2018; Dhingra et al., 2018) all require global constraints. In certain problems, however, RC may only hold locally around the global optimum. In this section, we explain how our framework can accommodate such cases, as long as the algorithms are initialized properly.

We first introduce the definition of Local Regularity Condition (LRC). The global RC in Definition 1 can be viewed as a special case of the following definition taking the radius of neighborhood as \( \epsilon = \infty \).

**Definition 3 (Local Regularity Condition).** A function \( f(\cdot) \) is said to satisfy the Local Regularity Condition LRC(\( \mu, \lambda, \epsilon \)) with positive constants \( \mu, \lambda \) and \( \epsilon \), if
\[
\langle \nabla f(z), z - x^* \rangle \geq \frac{\mu}{2} \| \nabla f(z) \|^2 + \frac{\lambda}{2} \| z - x^* \|^2
\]  
(30)

for all \( z \in N_\epsilon(x^*) := \{ z : \| z - x^* \| \leq \epsilon \| x^* \| \} \), where \( x^* = \arg \min_{z \in \mathbb{R}^n} f(z) \).

Noting LRC only holds in a small neighborhood, to show convergence under this property, we will first need to locate a proper initial guess in the local neighborhood, and then ensure all the following iterates
stay in this neighborhood. The initialization is typically obtained using the spectral method, see for example (Candès et al., 2015; Keshavan et al., 2010).

It remains to ensure all the following iterates will not exceed the local neighborhood satisfying LRC and \( y_k \in N_{\epsilon}(x^\star) \) for all \( k \), so that we can still transfer LRC to a quadratic bound of the feedback unit \( u_k = \nabla f(y_k) \) in the system (7) as previous analysis. We find out that this can be done as long as we make careful initialization as stated in the following theorem.

**Theorem 4.** Let \( x^\star \in \mathbb{R}^n \) be a minimizer of the loss function \( f(\cdot) \) which satisfies LRC(\( \mu, \lambda, \epsilon \)) with some positive constants \( \mu, \lambda, \epsilon \). Assume that \( P \succ 0 \) is a feasible solution of the LMI (10). If we have the first two initial iterates \( z_{-1}, z_0 \in N_{\epsilon/\sqrt{\text{cond}(P)}}(x^\star) \) with proper initialization, then \( y_k \in N_\epsilon(x^\star), \forall k \).

By the above theorem, we can make sure that the quadratic bound (9) can be applied at each iteration, and thus all the previous results still hold for the convergence analysis of AGD under LRC.

### 6 Conclusions

In this paper, we analyze the convergence of a general first-order methods (including GD, HB and NAG) under the Regularity Condition. Our main contribution lies in the analytical characterization of the convergence regions in terms of the algorithm parameters (\( \alpha, \beta \)) and the RC parameters (\( \mu, \lambda \)). Such convergence results do not exist in the current literature and offer useful insights in analysis and design of AGDs for a class of nonconvex optimization problems.

There are some opportunities for future work. First, convergence analysis of more optimization algorithms in nonconvex settings is to be explored under this framework. Second, the Regularity Condition in our setting does not include global ambiguity of the optimal points. However, RC with global ambiguity has shown up in many important signal processing problems such as phase retrieval (Candès et al., 2015), low-rank matrix completion (Sun and Luo, 2016; Ma et al., 2017) and blind deconvolution (Ma et al., 2017). The convergence of accelerated algorithms under this kind of RC is not well-understood and is left for future work.

### Acknowledgements

The work of H. Xiong and W. Zhang is partly supported by the National Science Foundation under grant CNS-1552838. The work of Y. Chi is supported in part by ONR under grant N00014-18-1-2142, by ARO under grant W911NF-18-1-0303, and by NSF under grants CCF-1806154 and ECCS-1818571.

### References


A Proof Outline

A.1 Proof of Proposition 1

If the key LMI holds with some $P > 0$, i.e.,
\[
\begin{bmatrix}
A^T PA - \rho^2 P & A^T PB \\
B^T PA & B^T PB
\end{bmatrix} + \begin{bmatrix}
C & D \\
0_{1 \times 2} & 1
\end{bmatrix}^T \begin{bmatrix}
-\lambda & 1 \\
1 & -\mu
\end{bmatrix} \begin{bmatrix}
C & D \\
0_{1 \times 2} & 1
\end{bmatrix} \preceq 0, \quad (31)
\]
multiply (31) by $\begin{bmatrix}
\phi_k - \phi_+ \\
u_k - u_+
\end{bmatrix}^T$ from left and $\begin{bmatrix}
\phi_k - \phi_+ \\
u_k - u_+
\end{bmatrix}$ from right, respectively and insert the Kronecker product. Then we obtain
\[
\begin{bmatrix}
\phi_k - \phi_+ \\
u_k - u_+
\end{bmatrix}^T \begin{bmatrix}
A^T PA - \rho^2 P & A^T PB \\
B^T PA & B^T PB
\end{bmatrix} \otimes I_n \begin{bmatrix}
\phi_k - \phi_+ \\
u_k - u_+
\end{bmatrix} + \begin{bmatrix}
y_k - y_+ \\
u_k - u_+
\end{bmatrix}^T \begin{bmatrix}
-\lambda I_n & I_n \\
I_n & -\mu I_n
\end{bmatrix} \begin{bmatrix}
y_k - y_+ \\
u_k - u_+
\end{bmatrix} \leq 0.
\]

We have argued that RC can be equivalently represented as a quadratic bound of the feedback term $u_k = \nabla f(y_k)$ as:
\[
\begin{bmatrix}
y_k - y_+ \\
u_k - u_+
\end{bmatrix}^T \begin{bmatrix}
-\lambda I_n & I_n \\
I_n & -\mu I_n
\end{bmatrix} \begin{bmatrix}
y_k - y_+ \\
u_k - u_+
\end{bmatrix} \geq 0. \quad (32)
\]
It can further imply that
\[
\begin{bmatrix}
\phi_k - \phi_+ \\
u_k - u_+
\end{bmatrix}^T \begin{bmatrix}
A^T PA - \rho^2 P & A^T PB \\
B^T PA & B^T PB
\end{bmatrix} \otimes I_n \begin{bmatrix}
\phi_k - \phi_+ \\
u_k - u_+
\end{bmatrix} \leq 0. \quad (33)
\]
Observe $\phi_{k+1} = (A \otimes I_n)\phi_k + (B \otimes I_n)u_k$. Hence we can further rearrange and simplify (33) as
\[
(\phi_{k+1} - \phi_+)^T (P \otimes I_n)(\phi_{k+1} - \phi_+) \leq \rho^2(\phi_k - \phi_+)^T (P \otimes I_n)(\phi_k - \phi_+).
\]
Such exponential decay with $P > 0$ for all $k$ can conclude Proposition 1.

A.2 Proof of Theorem 3

The goal is to find some $s \geq 0$, such that
\[
\begin{bmatrix}
1 - \rho^2 & -\alpha \\
-\alpha & \alpha^2
\end{bmatrix} + s \begin{bmatrix}
-\lambda & 1 \\
1 & -\mu
\end{bmatrix} = \begin{bmatrix}
1 - \rho^2 - s\lambda & s - \alpha \\
s - \alpha & \alpha^2 - s\mu
\end{bmatrix} \preceq 0. \quad (34)
\]
We check the principal minors of (34), then we have
\[
\begin{cases}
\rho^2 \geq 1 - s\lambda, \\
\alpha^2 - s\mu \leq 0, \\
\rho^2 \geq \frac{(1 - s\lambda)(\alpha^2 - s\mu) - (s - \alpha)^2}{\alpha^2 - s\mu}.
\end{cases} \quad (35)
\]
When analyzing the stability condition, we only need that there exists one rate $0 < \rho < 1$, it suffices to make $(1 - s\lambda)(\alpha^2 - s\mu) - (s - \alpha)^2 < 1$. Further we want this inequality has some feasible solution $s > 0$. Then we can eventually obtain the region of step size which can guarantee the convergence:
\[
\alpha < \frac{2 - 2\sqrt{1 - \mu\lambda}}{\lambda}.
\]
It remains to derive the convergence rate that GD with some reasonable $\alpha$ can achieve. This can be done by further checking (35) for some fixed $\alpha$.

$$\rho_{\text{optimal}}^2 = \min_{s \geq \alpha^2 / \mu} \max \left\{ \frac{1 - s\lambda}{\alpha^2 - s\mu} \right\}$$

$$= \min_{s \geq \alpha^2 / \mu} \left\{ \frac{(1 - s\lambda)(\alpha^2 - s\mu) - (s - \alpha)^2}{\alpha^2 - s\mu} \right\}$$

$$= \min_{s \geq \alpha^2 / \mu} \left\{ \frac{1 - \mu\lambda}{\mu^2} \frac{(s\mu - \alpha^2) + \alpha^2(1 - \mu\lambda) - \alpha^2}{\mu^2(\alpha^2 - s\mu)} \right\}$$

$$= \frac{2(\alpha - \mu \alpha)}{\mu^2} \left(1 - \frac{\alpha - \mu}{\mu^2} + \frac{(\alpha - \mu)^2}{\mu^2} + \frac{2(1 - \mu\lambda)}{\mu^2} \right).$$

The last equality can be reached by taking $s = \frac{\alpha - \mu}{\mu^2} + \frac{\alpha^2}{\mu^2}$.

### A.3 Proof of Lemma 2

We check the inner product of the input $u_k$ and output $y_k$ (we remind that $y_* = x^*$) of the dynamical system (22):

$$\langle u_k - u_*, y_k - y_* \rangle = -\alpha \nabla f(y_k) - \delta(y_k - y_*) \rangle.$$ 

$$= -\delta \|y_k - y_*\|^2 - \alpha \langle \nabla f(y_k), y_k - y_* \rangle$$

$$\leq -\delta \|y_k - y_*\|^2 - \frac{\alpha \mu}{\alpha} \|\nabla f(y_k)\|^2 - \frac{\alpha \lambda}{\alpha} \|y_k - y_*\|^2$$

$$= -\left[ \frac{\alpha \mu}{2} \|\nabla f(y_k)\|^2 + \frac{\delta^2 \mu}{2\alpha} \|y_k - y_*\|^2 + \frac{\delta \mu}{\alpha} \|y_k - y_*\|^2 - \frac{\alpha \lambda}{2} \|y_k - y_*\|^2 \right]$$

$$= -\frac{\mu}{2\alpha} \|u_k - u_*\|^2 - \frac{\delta \mu}{\alpha} \|u_k - u_*\|^2 - y_k - y_* \rangle - \frac{\alpha \lambda}{2} \|y_k - y_*\|^2.$$ 

By rearrangement, we have

$$- (2\alpha \delta + \lambda \alpha^2 + \mu \delta^2) \|y_k - y_*\|^2 - 2(\alpha + \mu \delta) \langle u_k - u_*, y_k - y_* \rangle - \mu \|u_k - u_*\|^2 \geq 0,$$

and thus conclude (23).

### A.4 Proof of Lemma 3

We check the KYPC($A', M'$) as listed for an AGD (22) characterized by $\alpha, \beta_1, \beta_2$:

1. $\det(e^{i\omega} I - A') \neq 0$ for $\omega \in \mathbb{R}$;

2. $A'$ is Schur stable;

3. The left upper corner of $M'$ in (23) is PSD.

### Condition 1:

We check

$$\det(e^{i\omega} I - A') = \left| e^{i\omega} - (1 + \beta_1 + \delta(1 + \beta_2)) \frac{\beta_1 + \delta \beta_2}{e^{i\omega}} \right| = (e^{i\omega})^2 - (1 + \beta_1 + \delta(1 + \beta_2)) e^{i\omega} + \beta_1 + \delta \beta_2$$

$$= \cos^2 \omega - \sin^2 \omega - (1 + \beta_1 + \delta(1 + \beta_2)) \cos \omega + \beta_1 + \delta \beta_2 + (2 \sin \omega \cos \omega - (1 + \beta_1 + \delta(1 + \beta_2)) \sin \omega).$$

By way of the opposite direction, let $\det(e^{i\omega} I - A') = 0$. Then we have

$$\begin{cases} 
\cos^2 \omega - \sin^2 \omega - (1 + \beta_1 + \delta(1 + \beta_2)) \cos \omega + \beta_1 + \delta \beta_2 = 0, \\
2 \sin \omega \cos \omega - (1 + \beta_1 + \delta(1 + \beta_2)) \sin \omega = 0.
\end{cases}$$
If $\sin \omega = 0$, $\cos \omega = \pm 1$. Then the first equality requires $(1 + \beta_1 + \delta (1 + \beta_2)) = \pm (1 + \beta_1 + \delta \beta_2)$; if $(1 + \beta_1 + \delta (1 + \beta_2)) = 2 \cos \omega, \cos^2 \omega - \sin^2 \omega - (1 + \beta_1 + \delta (1 + \beta_2)) \cos \omega + \beta_1 + \delta \beta_2 = -1 + \beta_1 + \delta \beta_2 = 0$.

To conclude, to satisfy the first condition, we obtain a restricted condition

$$(1 + \beta_1 + \delta (1 + \beta_2)) \neq \pm (1 + \beta), \beta_1 + \delta \beta_2 \neq 1. \quad (36)$$

**Condition 2:**

We want $A' = \begin{bmatrix} (1 + \beta_1 + \delta (1 + \beta_2)) & -(\beta_1 + \delta \beta_2) \\ 1 & 0 \end{bmatrix}$ to be Schur stable. Check the eigenvalues of $A'$.

$$|\lambda I - A'| = \left| \begin{array}{cc} \lambda - (1 + \beta_1 + \delta (1 + \beta_2)) & \beta_1 + \delta \beta_2 \\ -1 & \lambda \end{array} \right| = \lambda^2 - \lambda (1 + \beta_1 + \delta (1 + \beta_2)) + \beta_1 + \delta \beta_2 = 0.$$ 

Two roots of the above equality are $\lambda_{1,2} = \frac{(1 + \beta_1 + 4(1 + \beta_2)\pm \sqrt{(1 + \beta_1 + \delta (1 + \beta_2))^2 - 4(\beta_1 + \delta \beta_2)}}{2}$. To make sure that $A'$ is Schur stable, it suffices to let the magnitude of roots $|\lambda_{1,2}| < 1$.

When $(1 + \beta_1 + \delta (1 + \beta_2))^2 < 4(\beta_1 + \delta \beta_2), |\lambda_1| = |\lambda_2| = \frac{(1 + \beta_1 + \delta (1 + \beta_2))^2 + 4(\beta_1 + \delta \beta_2) - (1 + \beta_1 + \delta (1 + \beta_2))^2}{2} = \beta_1 + \delta \beta_2$, and thus we need $\beta_1 + \delta \beta_2 < 1$ in this case. When $(1 + \beta_1 + \delta (1 + \beta_2))^2 \geq 4(\beta_1 + \delta \beta_2)$, let

$$\begin{cases} \frac{(1 + \beta_1 + \delta (1 + \beta_2)) + \sqrt{(1 + \beta_1 + \delta (1 + \beta_2))^2 - 4(\beta_1 + \delta \beta_2)}}{2} < 1, \\ \frac{(1 + \beta_1 + \delta (1 + \beta_2)) - \sqrt{(1 + \beta_1 + \delta (1 + \beta_2))^2 - 4(\beta_1 + \delta \beta_2)}}{2} > -1. \end{cases}$$ 

Then we have

$$4(\beta_1 + \delta \beta_2) \leq (1 + \beta_1 + \delta (1 + \beta_2))^2 < (1 + \beta_1 + \delta \beta_2)^2.$$ 

To conclude, to satisfy the second condition, we obtain a restricted condition

$$-\frac{2(1 + \beta_1)}{1 + 2\beta_2} < \delta < 0. \quad (37)$$

**Condition 3:**

We want the left upper corner of $M'$ in (23): $- (2\alpha \delta + \lambda \alpha^2 + \mu \delta^2) \geq 0$. Then the restricted condition for the third condition is

$$-\delta - |\delta| \sqrt{1 - \mu \lambda} \leq \alpha \leq -\delta + |\delta| \sqrt{1 - \mu \lambda} \quad (38)$$

To conclude, by unifying all conditions (36)(37)(38), we can conclude that the parameters of an accelerated algorithm need to satisfy

$$0 < \alpha < \frac{2(1 + \beta_1)(1 + \sqrt{1 - \mu \lambda})}{\lambda (1 + 2\beta_2)}. \quad (39)$$

### A.5 Proof of Theorem 1

For the Heavy-ball method, we apply $\beta_1 = \beta, \beta_2 = 0$ and denote $u := \cos \omega$, then (29) can be written as:

$$h(u) := 4\mu_3 u^2 - 2(2\mu_3 + \mu \beta_2 - \alpha \beta + \mu - \alpha)u + 2\mu + 2\mu \beta_2^2 - 2\alpha - 2\alpha \beta + \lambda \alpha^2 \geq 0, \forall u \in [-1, 1]. \quad (40)$$

Observe that $h(u)$ is a quadratic function depending on $u$. We can check the minimal value of $h(\cdot)$ on $[-1, 1]$ by discussing its axis of symmetry denoted as $S = \frac{2\mu_3 + \mu \beta_2^2 - \alpha \beta + \mu - \alpha}{4\mu_3}$. 

1. When $S \geq 1$, $\alpha \leq \frac{\mu_3 (\beta - 1)^2}{\beta + 1}$. Then

$$h(u)_{\text{min}} = h(1) = \lambda \alpha^2 > 0.$$ 

Thus the feasible region in this case is:

$$\left\{ (\alpha, \beta) : \alpha \leq \frac{\mu_3 (\beta - 1)^2}{\beta + 1}, 0 < \beta < 1 \right\}. \quad (41)$$
2. When $S \leq -1$, $\alpha \geq \frac{\omega^{2}+6 \beta \mu+\mu}{1+\beta}$. We need
\[ h(u)_{\min} = h(-1) = \lambda \alpha^{2} - 4(1 + \beta)\alpha + 4\mu(1 + \beta)^{2} \geq 0. \]
Thus the feasible region in this case is:
\[
\left\{ (\alpha, \beta) : \alpha \geq \frac{2(\beta + 1)(1 + \sqrt{1 - \mu \lambda})}{\lambda}, 0 < \beta < 1 \right\} \cup \left\{ (\alpha, \beta) : \frac{\mu \beta^{2} + 6 \mu \beta + \mu}{\beta + 1} \leq \alpha \leq \frac{2(\beta + 1)(1 - \sqrt{1 - \mu \lambda})}{\lambda}, 0 < \beta < 1 \right\}. \tag{42}
\]

3. When $-1 < S < 1$, $\frac{\omega^{2}+6 \beta \mu+\mu}{1+\beta} < \alpha < \frac{\mu(1-\beta)^{2}}{1+\beta}$. We want $h(u)_{\min} = h \left( \frac{2\mu^{2} + \mu^{2} - \alpha \beta^{2} - \mu^{2}}{4\mu \beta} \right) \geq 0$. It is equivalent to solve
\[
(4\mu \beta^{2} - \beta^{2} - 1 - 2\beta)\alpha^{2} - (2\mu^{2} + 2\mu \beta^{2} - 2\mu \beta^{3} - 2\mu)\alpha + 4\mu^{2} \beta^{3} + 4\mu^{2} \beta - 6\mu^{2} \beta^{2} - \mu^{2} \beta^{4} - \mu^{2} \geq 0.
\]
Since $4\mu \beta^{2} - \beta^{2} - 1 - 2\beta < 4\beta - \beta^{2} - 1 - 2\beta \leq 0$, Thus the feasible region in this case is:
\[
\left\{ (\alpha, \beta) : \frac{\mu(\beta - 1)^{2}}{\beta + 1} \leq \frac{P_{2} - \sqrt{P_{2}^{2} - 4P_{1}P_{3}}}{2P_{1}}, 0 < \beta < 1 \right\} \cap \left\{ (\alpha, \beta) : \frac{\mu \beta^{2} + 6 \mu \beta + \mu}{\beta + 1} \leq \alpha \leq \frac{2(\beta + 1)(1 - \sqrt{1 - \mu \lambda})}{\lambda}, 0 < \beta < 1 \right\}, \tag{43}
\]
where $P_{1} = 4\mu \lambda \beta^{2} - \beta^{2} - 1 - 2\beta$, $P_{2} = 2\mu \beta + 2\mu \beta^{2} - 2\mu \beta^{3} - 2\mu$, $P_{3} = 4\mu^{2} \beta^{3} + 4\mu^{2} \beta - 6\mu^{2} \beta^{2} - \mu^{2} \beta^{4} - \mu^{2}$.

Taking the union of (41)(42)(43) gives the result of the FDI (40). Further intersecting the restricted condition obtained in Lemma 3, (24) can conclude the final region.

A.6 Proof of Theorem 2

Similar with the proof of Theorem 1, for the Nesterov’s accelerated gradient method, we apply $\beta_{1} = \beta_{2} = \beta$ and denote $u := \cos \omega$, then (29) can be written as:
\[
h(u) := (-4\mu \beta + 4 \alpha \beta)u^{2} + \left[ 2(\mu - \alpha)(1 + \beta)^{2} + 2(\lambda \alpha - 1)\alpha(1 + \beta) \right] u
- 2\mu(1 + \beta^{2}) + 2\alpha(1 + \beta)^{2} - \lambda \alpha^{2} \beta^{2} - 2\alpha(1 - \beta)\beta - \lambda \alpha^{2}(1 + \beta)^{2}
\leq 0, \forall u \in [-1, 1]. \tag{44}
\]

We check the maximal value of the quadratic function $h(\cdot)$ on $[-1, 1]$.

1. When $\alpha = \mu$, $h(u)_{\max} = f(-1) \leq 0$. Thus the feasible region in this case is:
\[
\left\{ (\alpha, \beta) : \alpha = \mu, 0 < \beta \leq \frac{-1 + \mu \lambda + \sqrt{1 - \mu \lambda}}{2(1 - \mu \lambda)} \right\}. \tag{45}
\]

2. When $\alpha > \mu$, we need to let $h(1) \leq 0, h(-1) \leq 0$. Since $h(1) = -\lambda \alpha^{2} < 0$, we only need to check $h(-1)$. Thus the feasible region in this case is:
\[
\left\{ (\alpha, \beta) : \alpha \geq \frac{2(\beta + 1)(1 + \sqrt{1 - \mu \lambda})}{\lambda(1 + 2\beta)}, 0 < \beta < 1 \right\} \cup \left\{ (\alpha, \beta) : \mu \beta \geq \alpha \geq \frac{2(\beta + 1)(1 - \sqrt{1 - \mu \lambda})}{\lambda(1 + 2\beta)}, 0 < \beta < 1 \right\}. \tag{46}
\]

3. When $\alpha < \mu$, we can check the maximal value of the quadratic function $h(u)$ by discussing the axis of symmetry $S = \frac{(\mu - \alpha)(1 + \beta)^{2} + (\lambda \alpha - 1)\alpha(1 + \beta)\beta}{4\mu \beta - 4 \alpha \beta}$.

(a) When $S \geq 1$, $h(u)_{\max} = -\lambda \alpha^{2} < 0$. Thus the feasible region in this case is:
\[
\left\{ (\alpha, \beta) : \alpha \leq \frac{1 - \beta + 2\beta^{2} - \sqrt{(1 - \beta + 2\beta^{2})^{2} - 4\mu \beta \beta(1 + \beta)(1 - \beta)^{2}}}{2\lambda \beta(\beta + 1)}, 0 < \beta < 1 \right\}. \tag{47}
\]
(b) When $S \leq -1$, $h(u)_{\text{max}} = h(-1) \leq 0$. Thus the feasible region in this case is:

$$\left\{ (\alpha, \beta) : 1 + 7\beta + 2\beta^2 - 4\mu \lambda \beta (1 + \beta)(1 + 6\beta + \beta^2) \leq 2\lambda \beta (\beta + 1) \leq \alpha < \mu, 0 < \beta < 1 \right\}.$$ (48)

(c) When $-1 < S < 1$, $h(u)_{\text{max}} = h(S) \leq 0$. Thus the feasible region in this case is:

$$\{ (\alpha, \beta) : L < \alpha < R, 0 < \beta < 1, g(S) \leq 0 \},$$ (49)

where $L = \frac{1 - \beta + 2\beta^2 - \sqrt{(1 - \beta + 2\beta^2)^2 - 4\mu \lambda \beta (1 + \beta)(1 + 6\beta + \beta^2)}}{2\lambda \beta (\beta + 1)}$, $R = \frac{1 + 7\beta + 2\beta^2 - \sqrt{(1 + 7\beta + 2\beta^2)^2 - 4\mu \lambda \beta (1 + \beta)(1 + 6\beta + \beta^2)}}{2\lambda \beta (\beta + 1)}$.

$g(S)$ is a 4th order inequality as noted in Remark 1, that is,

$$g(S) := 4\mu S^2 - 2(2\mu + \mu^2 - \alpha \beta + \mu - \alpha + 2\lambda \alpha) S + 2\mu + 2\mu^2 - 2\alpha - 2\alpha \beta + \lambda \alpha^2.$$

The result of the FDI (44) is the union of above regions (45)(46)(47)(48)(49). Further intersecting the restricted condition obtained in Lemma 3 can conclude the final region.

A.7 Proof of Theorem 4

Since $z_{-1}, z_{0} \in N_{x^*}(\epsilon/\sqrt{10 \text{cond}(P)})$, we know $\|\phi_0 - \phi_*\| < \frac{\epsilon}{\sqrt{10 \text{cond}(P)}}$. The exponential decay with $P \succ 0$:

$$(\phi_{k+1} - \phi_*)^T P(\phi_{k+1} - \phi_*) \leq \rho^2(\phi_k - \phi_*)^T P(\phi_k - \phi_*)$$ implies that $\|\phi_k - \phi_*\| \leq \sqrt{\text{cond}(P)} \rho^k \|\phi_0 - \phi_*\|$, which we have argued in Subsection 3.2. Therefore,

$$\|\phi_k - \phi_*\| \leq \sqrt{\text{cond}(P)} \rho^k \|\phi_0 - \phi_*\|$$

$$< \sqrt{\text{cond}(P)} \|\phi_0 - \phi_*\|$$

$$< \sqrt{\text{cond}(P)} \cdot \frac{\epsilon}{\sqrt{5 \text{cond}(P)}}$$

$$< \epsilon/\sqrt{5}.$$ Then

$$\|y_k - x^*\| = \left\| (1 + \beta_2)z_k^{(1)} - \beta_2 z_k^{(2)} - x^* \right\|$$

$$= \|C(\phi_k - \phi_*)\|$$

$$\leq \|C^T\| \|\phi_k - \phi_*\|$$

$$< \sqrt{(1 + \beta_2)^2 + \beta_2^2} \left( \epsilon/\sqrt{5} \right)$$

$$< \epsilon.$$