Policy Optimization in Reinforcement Learning: A Tale of Preconditioning and Regularization

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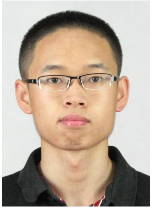

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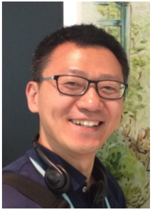

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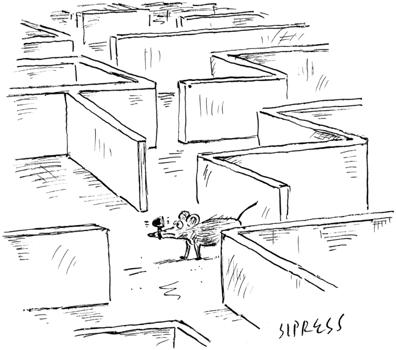
Reinforcement learning (RL)

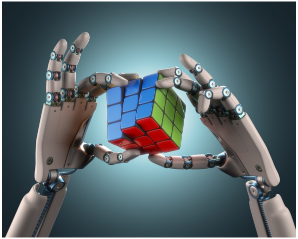
In RL, an agent learns by interacting with an environment.

- unknown environments
- delayed feedback or rewards
- trial-and-error
- sequential and online



"Recalculating ... recalculating ..."

Recent successes in RL

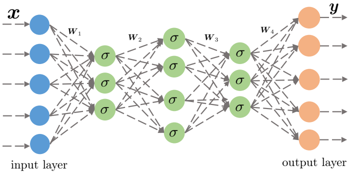


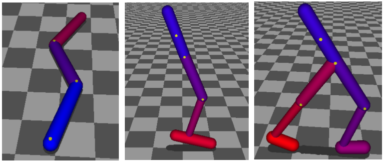
Policy optimization is a major driver to these successes.

Policy optimization in practice

$$\text{maximize}_\theta \quad \text{value}(\text{policy}(\theta))$$

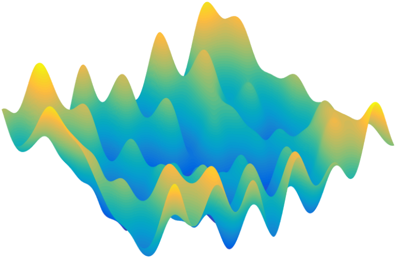
- directly optimize the policy, which is the quantity of interest;
- allow flexible differentiable parameterizations of the policy;
- work with both continuous and discrete problems.





Theoretical challenges: non-concavity

Little understanding on the global convergence of policy gradient methods until very recently, e.g. (Fazel et al., 2018; Bhandari and Russo, 2019; Agarwal et al., 2019; Mei et al. 2020), and many more.

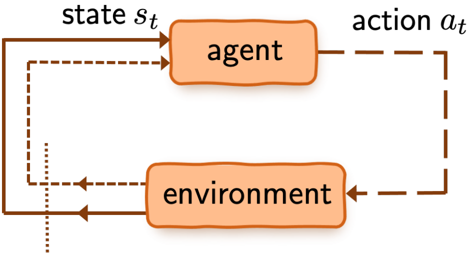


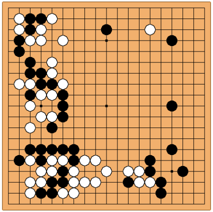
Our goal:

- understand finite-time convergence rates of popular heuristics;
- design fast-convergent algorithms that scale for finding policies with desirable properties.

Backgrounds: policy optimization in tabular Markov decision processes

Markov decision process (MDP)





- $\mathcal{S}$: state space
- $\mathcal{A}$: action space

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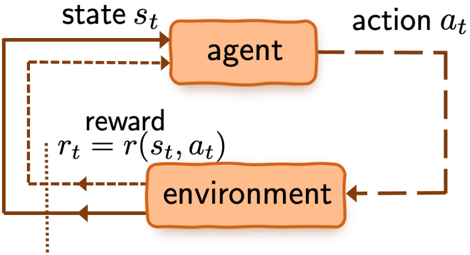

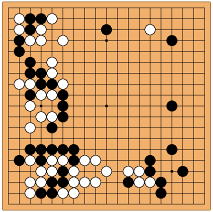

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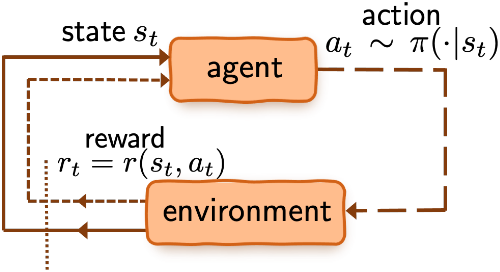

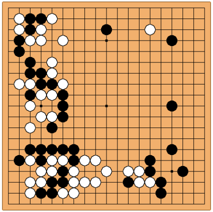

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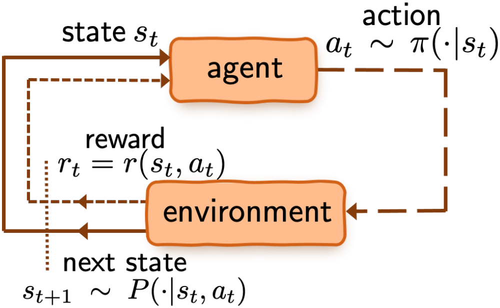

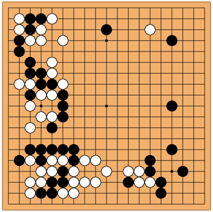

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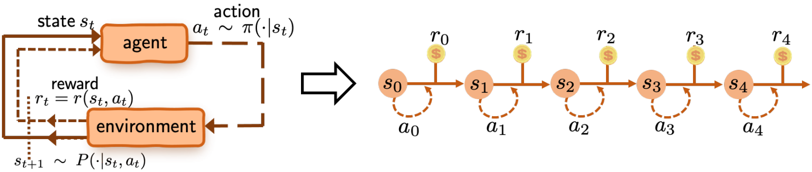
- $\pi(\cdot|s)$: policy (or action selection rule)
- $P(\cdot|s, a)$: transition probabilities

Value function and Q-function



Value function and **Q function** of policy π:

$$\forall s \in \mathcal{S} : \qquad V^{\pi}(s) := \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t r_t \,\middle|\, s_0 = s\right]$$

$$\forall (s,a) \in \mathcal{S} \times \mathcal{A} : \quad Q^{\pi}(s,a) := \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t r_t \,\middle|\, s_0 = s, a_0 = a\right]$$

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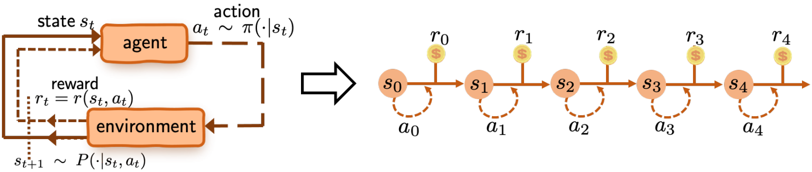

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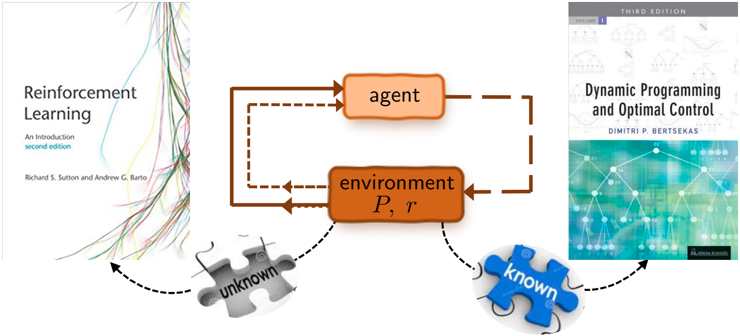
- $\gamma \in [0,1)$ is the discount factor; $\frac{1}{1-\gamma}$ is effective horizon
- Expectation is w.r.t. the sampled trajectory under π

Searching for the optimal policy



Goal: find the optimal policy $\pi^\star$ that maximize $V^\pi(s)$

- optimal value / Q function: $V^\star := V^{\pi^\star}$, $Q^\star := Q^{\pi^\star}$

Policy gradient methods

Given an initial state distribution $s \sim \rho$, find policy π such that

$$\text{maximize}_\pi \quad V^\pi(\rho) := \mathbb{E}_{s\sim\rho}\left[V^\pi(s)\right]$$

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softmax parameterization:

$$\pi_{\theta}(a|s) \propto \exp(\theta(s,a))$$

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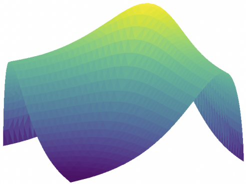
Policy gradient method (Sutton et al., 2000)

For $t = 0, 1, \cdots$

$$\theta^{(t+1)} = \theta^{(t)} + \eta \nabla_\theta V^{\pi_\theta^{(t)}}(\rho)$$

where η *is the learning rate.*

Global convergence of the PG method?



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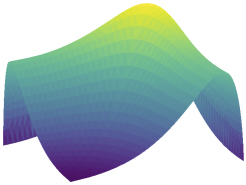

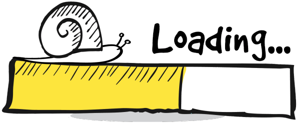

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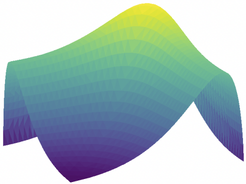

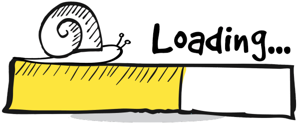

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$$c\left(|\mathcal{S}|, |\mathcal{A}|, \frac{1}{1-\gamma}, \cdots\right) O\left(\frac{1}{\varepsilon}\right) \text{ iterations}$$

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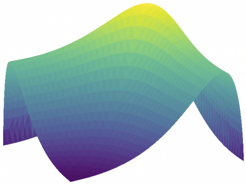

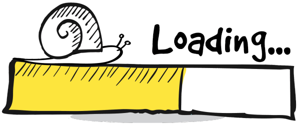

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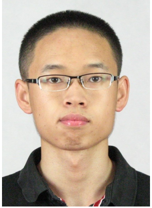
Is the rate of PG good, bad or ugly?

Softmax PG can take exponential time to converge



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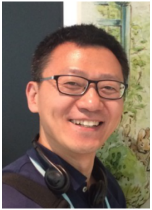

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A negative message

Theorem (Li, Wei, Chi, Gu, Chen, 2021)

There exists an MDP s.t. it takes softmax PG at least

$$\frac{1}{\eta}|\mathcal{S}|^{2^{\Theta(\frac{1}{1-\gamma})}} \text{ iterations}$$

to achieve $\|V^{(t)} - V^{\star}\|_{\infty} \leq 0.15$.

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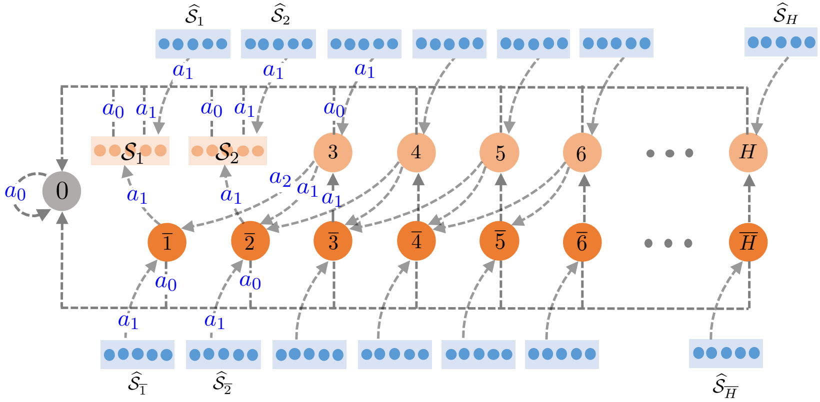
to achieve $\|V^{(t)} - V^{\star}\|_{\infty} \le 0.15$.

- Softmax PG can take (super)-exponential time to converge (in problems w/ large state space & long effective horizon)!
- Also hold for average sub-opt gap $\frac{1}{|\mathcal{S}|}\sum_{s\in\mathcal{S}}\big[V^{(t)}(s) - V^{\star}(s)\big]$.

MDP construction for our lower bound



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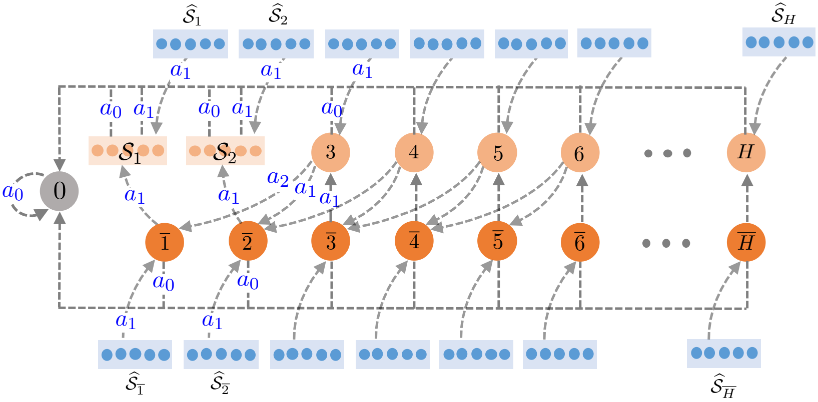

Key ingredients: for $3 \leq s \leq H \asymp \frac{1}{1-\gamma}$,

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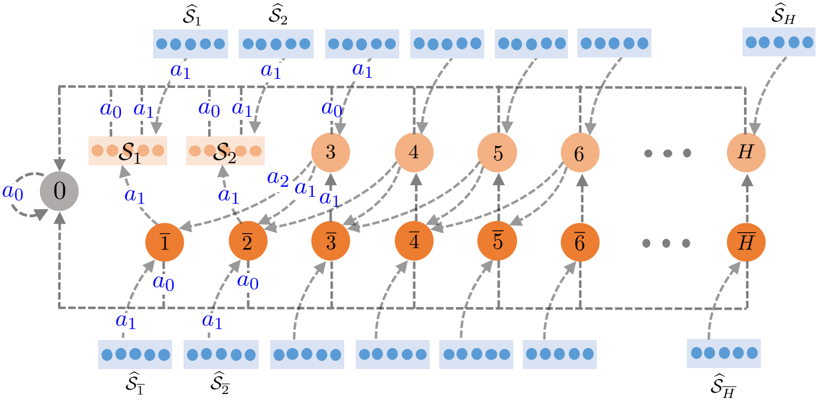
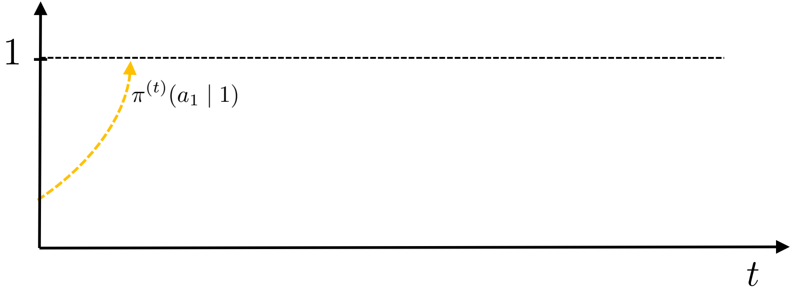


Key ingredients: for $3 \leq s \leq H \asymp \frac{1}{1-\gamma}$,

- $\pi^{(t)}(a_{\mathsf{opt}} \,|\, s)$ keeps decreasing until $\pi^{(t)}(a_{\mathsf{opt}} \,|\, s-2) \approx 1$

What is happening in our constructed MDP?



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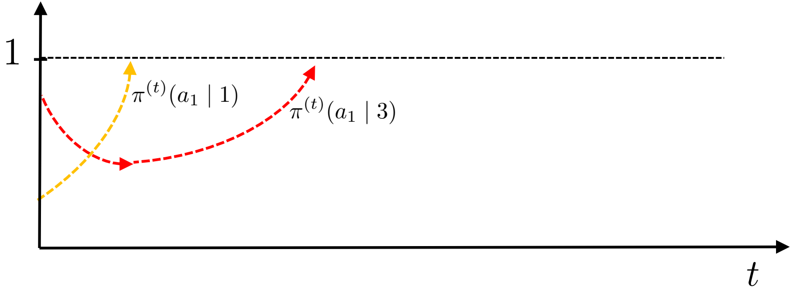

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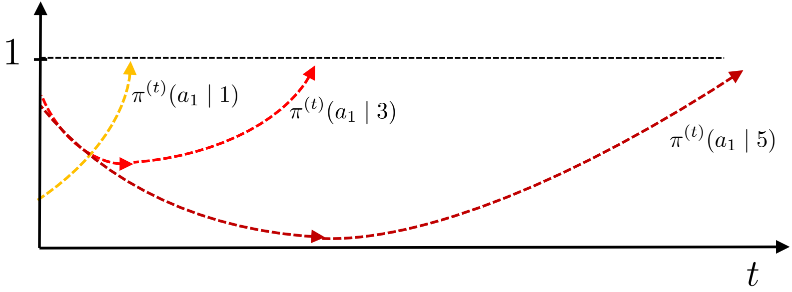

Convergence time for state s grows geometrically as s increases

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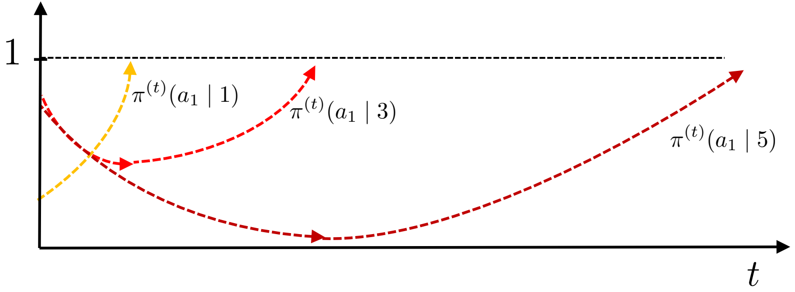

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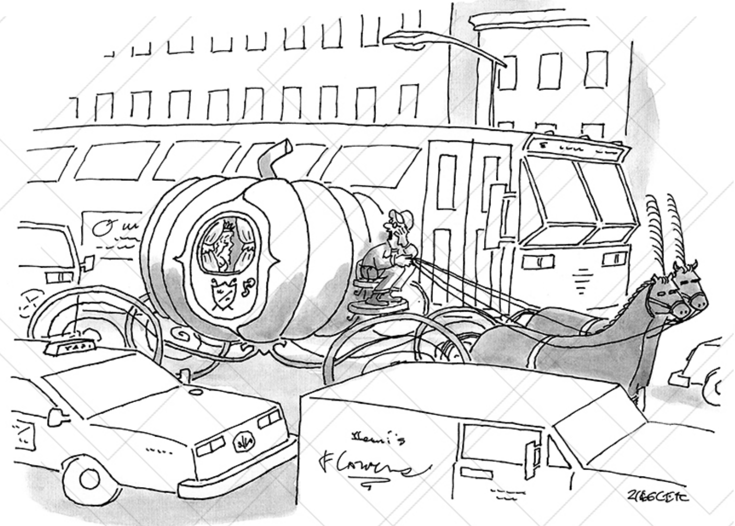
$$\text{convergence-time}(s) \gtrsim \big(\text{convergence-time}(s-2)\big)^{1.5}$$



"Seriously, lady, at this hour you'd make a lot better time taking the subway."

Accelerating the convergence via preconditioning and regularization

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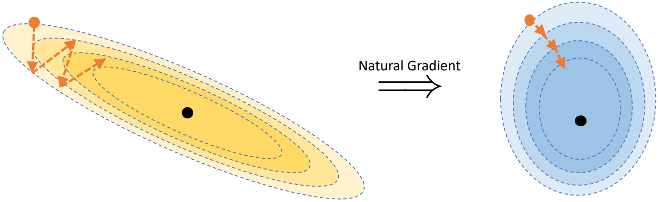
Booster #1: natural policy gradient



Natural policy gradient (NPG) method (Kakade, 2002)

For $t = 0, 1, \cdots$

$$\theta^{(t+1)} = \theta^{(t)} + \eta (\mathcal{F}_\rho^\theta)^\dagger \nabla_\theta V^{\pi_\theta^{(t)}}(\rho)$$

where η *is the learning rate and* $\mathcal{F}_\rho^\theta$ *is the Fisher information matrix:*

$$\mathcal{F}_\rho^\theta := \mathbb{E}\left[\big(\nabla_\theta \log \pi_\theta(a|s)\big) \big(\nabla_\theta \log \pi_\theta(a|s)\big)^\top \right].$$

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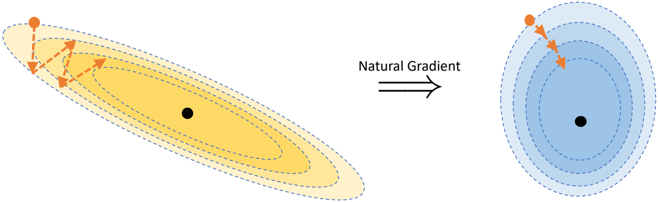

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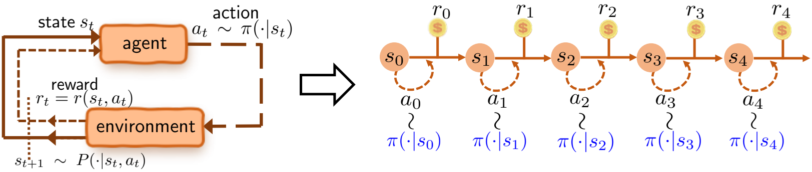
In fact, popular heuristic TRPO (Schulman et al., 2015) = NPG + line search.

Booster #2: entropy regularization



To encourage exploration, promote the stochasticity of the policy using the **"soft"** value function (Williams and Peng, 1991):

$$\forall s \in \mathcal{S}: \qquad V_\tau^\pi(s) := \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t \Big(r_t + \tau \mathcal{H}\big(\pi(\cdot|s_t)\big)\Big) \,\Big|\, s_0 = s\right]$$

where $\mathcal{H}$ is the Shannon entropy, and $\tau \geq 0$ is the reg. parameter.

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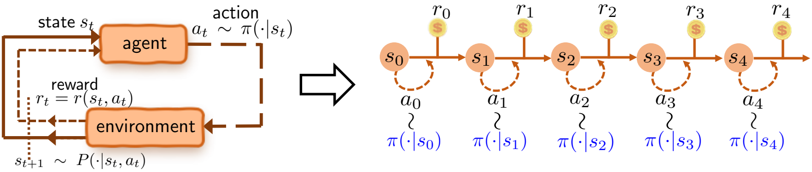

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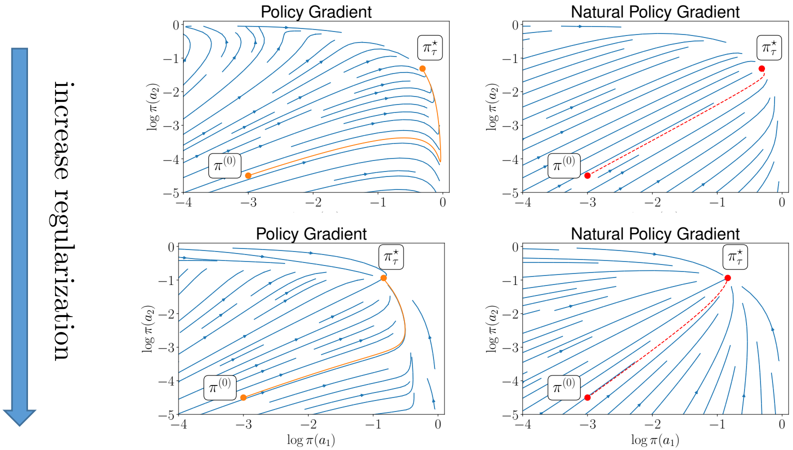
$$\text{maximize}_\theta \quad V_\tau^{\pi_\theta}(\rho) := \mathbb{E}_{s\sim\rho}\left[V_\tau^{\pi_\theta}(s)\right]$$

Entropy-regularized natural gradient helps!

Toy example: a bandit with 3 arms of rewards 1, 0.9 and 0.1.



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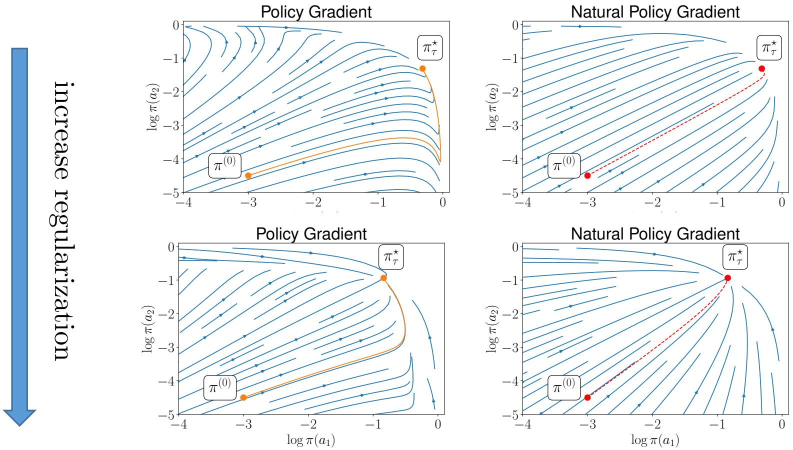
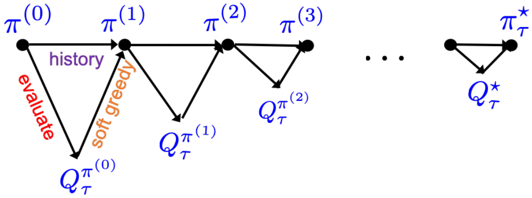


Can we justify the efficacy of entropy-regularized NPG?

Entropy-regularized NPG in the tabular setting



Entropy-regularized NPG (Tabular setting)

For $t = 0, 1, \cdots$, the policy is updated via

$$\pi^{(t+1)}(\cdot|s) \propto \underbrace{\pi^{(t)}(\cdot|s)}_{\text{current policy}}{}^{1-\frac{\eta\tau}{1-\gamma}} \underbrace{\exp(Q_\tau^{(t)}(s,\cdot)/\tau)}_{\text{soft greedy}}{}^{\frac{\eta\tau}{1-\gamma}}$$

where $Q_\tau^{(t)} := Q_\tau^{\pi^{(t)}}$ is the soft Q-function of $\pi^{(t)}$, and $0 < \eta \leq \frac{1-\gamma}{\tau}$.

- invariant with the choice of ρ
- Reduces to soft policy iteration (SPI) when $\eta = \frac{1-\gamma}{\tau}$.

Linear convergence with exact gradient

Exact oracle: perfect evaluation of $Q_\tau^{\pi^{(t)}}$ given $\pi^{(t)}$;

Theorem (Cen, Cheng, Chen, Wei, Chi '20)

For any learning rate $0 < \eta \le (1-\gamma)/\tau$, the entropy-regularized NPG updates satisfy

- **Linear convergence of soft Q-functions:**

$$\|Q_\tau^\star - Q_\tau^{(t+1)}\|_\infty \le C_1 \gamma \left(1 - \eta\tau\right)^t$$

for all $t \ge 0$, where $Q_\tau^\star$ is the optimal soft Q-function, and

$$C_1 = \|Q_\tau^\star - Q_\tau^{(0)}\|_\infty + 2\tau \left(1 - \frac{\eta\tau}{1-\gamma}\right) \| \log \pi_\tau^\star - \log \pi^{(0)} \|_\infty.$$

Implications

To reach $\|Q_\tau^\star - Q_\tau^{(t+1)}\|_\infty \le \epsilon$, the iteration complexity is at most

- **General learning rates ($0 < \eta < \frac{1-\gamma}{\tau}$):**

$$\frac{1}{\eta\tau} \log\left(\frac{C_1\gamma}{\epsilon}\right)$$

- **Soft policy iteration ($\eta = \frac{1-\gamma}{\tau}$):**

$$\frac{1}{1-\gamma} \log\left(\frac{\|Q_\tau^\star - Q_\tau^{(0)}\|_\infty \gamma}{\epsilon}\right)$$

Implications

To reach $\|Q_\tau^\star - Q_\tau^{(t+1)}\|_\infty \le \epsilon$, the iteration complexity is at most

- **General learning rates ($0 < \eta < \frac{1-\gamma}{\tau}$):**

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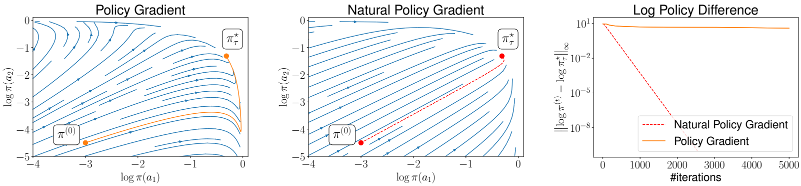
Global linear convergence of entropy-regularized NPG
at a rate independent of $|\mathcal{S}|$, $|\mathcal{A}|$!

Comparisons with entropy-regularized PG



(Mei et al., 2020) showed entropy-regularized PG achieves

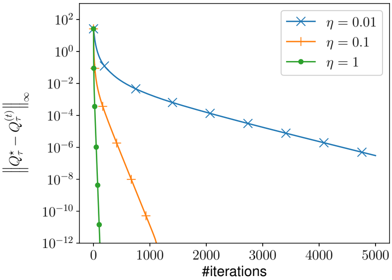
$$V_\tau^\star(\rho) - V_\tau^{(t)}(\rho) \le \left(V_\tau^\star(\rho) - V_\tau^{(0)}(\rho)\right)$$

$$\cdot \exp\left(-\frac{(1-\gamma)^4 t}{(8/\tau + 4 + 8\log|\mathcal{A}|)|\mathcal{S}|}\left\|\frac{d_\rho^{\pi_\tau^\star}}{\rho}\right\|_\infty^{-1} \min_s \rho(s) \underbrace{\left(\inf_{0\le k\le t-1} \min_{s,a} \pi^{(k)}(a|s)\right)^2}_{\text{can be exponential in } |\mathcal{S}| \text{ and } \frac{1}{1-\gamma}}\right)$$

Much faster convergence of entropy-regularized NPG at a **dimension-free** rate!

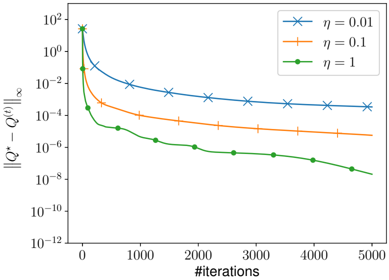
Comparison with unregularized NPG

Regularized NPG
$\tau = 0.001$



Linear rate: $\frac{1}{\eta\tau}\log\left(\frac{1}{\epsilon}\right)$
Ours

Vanilla NPG
$\tau = 0$



Sublinear rate: $\frac{1}{\min\{\eta,(1-\gamma)^2\}\epsilon}$
(Agarwal et al. 2019)

Comparison with unregularized NPG

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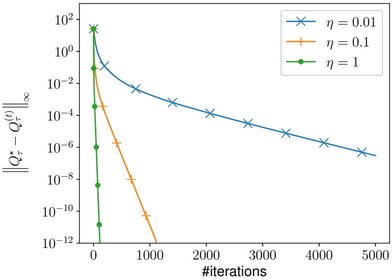

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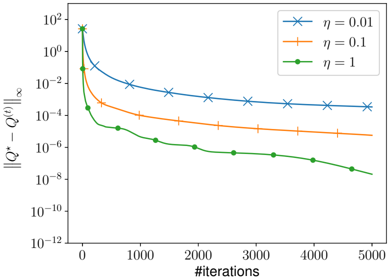

Sublinear rate: $\frac{1}{\min\{\eta,(1-\gamma)^2\}\epsilon}$

(Agarwal et al. 2019)

Entropy regularization enables fast convergence!

Entropy-regularized NPG with inexact gradients

Inexact oracle: inexact evaluation of $Q_\tau^{\pi^{(t)}}$ given $\pi^{(t)}$, which returns $\widehat{Q}_\tau^{(t)}$ that

$$\big\|\widehat{Q}_\tau^{(t)} - Q_\tau^{(t)}\big\|_\infty \le \delta,$$

e.g., using sample-based estimators (Williams, 1992).

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Inexact entropy-regularized NPG:

$$\pi^{(t+1)}(a|s) \;\propto\; \big(\pi^{(t)}(a|s)\big)^{1-\frac{\eta\tau}{1-\gamma}} \exp\Big(\frac{\eta \widehat{Q}_\tau^{(t)}(s,a)}{1-\gamma}\Big)$$

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Question: Robustness of entropy-regularized NPG?

Linear convergence with inexact gradients

Theorem (Cen, Cheng, Chen, Wei, Chi '20; improved)

For any learning rate $0 < \eta \le (1-\gamma)/\tau$, the entropy-regularized NPG updates achieve the same iteration complexity as the exact case, as long as

$$\delta \le \frac{1-\gamma}{\gamma} \cdot \min\left\{ \frac{\epsilon}{4}, \sqrt{\frac{\epsilon\tau}{2}} \right\}$$

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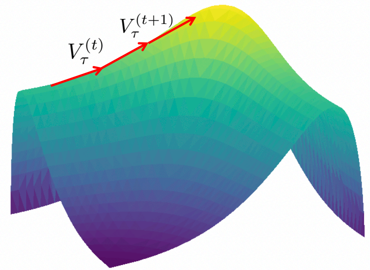
Statistical implication: how many samples are sufficient to find an ϵ-optimal policy of the unregularized MDP?

$$\widetilde{\mathcal{O}}\left(\frac{|\mathcal{S}||\mathcal{A}|}{(1-\gamma)^7\epsilon^2}\right) \text{ samples}$$

Recipe: set $\tau = \frac{(1-\gamma)\epsilon}{\log|\mathcal{A}|}$; use fresh samples for policy evaluation (Li et al., 2020).

A glimpse of the analysis

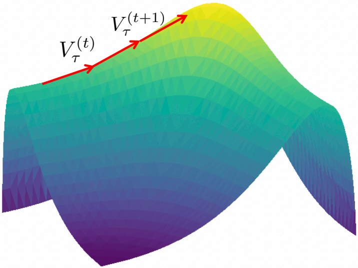
A key lemma: monotonic performance improvement



$$V_\tau^{(t+1)}(\rho) - V_\tau^{(t)}(\rho) = \mathbb{E}_{s\sim d_\rho^{(t+1)}}\Bigg[\left(\frac{1}{\eta} - \frac{\tau}{1-\gamma}\right)\underbrace{\mathsf{KL}\Big(\pi^{(t+1)}(\cdot|s)\,\|\,\pi^{(t)}(\cdot|s)\Big)}_{\text{KL divergence}} + \frac{1}{\eta}\underbrace{\mathsf{KL}\left(\pi^{(t)}(\cdot|s)\,\|\,\pi^{(t+1)}(\cdot|s)\right)}_{\text{KL divergence}}\Bigg]$$

discounted state visitation distribution

A key lemma: monotonic performance improvement



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$$+\frac{1}{\eta}\underbrace{\mathsf{KL}\left(\pi^{(t)}(\cdot|s)\,\|\,\pi^{(t+1)}(\cdot|s)\right)}_{\text{KL divergence}}\Bigg]$$

discounted state visitation distribution

Implication: monotonic improvement of $V_\tau(s)$ and $Q_\tau(s,a)$.

Recall: Bellman's optimality principle

Bellman operator

$$\mathcal{T}(Q)(s,a) := \underbrace{r(s,a)}_{\text{immediate reward}} + \gamma \underset{s' \sim P(\cdot|s,a)}{\mathbb{E}} \underbrace{\left[\max_{a' \in \mathcal{A}} Q(s',a')\right]}_{\text{next state's value}}$$

- one-step look-ahead

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Bellman equation: $Q^\star$ is *unique* solution to

$$\mathcal{T}(Q^\star) = Q^\star$$

γ-**contraction of Bellman operator:**

$$\|\mathcal{T}(Q_1) - \mathcal{T}(Q_2)\|_\infty \le \gamma \|Q_1 - Q_2\|_\infty$$

Richard Bellman

A key operator: soft Bellman operator

Soft Bellman operator

$$\mathcal{T}_\tau(Q)(s,a) := \underbrace{r(s,a)}_{\text{immediate reward}} + \gamma \mathop{\mathbb{E}}_{s' \sim P(\cdot|s,a)} \left[\max_{\pi(\cdot|s')} \mathop{\mathbb{E}}_{a' \sim \pi(\cdot|s')} \Big[\underbrace{Q(s',a')}_{\text{next state's value}} - \underbrace{\tau \log \pi(a'|s')}_{\text{entropy}} \Big] \right],$$

A key operator: soft Bellman operator

Soft Bellman operator

$$\mathcal{T}_\tau(Q)(s,a) := \underbrace{r(s,a)}_{\text{immediate reward}} + \gamma \mathop{\mathbb{E}}_{s' \sim P(\cdot|s,a)} \left[\max_{\pi(\cdot|s')} \mathop{\mathbb{E}}_{a' \sim \pi(\cdot|s')} \Big[\underbrace{Q(s',a')}_{\text{next state's value}} - \underbrace{\tau \log \pi(a'|s')}_{\text{entropy}} \Big] \right],$$

Soft Bellman equation: $Q^\star_\tau$ is *unique* solution to

$$\mathcal{T}_\tau(Q^\star_\tau) = Q^\star_\tau$$

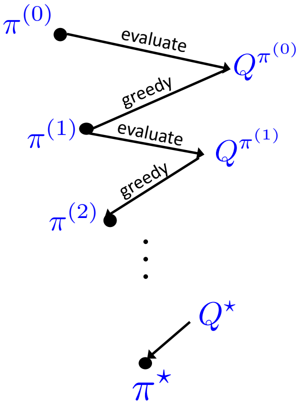
γ**-contraction of soft Bellman operator:**

$$\|\mathcal{T}_\tau(Q_1) - \mathcal{T}_\tau(Q_2)\|_\infty \le \gamma \|Q_1 - Q_2\|_\infty$$

Richard Bellman

Analysis of soft policy iteration ($\eta = \frac{1-\gamma}{\tau}$)

Policy iteration



Bellman operator

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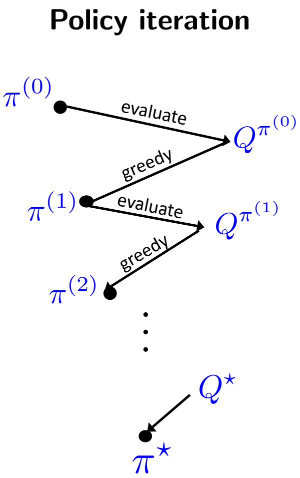
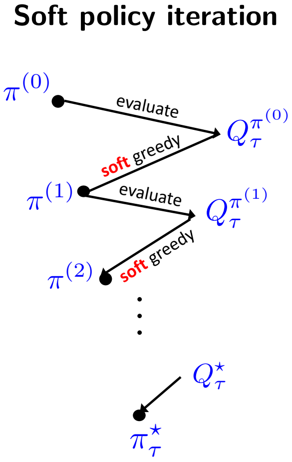


Bellman operator



Soft Bellman operator

A key linear system: general learning rates

Let $x_t := \begin{bmatrix} \|Q_\tau^\star - Q_\tau^{(t)}\|_\infty \\ \|Q_\tau^\star - \tau \log \xi^{(t)}\|_\infty \end{bmatrix}$ and $y := \begin{bmatrix} \|Q_\tau^{(0)} - \tau \log \xi^{(0)}\|_\infty \\ 0 \end{bmatrix}$,

where $\xi^{(t)} \propto \pi^{(t)}$ is an auxiliary sequence, then

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where $\xi^{(t)} \propto \pi^{(t)}$ is an auxiliary sequence, then

$$x_{t+1} \leq A x_t + \gamma \left(1 - \frac{\eta\tau}{1-\gamma}\right)^{t+1} y,$$

where

$$A := \begin{bmatrix} \gamma \\ 1 \end{bmatrix} \cdot \begin{bmatrix} \frac{\eta\tau}{1-\gamma} & 1 - \frac{\eta\tau}{1-\gamma} \end{bmatrix}$$

is a rank-1 matrix with a non-zero eigenvalue $\underbrace{1 - \eta\tau}_{\text{contraction rate!}}$.

Beyond entropy regularization

Shicong Cen — CMU

Wenhao Zhan — Princeton

Yuxin Chen — Princeton

Jason Lee — Princeton

The ever-important role of regularization

Leverage regularization to promote structural properties of the learned policy.

cost-sensitive RL

weighted 1-norm

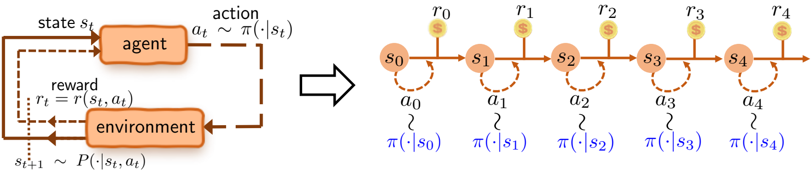
sparse exploration

Tsallis entropy

constrained and safe RL

log-barrier

Regularized RL in general form

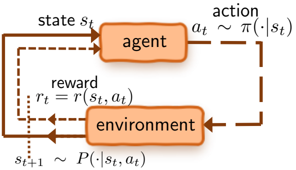


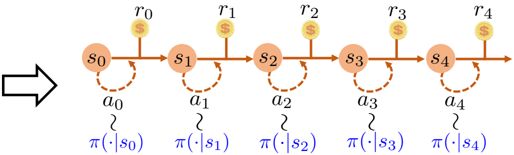
The regularized value function is defined as

$$\forall s \in \mathcal{S}: \qquad V_\tau^\pi(s) := \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t \Big(r_t - \tau h_{s_t}(\pi(\cdot|s_t))\Big) \,\Big|\, s_0 = s\right],$$

where h_s is convex (and possibly nonsmooth) w.r.t. $\pi(\cdot|s)$.

Regularized RL in general form





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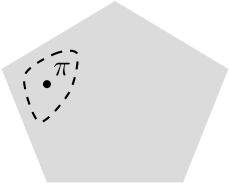
where h_s is convex (and possibly nonsmooth) w.r.t. $\pi(\cdot|s)$.

$$\text{maximize}_\pi \quad V_\tau^\pi(\rho) := \mathbb{E}_{s\sim\rho}\left[V_\tau^\pi(s)\right]$$

Detour: a mirror descent view of entropy-regularized NPG



Entropy-regularized NPG = mirror descent with KL divergence (Lan, 2021; Shani et al., 2020):

$$\pi^{(t+1)}(\cdot|s) = \underset{p\in\Delta(\mathcal{A})}{\operatorname{argmin}} \big\langle -Q_\tau^{(t)}(s,\cdot),\, p\big\rangle - \tau\mathcal{H}(p) + \frac{1}{\eta}\mathsf{KL}\big(p\|\pi^{(t)}(\cdot|s)\big)$$

$$\propto \underbrace{\pi^{(t)}(\cdot|s)}_{\text{current policy}}{}^{\frac{1}{1+\eta\tau}} \underbrace{\exp(Q_\tau^{(t)}(s,\cdot)/\tau)}_{\text{soft greedy}}{}^{\frac{\eta\tau}{1+\eta\tau}}$$

for all $s \in \mathcal{S}$.

Generalized policy mirror descent (GPMD)

Definition (Generalized Bregman divergence, Kiwiel 1997)

The generalized Bregman divergence w.r.t. to a convex $h : \Delta(\mathcal{A}) \mapsto \mathbb{R}$ is defined as:

$$\begin{aligned} D_h(p, q; g) &= h(p) - h(q) - \langle g, p - q \rangle \\ &= h(p) - h(q) - \langle g - c \cdot \mathbf{1}, p - q \rangle, \end{aligned}$$

for $p, q \in \Delta(\mathcal{A})$, where $g \in \partial h(q)$ and $c \in \mathbb{R}$.

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A natural idea

For $t = 0, 1, \cdots,$

$$\pi^{(t+1)}(\cdot|s) = \operatorname*{argmin}_{p \in \Delta(\mathcal{A})} \langle -Q_\tau(s, \cdot), p \rangle + \tau h_s(p) + \frac{1}{\eta} D_{h_s}(p, \pi^{(t)}(\cdot|s); \partial h_s(\pi^{(t)}(\cdot|s)))$$

PMD with Generalized Bregman Divergence (**GPMD**)

Plugging in a recursive surrogate $\{\xi^{(t)}\}$ of $\partial h_s(\pi^{(t)}(\cdot|s))$, we obtain the formal algorithm.

Generalized policy mirror descent (GPMD) method

For $t = 0, 1, \cdots$, *update*

$$\pi^{(t+1)}(\cdot|s) = \operatorname*{argmin}_{p \in \Delta(\mathcal{A})} \langle -Q_\tau(s, \cdot), p \rangle + \tau h_s(p) + \frac{1}{\eta} D_{h_s}(p, \pi^{(t)}(\cdot|s); \xi^{(t)}(s, \cdot)),$$

and

$$\xi^{(t+1)}(s, \cdot) = \frac{1}{1 + \eta\tau} \xi^{(t)}(s, \cdot) + \frac{\eta}{1 + \eta\tau} Q_\tau^{(t)}(s, \cdot).$$

The subproblem does not admit closed-form solution in general.

Linear convergence with exact gradient

Exact oracle: perfect evaluation of $Q_\tau^{\pi^{(t)}}$ given $\pi^{(t)}$; exact solution to subproblems.

— *Read our paper for the inexact case!*

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Theorem (Zhan*, Cen*, Huang, Chen, Lee, Chi '21)

For any learning rate $\eta > 0$, *the GPMD updates satisfy*

- **Linear convergence of soft Q-functions:**

$$\|Q_\tau^\star - Q_\tau^{(t+1)}\|_\infty \le C_1 \gamma \left(1 - \frac{\eta\tau(1-\gamma)}{1+\eta\tau}\right)^t$$

where $C_1 = \|Q_\tau^\star - Q_\tau^{(0)}\|_\infty + \frac{2}{1+\eta\tau}\|Q_\tau^\star - \tau\xi^{(0)}\|_\infty$.

Implications

To reach $\|Q_\tau^\star - Q_\tau^{(t+1)}\|_\infty \le \epsilon$, the iteration complexity is at most

- **General learning rates ($\eta > 0$):**

$$\frac{1+\eta\tau}{\eta\tau(1-\gamma)} \log\left(\frac{C_1\gamma}{\epsilon}\right)$$

- **Regularized policy iteration ($\eta = \infty$):**

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Global linear convergence of GPMD at a **dimension-free** rate!

Comparison with PMD (Lan, 2021)

Policy mirror descent (PMD) method (Lan, 2021)

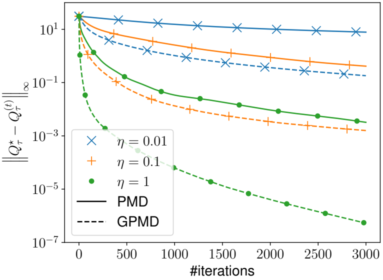
For $t = 0, 1, \cdots,$

$$\pi^{(t+1)}(\cdot|s) = \operatorname*{argmin}_{p \in \Delta(\mathcal{A})} \langle -Q_\tau(s, \cdot), p \rangle + \tau h_s(p) + \frac{1}{\eta} \mathsf{KL}(p || \pi^{(t)}(\cdot|s))$$

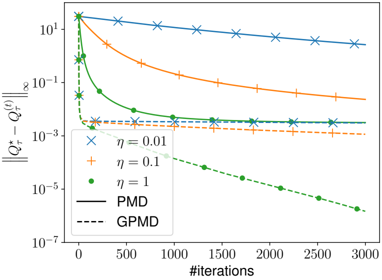
- Linear convergence is established only when h_s is stronger than entropy regularization ($h_s + \mathcal{H}$ is convex).
- In contrast, GPMD converges linearly for general convex and nonsmooth h_s!

Numerical examples

h_s = **Tsallis Entropy**



h_s = **Log Barrier**



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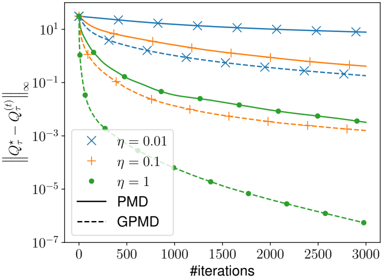

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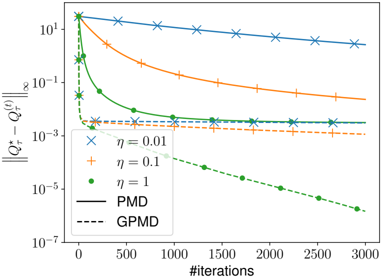


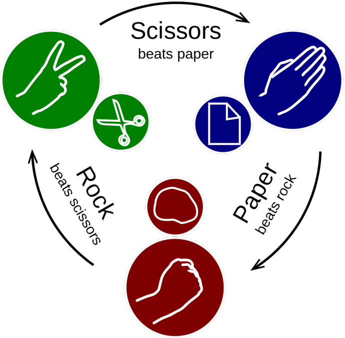
GPMD achieves faster convergence than PMD!

Bonus track: entropy-regularized games

Shicong Cen
CMU

Yuting Wei
CMU

Zero-sum entropy-regularized two-player matrix game



			
	0	-1	1
	1	0	-1
	-1	1	0

Quantal response equilibrium (McKelvey and Palfrey, 1995)

$$\max_{\mu \in \Delta(\mathcal{A})} \min_{\nu \in \Delta(\mathcal{B})} \mu^\top A \nu + \tau \mathcal{H}(\mu) - \tau \mathcal{H}(\nu)$$

- Basic building block for solving value iteration in zero-sum two-player Markov games.

Motivation: an implicit update method

Implicit update (IU) method

For $t = 0, 1, \cdots,$

$$\begin{cases} \mu^{(t+1)} \propto [\mu^{(t)}]^{1-\eta\tau} \exp\left([A\nu^{(t+1)}]/\tau\right)^{\eta\tau} \\ \nu^{(t+1)} \propto [\nu^{(t)}]^{1-\eta\tau} \exp\left(-[A^\top \mu^{(t+1)}]/\tau\right)^{\eta\tau} \end{cases}$$

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Theorem (Cen, Wei, Chi, 2021)

Suppose that $0 < \eta \leq 1/\tau$*, then for all* $t \geq 0$*,*

$$\mathsf{KL}\left(\zeta_\tau^\star \,\|\, \zeta^{(t)}\right) \leq (1 - \eta\tau)^t \mathsf{KL}\left(\zeta_\tau^\star \,\|\, \zeta^{(0)}\right),$$

where $\mathsf{KL}\left(\zeta_\tau^\star \,\|\, \zeta^{(t)}\right) = \mathsf{KL}\left(\mu_\tau^\star \| \mu^{(t)}\right) + \mathsf{KL}\left(\nu_\tau^\star \| \nu^{(t)}\right)$.

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Can we make this practical?

From implicit updates to policy extragradient methods

Predictive update (PU) method

For $t = 0, 1, \cdots,$

2. *update:*

$$\begin{cases} \mu^{(t+1)} \propto [\mu^{(t)}]^{1-\eta\tau} \exp\left([A\bar{\nu}^{(t+1)}]/\tau\right)^{\eta\tau} \\ \nu^{(t+1)} \propto [\nu^{(t)}]^{1-\eta\tau} \exp\left(-[A^\top \bar{\mu}^{(t+1)}]/\tau\right)^{\eta\tau} \end{cases}$$

From implicit updates to policy extragradient methods

Predictive update (PU) method

For $t = 0, 1, \cdots$,

1. *extrapolate/predict:*

$$\begin{cases} \bar{\mu}^{(t+1)} \propto [\mu^{(t)}]^{1-\eta\tau} \exp\left([A\nu^{(t)}]/\tau\right)^{\eta\tau} \\ \bar{\nu}^{(t+1)} \propto [\nu^{(t)}]^{1-\eta\tau} \exp\left(-[A^\top \mu^{(t)}]/\tau\right)^{\eta\tau} \end{cases}$$

2. *update:*

$$\begin{cases} \mu^{(t+1)} \propto [\mu^{(t)}]^{1-\eta\tau} \exp\left([A\bar{\nu}^{(t+1)}]/\tau\right)^{\eta\tau} \\ \nu^{(t+1)} \propto [\nu^{(t)}]^{1-\eta\tau} \exp\left(-[A^\top \bar{\mu}^{(t+1)}]/\tau\right)^{\eta\tau} \end{cases}$$

From implicit updates to policy extragradient methods

Optimistic multiplicative weights update (OMWU) method

For $t = 0, 1, \cdots,$

1. *extrapolate/predict:*

$$\begin{cases} \bar{\mu}^{(t+1)} \propto [\mu^{(t)}]^{1-\eta\tau} \exp\left([A\bar{\nu}^{(t)}]/\tau\right)^{\eta\tau} \\ \bar{\nu}^{(t+1)} \propto [\nu^{(t)}]^{1-\eta\tau} \exp\left(-[A^\top \bar{\mu}^{(t)}]/\tau\right)^{\eta\tau} \end{cases}$$

2. *update:*

$$\begin{cases} \mu^{(t+1)} \propto [\mu^{(t)}]^{1-\eta\tau} \exp\left([A\bar{\nu}^{(t+1)}]/\tau\right)^{\eta\tau} \\ \nu^{(t+1)} \propto [\nu^{(t)}]^{1-\eta\tau} \exp\left(-[A^\top \bar{\mu}^{(t+1)}]/\tau\right)^{\eta\tau} \end{cases}$$

Last-iterate convergence

- **Entropy-regularized matrix game:** To get an ϵ-optimal solution to the regularized problem (ϵ-**QRE**), the iteration complexity is at most

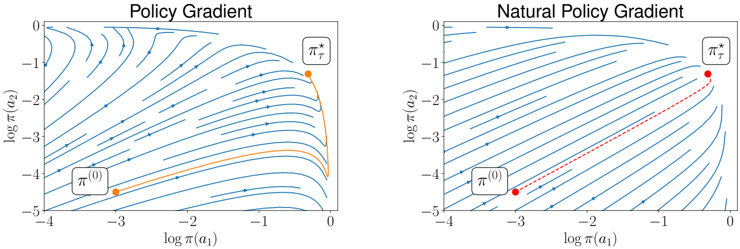
$$\widetilde{O}\left(\left(1+\frac{\|A\|_\infty}{\tau}\right)\log\frac{1}{\epsilon}\right).$$

- **Unregularized matrix game:** To get an ϵ-optimal solution to the unregularized problem (ϵ-**NE**), the iteration complexity is at most

$$\widetilde{O}\left(\frac{\|A\|_\infty}{\epsilon}\right).$$

No need to assume unique Nash equilibrium!

Concluding remarks



fast global linear convergence of RL enabled by **regularization + preconditioning**

Future directions:

- function approximation
- sample complexities
- Markov games
- multi-agent RL

Thanks!

- Fast Global Convergence of Natural Policy Gradient Methods with Entropy Regularization, *Operations Research*; arXiv: 2007.06558.
- Softmax Policy Gradient Methods Can Take Exponential Time to Converge, COLT 2021; arXiv: 2102.11270.
- Policy Mirror Descent for Regularized Reinforcement Learning: A Generalized Framework with Linear Convergence, arXiv: 2105.11066.
- Fast Policy Extragradient Methods for Competitive Games with Entropy Regularization, arXiv: 2105.15186.

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