Range Sidelobe Suppression in a Desired Doppler Interval

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Advances in Radar Hardware

**Multiple degrees of freedom:** New generation of radar transmitters allows for transmission of different waveforms across

- Time
- Space
- Frequency
- Polarization

**Question:** How to utilize new hardware capabilities to sense more accurately and with less complexity?

**Today:** Clearing out range sidelobes in a desired Doppler interval by sequencing Golay complementary waveforms in time.
Outline

1. Phase-coded Golay Complementary Waveforms
   - Sensitivity to Doppler shifts

2. PTM Pulse Trains
   - Resilience to Modest Doppler Shifts
   - Resilience to High Doppler Shifts

3. Conclusion
Golay Complementary Pair

Golay pair \((x, y)\) of length \(L\) has perfect autocorrelation property:

\[
C_x(k) + C_y(k) = 2L\delta(k)
\]
Phase-coded Waveforms

- Baseband waveforms phase coded by Golay pair \((x, y)\):

\[
s_x(t) = \sum_{\ell=0}^{L-1} x(\ell) \Omega(t - \ell T_c), \quad s_y(t) = \sum_{\ell=0}^{L-1} y(\ell) \Omega(t - \ell T_c)
\]

\(\Omega(t)\) is a unit energy pulse shape of duration \(T_c\).

- Ambiguity function of \(S(t) = s_x(t) + s_y(t - T)\):

\[
\chi_S(\tau, \nu) = \chi_{s_x}(\tau, \nu) + e^{j\nu T} \chi_{s_y}(\tau, \nu)
\]

\(T\) is the PRI duration.

- For zero Doppler it is free of range sidelobes:

\[
\chi_S(\tau, 0) = 2L\chi_{\Omega}(\tau, 0)
\]

But very sensitive to Doppler shifts!
Sensitivity to Doppler Shifts
Five Targets Scenario - Conventional Approach
Five Targets Scenario - PTM in action
P-Pulse Train

- Transmit phase coded Golay pairs in accordance to $\mathcal{P}$:

$$ Z_{\mathcal{P}}(t) = \frac{1}{2} \sum_{n=0}^{N-1} \left[ (1 + p_n) s_x(t - nT) + (1 - p_n) s_y(t - nT) \right] $$

- Ambiguity function of the $\mathcal{P}$-pulse train:

$$ \chi_{Z_{\mathcal{P}}}(k, \theta) = \frac{1}{2} \left[ C_x(k) + C_y(k) \right] \sum_{n=0}^{N-1} e^{jn\theta} + \frac{1}{2} \left[ C_x(k) - C_y(k) \right] \sum_{n=0}^{N-1} p_n e^{jn\theta} $$

- Sidelobe free:

- Range sidelobes

- Shape the spectrum of $\mathcal{P}$:

$$ S_\mathcal{P}(\theta) = \sum_{n=0}^{N-1} p_n e^{jn\theta} $$

Goal: design high order spectral nulls around the desired Doppler!
Resilience to Modest Doppler Shifts

- Zero-force the Taylor expansions around $\theta = 0$ up to order $M$:

$$f_{P}^{(t)}(\theta_0) = j^{t} \sum_{n=0}^{N-1} n^{t} p_n = 0, \quad t = 0, \cdots, M$$

**Theorem (Pezeshki et al., 2007)**

To zero-force up to $M$ Taylor moments, coordinate the transmission of a Golay pair $(x, y)$ according to the length $N = 2^{M+1}$ PTM sequence, with 1 locations corresponding to $x$ and $-1$ locations corresponding to $y$. 
PTM Sequences

Definition (Prouhet-Thue-Morse)

The Prouhet-Thue-Morse (PTM) sequence \( \mathcal{P} = (p_k)_{k \geq 0} \) over \( \{-1, 1\} \) is defined by the following recursions:

1. \( p_0 = 1 \)
2. \( p_{2k} = p_k \)
3. \( p_{2k+1} = \overline{p}_k = -p_k \)

for all \( k > 0 \).

- 2nd order: PTM sequence of length \( 8 = 2^{2+1} \)

\[
\begin{array}{cccccccc}
  x & y & y & x & y & x & x & y \\
  1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\
\end{array}
\]
PTM pulse trains in action
Resilience to High Doppler Shifts

- Zero-force the Taylor expansions around $\theta = \theta_0$ up to order $M$:

$$f^{(t)}_P(\theta_0) = j^t \sum_{n=0}^{N-1} n^t p_n e^{j\theta_0} = 0, \quad t = 0, \cdots, M$$

- Focus on Rational Doppler shifts:

$$\theta_0 = \frac{2\pi l}{m}, \quad m \neq 1, \text{ and } m, l \text{ coprime.}$$

- Let $\{b_r\}_{r=0}^{s-1} = \{p_{rm+i}\}_{r=0}^{s-1}$ satisfy the PTM condition

$$\sum_{r=0}^{s-1} r^u b_r = 0, \text{ for all } 0 \leq u \leq t - 1$$
Oversampled PTM Sequences

Theorem (Oversampled PTM Sequences)

Let \( P = \{p_n\}_{n=0}^{2^M m - 1} \) be the \((2^M, m)\)-PTM sequence, that is to say that \( \{p_{rm+i}\}_{r=0}^{2^M-1}, i = 0, \cdots, m - 1 \) is a PTM sequence of length \( 2^M \). Then the spectrum \( S_P(\theta) \) of \( P \) has \( M \)th-order nulls at all \( \theta_0 = 2\pi l / m \) where \( l \) and \( m \neq 1 \) are co-prime integers.

Example: \( M = 3, m = 3 \rightarrow \{p_n\} = 000111111000 \cdots \)

Corollary (Simultaneous Nulls)

Let \( P \) be the \((2^M, m)\)-PTM sequence. Then the spectrum \( S_P(\theta) \) of \( P \) has

1. an \((M - 1)\)th-order null at \( \theta_0 = 0 \).
2. \((M - h - 1)\)th-order nulls at all \( \theta_0 = 2\pi l / (2^h m) \), where \( l \) and \( m \neq 1 \) are co-prime, and \( 1 \leq h \leq M - 1 \).
The spectral of \((2^3, 2)\)- and \((2^2, 2)\)-PTM sequences
Oversampled PTM pulse trains in action
Conclusion

- Doppler resilient pulse train of Golay complementary waveforms are constructed by coordinating the transmission of a Golay pair of phase coded waveforms in time according to the 1's and −1's in a PTM sequence or its oversampled versions.

- The magnitude of the range sidelobes of the pulse train ambiguity function of the constructed pulse trains are proportional to the magnitude spectra of \((2^M, m)\)-PTM sequences, which have high-order nulls in a desired Doppler band.

- Numerical examples demonstrate the annihilation of range sidelobes in the ambiguity functions of \((2^M, m)\)-PTM pulse trains.
Thank you!
Why PTM Sequence?

- Look at the calculations for zero-forcing the 1st and 2nd order moments.
- Key is partitioning of $\mathcal{S} = \{0, 1, \ldots, 7\}$ into disjoint subsets 
  $\mathcal{S}_0 = \{0, 3, 5, 6\}$ and $\mathcal{S}_1 = \{1, 2, 4, 7\}$ that satisfy

  $$(0^m + 3^m + 5^m + 6^m) - (1^m + 2^m + 4^m + 7^m) = 0, \quad \text{for } m = 1, 2.$$ 

- **Prouhet’s Problem:** Let $\mathcal{S} = \{0, 1, \ldots, N - 1\}$. Given $M$, is it possible to partition $\mathcal{S}$ into two disjoint subsets $\mathcal{S}_0$ and $\mathcal{S}_1$ such that

  $$\sum_{r \in \mathcal{S}_0} r^m = \sum_{q \in \mathcal{S}_1} q^m$$

  for all $0 \leq m \leq M$?

**Solution:** Possible when $N = 2^{M+1}$. The partitions are identified by the PTM sequence.
Reed-Müller Pulse Trains: Sidelobe Suppression

**Question:** Can we clear out range sidelobes in other Doppler intervals?

**First order Reed-Müller code** $RM(1, M)$ consists of $2^M$ code words of the form

$$r_b(n) = \sum_{m=0}^{2^M-1} b_m n_m \quad \text{for } n = 0, \ldots, 2^M - 1$$

where $n_m$ denotes the $m$th binary digit of $n$.

**Walsh functions** are the exponentiated Reed-Müller codes

$$w_b(n) = (-1)^{r_b(n)} \quad \text{for } n = 0, \ldots, 2^M - 1$$

**PTM sequence** is equal to $r_b(n)$ with $b = (1, 1, \ldots, 1)$, and $(-1)^{r_b(n)}$ is its corresponding Walsh function.