Compressive Spectrum Estimation using Quantized Measurements

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Abstract—In this paper, we aim to recover a spectrally-sparse signal from the quadrants of complex random linear measurements. We propose a new algorithm based on atomic norm regularization, which is equivalent to proximal mapping of a properly designed surrogate signal with respect to the atomic norm. Moreover, the frequencies can be localized without knowing the model order a priori via a dual polynomial approach. It is shown that under the Gaussian measurement model, the signal can be reconstructed accurately with high probability, as soon as the number of quantized measurements exceeds the order of $k \log n$, where $k$ is the number of frequencies and $n$ is the signal dimension. Our results can be extended to more general nonlinear measurements using generalized linear models.

Index Terms—line spectrum estimation, quantization, atomic norms, compressive sensing

I. INTRODUCTION

Emerging applications in wireless communications, cognitive radio, and radar systems deal with signals of wideband or ultrawideband. Spectrum sensing or signal acquisition in this regime is a fundamental challenge in signal processing, since the well-known Shannon-Nyquist sampling rate may become prohibitively high in practice. It is highly desirable to come up with alternative approaches that have less demanding requirements in terms of sampling rates. Compressed sensing (CS) [1], [2] allows sub-Nyquist sampling [3]–[5] when the wideband spectrum is approximately sparse.

In analog-to-digital conversion (ADC), quantization is another necessary step which maps the analog samples into a finite number of bits for digital processing. The figure-of-merit of ACs examines both the sampling rate and the effective number of bits (ENOB), which is the number of bits per sample. Typically, the ENOB is smaller to compensate for high sampling rate [6]. In bandwidth-constrained network sensing scenarios, it is also of interest to transmit quantized messages to reduce the communication overhead [7]–[9].

In this paper, we study spectrum estimation of sparse bandlimited signals from the signs of random linear measurements. When the measurements are complex-valued, the quantization is based on the quadrants. To avoid basis mismatch [10], atomic norm minimization [11]–[14] has been proposed to promote spectral sparsity via convex optimization without discretizing the frequencies onto a finite grid. We propose a convex program to recover the signal by finding the proximal projection of a surrogate signal, formed by linear combinations of the sign-modulated measurement vectors, with respect to the atomic norm. Moreover, the frequencies can be localized without knowing the model order a priori, by examining the peak of a dual polynomial constructed from the dual solution. When the measurement vectors are composed of i.i.d. complex Gaussian entries, under a mild separation condition, it is shown that the signal can be reconstructed accurately, up to a scaling difference, from $O(k \log n)$ measurements, where $k$ is the level of spectral sparsity, and $n$ is the signal dimension. Furthermore, we quantify how the performance relates to the signal-to-noise ratio before quantization.

Our work is closely related to 1-bit compressed sensing [15]–[22], which aims to recover a sparse signal from signs of random linear measurements. In particular, Plan and Vershynin [18]–[21] generalize this idea to reconstructing signals that belong to some set with small Gaussian width. Our work is concerned with a special class of spectrally-sparse signals that can be viewed as sparse in an infinite-dimensional dictionary.

The rest of this paper is organized as follows. Section II provides the signal model and problem formulation. Section III presents the proposed algorithm with performance guarantees. Numerical experiments are provided in Section IV. We conclude in Section V.

II. SIGNAL MODEL AND PROBLEM FORMULATION

Let $x^* \in \mathbb{C}^n$ be a line spectrum signal, which is composed of a small number of spectral lines, defined as

$$x^* = \sum_{i=1}^{k} c_i v(f_i),$$  

(1)

where $k$ is the number of frequencies, $c_i \in \mathbb{C}$, $f_i \in [0, 1]$, and

$$v(f_i) = [1, e^{j2\pi f_i}, \ldots, e^{j2\pi(n-1)f_i}]^T.$$  

Denote the set of frequencies as $\mathcal{F} = \{f_i\}_{i=1}^{k}$. Define the quantized measurements as

$$y_i = \text{sign}(\langle a_i, x^* \rangle + \sigma \epsilon_i), \quad i = 1, \ldots, m,$$  

(2)

where $m$ is the number of measurements, $y_i \in \{\pm 1 \pm j\}$ is drawn from the QPSK constellation based on which quadrant the complex-valued signal $y_{i, \text{UQ}} = \langle a_i, x^* \rangle + \sigma \epsilon_i$ resides. The measurement vectors $a_i$’s are composed of i.i.d. standard complex Gaussian entries, $CN(0,1)$, $\epsilon_i \sim CN(0,1)$ are i.i.d. and $\sigma$ is the noise level. Our goal is to recover $x^*$. 

and the set of frequencies, from the quantized measurements \( y = \{y_i\}_{i=1}^m \), without knowing the sparsity level \( k \).

III. PROPOSED APPROACH

Atomic norm minimization [11]–[14] is recently demonstrated as an effective convex regularizer to motivate spectral sparsity without discretization. Let \( \mathcal{A} = \{ \mathbf{v}(f) : f \in [0, 1) \} \) denote the atomic set, then \( \|x\|_A \) is defined as
\[
\|x\|_A := \inf \left\{ \sum_i |\alpha_i| : z = \sum_i \alpha_i \mathbf{v}(f_i) \right\},
\]
which can be viewed as a continuous analog of the \( \ell_1 \) norm. Appealingly, \( \|x\|_A \) admits a semidefinite programming characterization, which can be computed efficiently using off-the-shelf solvers:
\[
\|x\|_A = \min_{u \in \mathbb{C}^n} \left\{ \frac{1}{2} \text{Tr}(T(u)) + \frac{t}{2} \left\lfloor \frac{T(u)}{x^H \mathbf{u}} t \right\rfloor \geq 0 \right\},
\]
where \( T(u) \in \mathbb{C}^{n \times n} \) denotes the Hermitian Toeplitz matrix with \( u \) as the first column.

A. Proposed Algorithm

Inspired by [21], we construct a surrogate signal from the quantized measurements as
\[
s = \frac{1}{m} \sum_{i=1}^m y_i a_i, \tag{3}
\]
which is an unbiased estimator of \( x^* \) up to a scaling difference, i.e.
\[
\mathbb{E}[s] = \lambda \frac{x^*}{\|x^*\|_2},
\]
where
\[
\lambda = \frac{2\|x^*\|_2}{\sqrt{\pi} (\sigma^2 + \|x^*\|_2^2)} = \frac{2}{\sqrt{\pi (1/\text{SNR} + 1)}} \tag{4}
\]
depends on the signal-to-noise ratio before quantization \( \text{SNR} = \|x^*\|^2/\sigma^2 \). Fig. 1 depicts \( \lambda \) as a function of SNR.

We propose the following atomic norm regularized algorithm,
\[
\hat{x} = \text{argmin}_{x \in \mathbb{C}^n} \left\{ \frac{1}{2} \|x - s\|_2^2 + \tau \|x\|_A \right\}, \tag{5}
\]
which is the proximal mapping of the surrogate signal \( s \) with respect to the atomic norm, where \( \tau > 0 \) is the regularization parameter. The proposed algorithm can be solved efficiently via the following semidefinite program:
\[
\min_{u \in \mathbb{C}^n} \frac{1}{2} \|x - s\|_2^2 + \tau (u_0 + t)
\]
s.t.
\[
\left[ \begin{array}{c} T(u) \\ x \end{array} \right] \geq 0,
\]
where \( u_0 \) the first entry of \( u \). One appealing feature of atomic norm minimization is that the set of frequencies can be recovered via the dual polynomial approach [23]. Namely, denote the dual variable as \( \hat{q} = (s - \hat{x})/\tau \), then the dual polynomial is defined as
\[
Q(f) = \langle \hat{q}, \mathbf{v}(f) \rangle.
\]
The set of frequencies can be localized as
\[
\hat{\mathcal{F}} = \{ f : |Q(f)| = 1 \}.
\]
We refer interested readers to the details in [11].

B. Performance Guarantee

The performance of atomic norm minimization relies critically on the separation condition, which is defined as
\[
\Delta = \min_{k \neq j} |f_k - f_j| \geq \frac{4}{n}. \tag{6}
\]
where \( |f_k - f_j| \) is evaluated as the wrap-around difference on the unit modulus. By [23], the optimal condition for (5) is given by the lemma below.

**Lemma 1** (Optimality conditions [23]). \( \hat{x} \) is the solution of (5) if and only if \( |s - \hat{x}|_A^* \leq \tau \), and \( \langle s - \hat{x}, \hat{x} \rangle = \tau \|\hat{x}\|_A^* \).

Denote the deviation \( w = s - \mathbb{E}[s] \), where \( \mathbb{E}[w] = 0 \). The dual norm \( \|w\|_{A^*} \) is given as
\[
\|w\|_{A^*} = \sup_{\|x\|_A \leq 1} \langle w, x \rangle = \sup_{f \in [0, 1]} |\langle w, \mathbf{v}(f) \rangle|.
\]
From [23], we wish to set \( \tau \geq \eta \|w\|_{A^*}^* \) for some constant \( \eta > 1 \). It is therefore critical to bound \( \|w\|_{A^*}^* \), which is provided by the following lemma.

**Lemma 2.** With high probability, we have
\[
\|w\|_{A^*}^* \leq C \sqrt{\frac{n \log n}{m}},
\]
where \( C \) is some universal constant.

The separation condition guarantees the existence of an interpolation dual polynomial which allows the application of [24],
Lemma 1]. To sum up, we have the performance guarantee of the proposed algorithm in (5), stated below.

**Theorem 1.** Set \( \tau := \eta \sqrt{n} \log n / m \) for some constant \( \eta \).
Under the separation condition, the solution \( \hat{x} \) satisfies
\[
\frac{\| \hat{x}/\lambda - x^* \|_2}{\| x^* \|_2} \lesssim \frac{1}{\lambda} \sqrt{\frac{k \log n}{m}}
\]
with high probability.

We omit the full proof due to page limits. Theorem 1 suggests that the proposed algorithm accurately recovers the signal as soon as \( m \) is on the order of \( k \log n \), which is order-wise near-optimal, since at least an order of \( k \log(n/k) \) measurements are needed in order to recover a sparse signal in the DFT basis [19]. Moreover, the theorem also suggests that the normalized reconstruction error is inverse proportional to \( \lambda \), which plays the role of SNR after quantization and is a nonlinear function of the SNR before quantization. In the low SNR regime, \( \lambda \) scales as \( 1/\sqrt{\text{SNR}} \), and the performance is comparable to that using unquantized measurements. However, in the high SNR regime, there is a saturation phenomenon, as evidenced by Fig. 1, and the performance does not improve as much as we increase SNR. These results are qualitatively in line with existing work on one-bit CS [19].

**Remark:** More generally, Theorem 1 can be extended to the generalized linear model, as long as the measurements \( y_i \)'s are i.i.d. and satisfy
\[
\mathbb{E}[y_i | \alpha_i] = g(\langle \alpha_i, x^* \rangle)
\]
for some link function \( g(\cdot) \), and accordingly \( \lambda = \mathbb{E}[g(\theta)^2] \)
where the expectation is taken with respect to \( \theta \sim \mathcal{CN}(0, \| x^* \|_2^2) \). This allows us to model more complex quantization schemes, e.g. higher-bit quantization schemes, as well as quantization errors.

### IV. Numerical Experiments

Let \( n = 64 \) and \( k = 3 \). The set of frequencies is located at \( \mathcal{F} = \{0.3, 0.325, 0.8\} \), where the first two frequencies are separated barely more than \( 1/n \), the Rayleigh limit. The number of bits is set as \( m = 1000 \), where the measurement vectors are generated with i.i.d. \( \mathcal{CN}(0, 1) \) entries. Fig. 2 shows the amplitude of the constructed dual polynomial by solving (5), where its peaks can be used to localize the frequencies. It can be seen that it matches accurately with the ground truth.

Define the signal-to-noise ratio (SNR) before quantization as \( \text{SNR}_{\text{db}} = 20 \log_{10}(\| x_2 \| / \sigma) \)dB. The normalized reconstruction error is defined as \( 1 - \| \hat{x} - x^* \|^2 / (\| x \|^2 \| x^* \|^2) \), where \( \hat{x} \) is the reconstructed signal. Fig. 3 shows the normalized reconstruction error with respect to the number of measurements at different SNRs before quantization. It can be seen that the reconstruction accuracy improves as we increase the SNR as well as the number of measurements, validating the theoretical analysis.

Next we compare the performance of signal reconstruction using atomic norm with unquantized measurements \( y_{UQ} = \{y_i, uQ \}_{i=1}^m \), by running the algorithm:
\[
\hat{x} = \arg\min_{x \in \mathbb{C}^n} \frac{1}{2} \| y_{UQ} - Ax \|^2 + \tau \| x \|_A,
\]
where \( A = [a_1, \ldots, a_m]^T \). Fig. 4 shows the normalized reconstruction error at different SNRs with comparisons to that using the QPSK quantized measurements. It can be seen that at low SNR, using quantized measurements can potentially achieve better reconstruction quality with much fewer measurement budgets in bits. Also it can be seen that improving the SNR before quantization does not have as strong impact as for the unquantized case.

### V. Conclusions

In this paper, we developed a novel algorithm based on atomic norm minimization for spectral compressed sensing.
using 1-bit quantized measurements. Under a mild separation condition, we establish that we can accurately recover a spectrally-sparse signal from the signs of $O(k \log n)$ random linear measurements. The approach also allows accurate frequency localization without knowing the model order. Our future work will examine other quantization schemes (e.g., sigma-delta quantization) and compare the performance of the proposed algorithm with estimation-theoretic lower bounds.

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