18-799F Algebraic Signal Processing Theory
Spring 2007
Solutions: Midterm

1. (a) True: every Euclidean domain is a principal ideal domain.
   (b) True: in an abelian group every element commutes with all others, so any subgroup is automatically normal.
   (c) $(\mathbb{F}^{n\times n}, \times)$, where $\mathbb{F}$ is a field.
   (d) $(\mathbb{Z}/n\mathbb{Z})^\times = \mathbb{Z}/n\mathbb{Z} \setminus \{0\}$ iff n is prime.
   (e) $(\mathbb{Z}, +, \times)$, $\mathbb{C}[x]$, $\mathbb{C}[x]/(x^n - 1)$.
   (f) Such function doesn't exist: since f is injective, $|S| = |\mathrm{im} f|$, so $\mathrm{im} f = S$, i.e. f is surjective.
   (g) $12\mathbb{Z} + 9\mathbb{Z} = \gcd(12, 9)\mathbb{Z} = 3\mathbb{Z}$.
   (h) $\mathbb{C} = \mathbb{R} + i\mathbb{R}$ is an $\mathbb{R}$-vector space of dimension 2.
   (i) If $A, B \in \mathbb{C}^{n\times n}$ represent the same linear mapping, then there exists nonsingular matrix $P \in \mathbb{C}^{n\times n}$, such that $A = PBP^{-1}$. (Moreover, A and B have the same determinant, spectrum, i.e. collection of eigenvalues, trace, and characteristic polynomial).
   (j) For field A, an A-module is an A-vector space.

2. ϕ is a homomorphism of algebras because for any $s(x) = \sum_{k=0}^{n} s_k x^k, t(x) = \sum_{j=0}^{m} t_j x^j \in \mathbb{R}[x]$ and $\alpha, \beta \in \mathbb{R}$:

   (i) $\phi(\alpha s(x) + \beta t(x)) = \alpha s_0 + \beta t_0 = \alpha\phi(s(x)) + \beta\phi(t(x))$;
   (ii) $\phi(s(x)t(x)) = s_0 t_0 = \phi(s(x))\phi(t(x))$.

   $\ker\phi = \{s(x) \in \mathbb{R}[x] \mid s_0 = 0\} = x\mathbb{R}[x]$. Since it is a kernel of a ring homorphism, $\ker\phi$ is an ideal in $\mathbb{R}[x]$.

   ψ is not a homomorphism since in general $\psi(s(x)t(x)) = s_0 t_1 + s_1 t_0 \neq s_1 + t_1 = \psi(s(x)) + \psi(t(x))$; e.g. $\psi(x(x-1)) = 1 \neq 1 + 1 = \psi(x) + \psi(x-1)$.

3. (a) Let $w_3 = e^{\frac{i2\pi}{3}}$. Then
$$\mathrm{DFT}_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & w_3 & w_3^2 \\ 1 & w_3^2 & w_3 \end{pmatrix}.$$
   (b) Let
$$\begin{array}{rrcl} \phi: & \mathbb{C}[x]/(x^3-1) & \to & \mathbb{C}[x]/(x-1) \oplus \mathbb{C}[x]/(x^2+x+1) \\ \psi: & \mathbb{C}[x]/(x^2+x+1) & \to & \mathbb{C}[x]/(x-w_3) \oplus \mathbb{C}[x]/(x-w_3^2) \end{array}$$
   Since the basis of $\mathbb{C}[x]/(x-1) \oplus \mathbb{C}[x]/(x^2+x+1)$ is $\{(1,0),(0,1),(0,x)\}$,
$$\begin{aligned} \phi(1) &= (1,1) \\ \phi(x) &= (1,x) \\ \phi(x^2) &= (1,-1-x) \end{aligned}$$
   and the corresponding matrix is
$$B_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$
   The basis of $\mathbb{C}[x]/(x-w_3) \oplus \mathbb{C}[x]/(x-w_3^2)$ is $\{(1,0),(0,1)\}$. Then
$$\begin{aligned} \psi(1) &= (1,1) \\ \psi(x) &= (w_3, w_3^2) \end{aligned}$$

and the corresponding matrix is

$$B_2 = I_1 \oplus \begin{pmatrix} 1 & w_3 \\ 1 & w_3^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & w_3 \\ 0 & 1 & w_3^2 \end{pmatrix}.$$

Since the roots are in the same order as in the CRT case, we don't need to introduce any permutation matrix. Hence, $\mathrm{DFT}_3 = B_2 B_1$, and

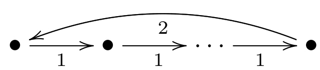
$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & w_3 & w_3^2 \\ 1 & w_3^2 & w_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & w_3 \\ 0 & 1 & w_3^2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

(c) (i) Computing $y = \mathrm{DFT}_3 z$ requires 4 (non-trivial) complex multiplications and 6 additions.
(ii) Computing $y = B_2 B_1 z = B_2(B_1 z)$ requires 4 additions for B_1, and 2 multiplications and 2 additions for B_2 - a total of 6 additions and 2 multiplications.

4. (a)

$$\phi(x) = \begin{pmatrix} 0 & \cdots & \cdots & 0 & 2 \\ 1 & \ddots & & \vdots & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & 0 \\ 0 & \cdots & 0 & 1 & 0 \end{pmatrix}.$$

The visualization is



(b) Let I_k denote an identity matrix of size $k \times k$. Since $\phi(x^k)$ is a block matrix of the structure

$$\phi(x^k) = \begin{pmatrix} 0 & 2I_k \\ I_{n-k} & 0 \end{pmatrix},$$

then for $h = \sum_{k=0}^{n-1} h_k x^k \in \mathcal{A}$:

$$\phi(h) = \begin{pmatrix} h_0 & 2h_{n-1} & \cdots & 2h_2 & 2h_1 \\ h_1 & \ddots & & \vdots & \vdots \\ h_2 & \ddots & \ddots & 2h_{n-1} & \vdots \\ \vdots & \ddots & \ddots & h_0 & 2h_{n-1} \\ h_{n-1} & \cdots & h_2 & h_1 & h_0 \end{pmatrix}$$

(c) Let $\alpha = \{\alpha_k = \sqrt[n]{2} e^{\frac{ki2\pi}{n}} = \sqrt[n]{2} w_n^k\}_{0 \le k < n}$ be the set of roots of polynomial $x^n - 2$. The spectral decomposition of $\mathcal{M}$ is

$$\begin{array}{rcl} \mathbb{C}[x]/(x^n - 2) & \to & \bigoplus_{k=0}^{n-1} \mathbb{C}[x]/(x - \alpha_k) \\ s(x) & \mapsto & (s(\alpha_0), \dots, s(\alpha_{n-1})) \end{array}$$

(d) $\mathcal{F} = \mathcal{P}_{b,\alpha} = [(\sqrt[n]{2} w_n^k)^l]_{0 \le k,l < n}$.

(e) Since all roots of $p(x) = x^n - 2$ are distinct, $\mathcal{F} = \mathcal{P}_{b,\alpha}$ completely diagonalizes $\phi(h)$.

(f) Since $\mathcal{F}\phi(h)\mathcal{F}^{-1} = \mathrm{diag}(h(\alpha_0), \dots, h(\alpha_{n-1}))$, so the eigenvalues of $\phi(h)$ are the same as those of the matrix on the right side of the equation, i.e. $h(\alpha_0), \dots, h(\alpha_{n-1})$. On the other hand, the collection $(h(\alpha_0), \dots, h(\alpha_{n-1}))$ is exactly the frequency response of h.

(g) $\mathcal{F} = \mathrm{DFT}_n \cdot \mathrm{diag}(\sqrt[n]{2}^l, 0 \le l < n))$.