Recap: - Spatial quincunx model

$\mathcal{A} = \mathcal{M} = \mathbb{C}[x,y]/\langle T_4(x), T_4(y)\rangle \qquad \mathcal{B} = \mathcal{N} = ?$

- $\mathcal{B}$ as subalgebra of $\mathcal{A}$: $\langle T_2(x), T_2(y), T_1(x)T_1(y)\rangle_{alg}$
- $\mathcal{B}$ in "canonical form:" $\mathbb{C}[u,v,w]/\langle T_{4/2}(u), T_{4/2}(v), 4w^2-(u+1)(v+1)\rangle$

Continuous signal models and sampling

Example: cont. inf. time

$\mathcal{A} = L^1(\mathbb{R})$, $\mathcal{M} = L^2(\mathbb{R})$ $\qquad L_p(\mathbb{R}) = \{s(t) \mid \int_{-\infty}^{\infty} |s(t)|^p dt < \infty\}$

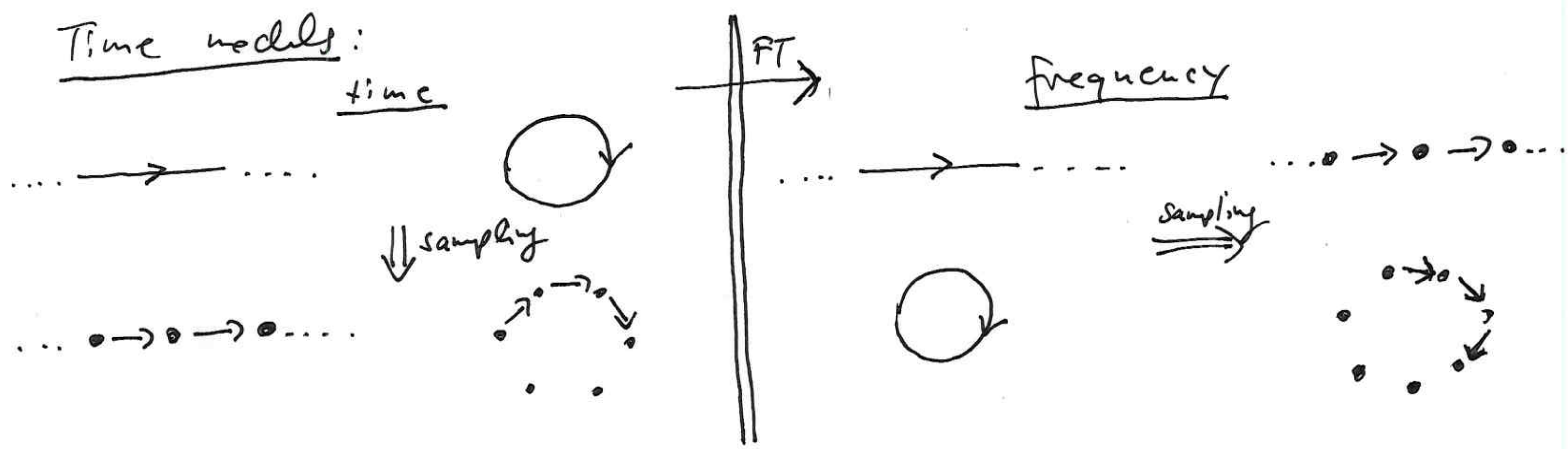
operation: $h(t) \in \mathcal{A}$, $s(t) \in \mathcal{M}$

$$h(t) * s(t) = \int_{-\infty}^{\infty} h(\tau)\, s(t-\tau)\, d\tau$$

$$\Phi: L^2(\mathbb{R}) \to \mathcal{M}$$
$$s(t) \mapsto s(t)$$

visualization: 

Time models:



Sampling: Goal: continuous signal model

→ discrete "

Procedure: $(\mathcal{A}, \mathcal{M}, \Phi)$ cont. model (sketch)

example inf. time

1.) select "shift" $x \in \mathcal{A}$ $\qquad x = \delta(1)$

select sampling point $s_0 = s(t_0)$ $\qquad s_0 = s(0)$

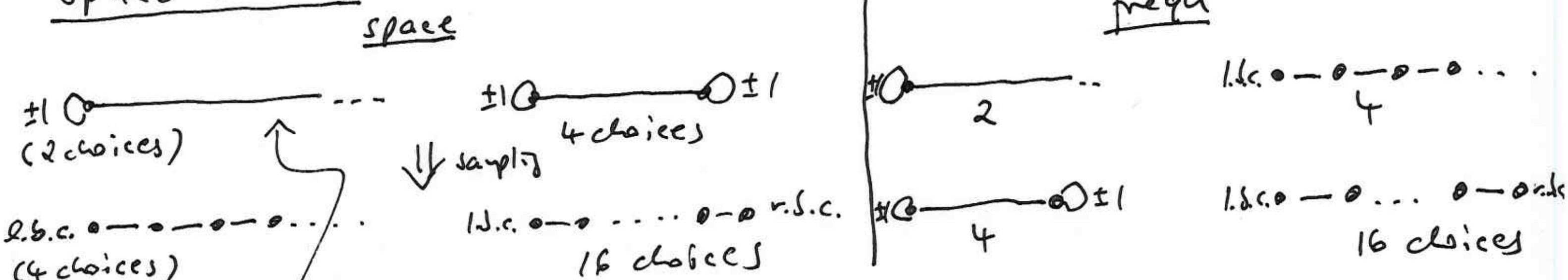
2.) $\mathcal{A}_s = \langle x \rangle_{alg}$ $\qquad \mathcal{A}_s = \langle x \rangle_{alg}$

~~$\mathcal{M}_s = \langle t_0 \rangle_{\mathcal{A}_s\text{-module}}$~~ $\qquad \mathcal{M}_s = \{ (s_n)_{n \in \mathbb{Z}} \}$

construct $\mathcal{M}_s$ so sampling points are closed under $\mathcal{A}_s$

3.) derive sampling theorem / Nyquist frequ.

Space models



$\mathcal{A} = L^1(\mathbb{N})$

$\mathcal{M} = L^2(\mathbb{N})$ + symm/asymm. b.c.'s

$h(t) * s(t) = \int_0^\infty h(\tau) \frac{1}{2} (s(t-\tau) + s(t+\tau)) d\tau$

Sampling

example

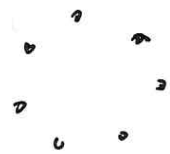
- assume $x = \delta(1)$ 

- Q: how many choices for initial sampling point?

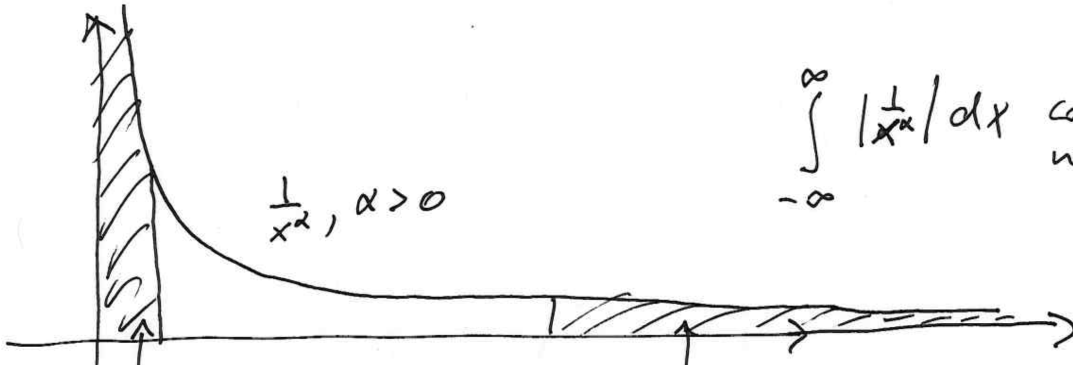
A: two: $t_0 = 0$, $t_0 = \frac{1}{2}$ $\Longrightarrow$ 4 choices

Similarly one should be able to define 2-D cont. signal models, sampling theorems, Nyquist, etc. ...

A word about L-spaces

	infinite	compact
cont.	$L^1 \not\subseteq L^2$, $L^1 \not\supseteq L^2$	 $L^2 \subseteq L^1$
discrete	$L^1 = \ell^1 \subseteq \ell^2 = L^2$	 $L^2 = L^1$

intuition:



$\int_{-\infty}^{\infty} \left|\frac{1}{x^\alpha}\right| dx$ converges nowhere

converges for $\alpha < 1$ (small p better)

converges for $\alpha > 1$ (high p better) = less restrictive

$\underbrace{}_{\text{problem disappears for discrete}}$ $\underbrace{}_{\text{problem disappears for compact}}$

Duality

- we visualized the spectrum above without signal model
- e.g. vis. of finite time = vis. of its spectrum

suggests: $DFT \approx DFT^{-1}$ ($\approx$ = closely related to)

Intuition: example fin. time

$$\mathbb{C}[x]/(x^n-1) \underset{k}{\cong} \bigoplus \mathbb{C}[x]/x-\omega_n^k$$

spectrum:

$$\left.\begin{array}{l} \text{- take } \omega_n \text{ as shift (generates an alg)} \\ \text{- take } \omega_n^0, \dots, \omega_n^{n-1} \text{ as basis (spans all)} \\ \quad \omega_n \circ \omega_n^k = \omega_n^{k+1 \bmod n} \to \text{directed circle} \end{array}\right\} \text{dual model}$$

now space:

$$\mathbb{C}[x]/T_n(x) \cong \bigoplus_{k} \mathbb{C}[x]/x - \cos\frac{k+\frac{1}{2}}{n}\pi \qquad (\leftrightarrow \text{DCT-3})$$

spectrum: – take $c_1 = \cos\frac{\pi}{n}$ as shift

– take $\underbrace{\cos\frac{\frac{1}{2}}{n}\pi}_{= c_0}, \dots, \underbrace{\cos\frac{n-\frac{1}{2}}{n}\pi}_{= c_{n-1}}$ as basis

$$c_1 \cdot c_k = \frac{1}{2}(c_{k-1} + c_{k+1}) \qquad \text{(space structure)}$$

$$c_{-1} = \cos\frac{-\frac{1}{2}}{n}\pi = c_0$$

$$c_n = \cos\frac{n+\frac{1}{2}}{n}\pi = c_{n-1}$$

$\rightarrow$ $C_0 — \circ — \circ \dots \circ — \circ$ $\quad (\leftrightarrow \text{DCT-2})$

$\Rightarrow$ $\text{DCT-3}^{-1} \approx \text{DCT-2}$

Downsampling

inf. time: $(s_n)_{n\in\mathbb{Z}} \longrightarrow (s_{2n})_{n\in\mathbb{Z}}$

in $\mathcal{M}$: $s = \sum s_n z^{-n}$ $\longrightarrow$ $\sum s_{2n} z^{-n}$ (every other omitted)

$\cong$

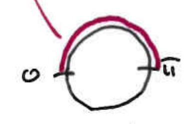
$\longrightarrow$ $\sum s_{2n} z^{-2n}$ (every other sample zero)

spec: $\sum s_n e^{-i\ell n}$

fct. of $\ell \in [0, 2\pi)$:

spec: $\sum s_{2n} e^{-i\ell n}$

spec: $\sum s_{2n} e^{-i\ell/2 \cdot n}$



+ period π

downsampling (sketch): $(\mathcal{A}, \mathcal{M}, \Phi)$ given

1.) pick subalgebra $\mathcal{B} \leq \mathcal{A}$

pick $\mathcal{B}$-module $\mathcal{N} \leq \mathcal{M}$

(if $\mathcal{A} = \mathcal{M}$, $\mathcal{N} = \mathcal{B}$ will do)

2.) downsampling = projection $\mathcal{M} \rightarrow \mathcal{N}$

3.) derive ~~aliasing~~ aliasing etc...

example inf. time

$\mathcal{B} = \{\sum h_{2n} x^{2n}\} = \langle x^2 \rangle$

$\mathcal{N} = \{\sum s_{2n} x^{2n}\}$

$\mathcal{M} = \mathcal{N} \oplus x\mathcal{N}$

defines a projection

polyphase decomposition

(= "induction")

downsampling in finite models

general case:

$\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/p(x)$, subalgebras ?

Theorem: The subalgebras $\mathcal{B}$ of $\mathcal{A}$ are of the form $\mathcal{B} = \langle r(x) \rangle_{alg}$, $r(x) \in \mathcal{A}$. $\mathcal{B} \subsetneq \mathcal{A}$ iff $|\{r(\alpha_k)\}| = m < n$.
It is $\dim \mathcal{B} = m$.

intuition:
$$\Delta : \mathbb{C}[x]/p(x) \to \bigoplus_k \mathbb{C}[x]/x-\alpha_k$$
$$\underbrace{q(r(x))}_{\text{generic element in } \mathcal{B}} \mapsto \underbrace{\left(q(r(\alpha_k))\right)_{0 \le k < n}}_{\text{only } m \text{ different values} \to \text{ aliasing}}$$

Example time:

$\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/x^8-1$, $\alpha_k = \omega_8^k$, $\mathcal{B} = \langle x^2 \rangle_{alg} = \langle 1, x^2, x^4, x^6 \rangle_{vs} = \mathcal{N}$

$$\Delta : q(x^2) \mapsto (\underbrace{q(\omega_8), q(\omega_8^2), \ldots}_{\text{first half}} = \underbrace{\ldots, q(\omega_8^{14})}_{\text{second half}})$$

polyphase dec: $\mathcal{M} = \mathcal{N} \oplus x\mathcal{N}$

many other $r(x)$ are possible. Try $r(x) = \frac{x + x^{n-1}}{2} = \frac{x + x^{-1}}{2}$

$$\mathcal{B} = \langle \tfrac{x+x^{-1}}{2} \rangle_{alg} = \langle 1, \underset{y}{\tfrac{x+x^{-1}}{2}}, \underset{T_2(y)}{\tfrac{x^2+x^{-2}}{2}}, \underset{T_3(y)}{\tfrac{x^3+x^{-3}}{2}}, \underset{T_4(y)}{\tfrac{x^4+x^{-4}}{2}} \rangle_{vs} = \mathcal{N}$$

x^4

polyphase dec: $\mathcal{M} = \underbrace{\mathcal{N}}_{\dim = 5} \oplus \underbrace{\tfrac{x-x^{-1}}{2}\mathcal{N}}_{\dim = 3}$

Subalgebras by decomposition

$\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/p(x)\,, \quad p(x) = q(r(x))$

degs: $\quad n \quad k \quad m \quad , \quad n = km$

$\Rightarrow\ \mathcal{B} = \langle r(x) \rangle_{alg} \leq \mathcal{A}\,, \quad \mathcal{B} \cong \mathbb{C}[y]/q(y)\,, \quad \dim \mathcal{B} = k$

$r(x)$ maps n zeros of $p \rightarrow k$ zeros of q

First step in Cooley-Tukey: $\quad q_0 \cdots q_{k-1}$ basis of $\mathbb{C}[y]/q(y)$

base change $b \longrightarrow b'$

$$b' = \begin{pmatrix} r_0(x)\, q_0(r(x)), & \cdots & , r_{m-1}(x)\, q_0(r(x)) \\ r_0(x)\, q_1(r(x)), & \cdots & , r_{m-1}(x)\, q_1(r(x)) \\ \cdots & \cdots & \cdots \\ \underbrace{r_0(x)\, q_{m-1}(r(x))}_{\substack{\text{spans} \\ r_0\mathcal{B} = \mathcal{B}}}, & \cdots & , \underbrace{r_{m-1}(x)\, q_{m-1}(r(x))}_{\substack{\text{spans} \\ r_{m-1}\mathcal{B}}} \end{pmatrix}$$

assume $r_0 = 1$

polyphase dec: $\mathcal{A} \cong r_0\mathcal{B} \oplus \ldots \oplus r_{m-1}\mathcal{B}$

All the above is compatible with arbitrary finite, shift-invariant signal models: 1-D, 2-D, etc.