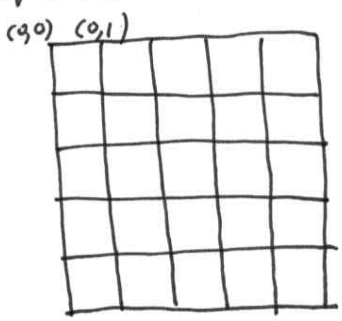
Recap: Lecture 27

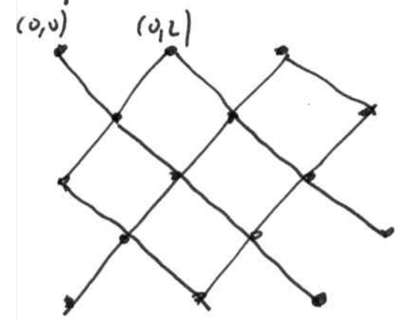
- tensor product of vector spaces, algebras, signal models
- dimensions multiply, signal values are naturally placed on an array
- square arrays
- separable: $\mathbb{C}[x,y]/\langle p(x), q(y)\rangle$

## Spatial Quincunx model (nonseparable)

separable model




quincunx model




$\mathcal{A} = \mathcal{M} = \mathbb{C}[x,y]/\langle T_n(x), T_n(y)\rangle$

$b = \{T_i(x)T_j(x) \mid 0 \le i,j < n\}$

$\dim = n^2$

$\mathcal{B} = \mathcal{N} = \langle T_i(x)T_j(x) \mid 2 \mid i+j\rangle_{VS}$

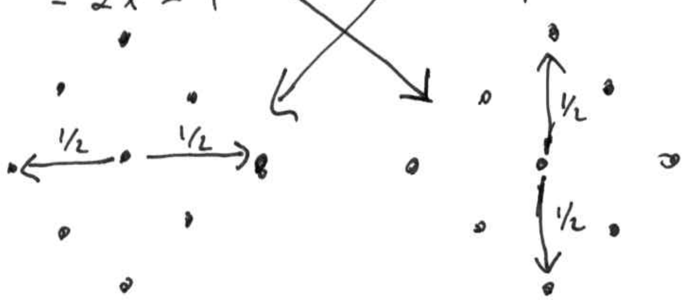
$\dim = \frac{n^2}{2}$

last time: $\mathcal{B}$ is an algebra = closed under mult

## Generators and Relations

$u = T_2(x) = 2x^2 - 1$

$v = T_2(y) = 2y^2 - 1$




$0 = T_n(x) = T_{n/2}(T_2(x)) = T_{n/2}(u)$

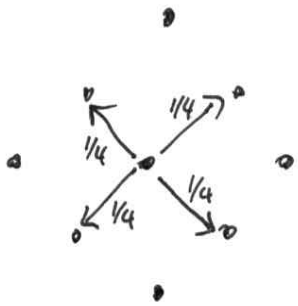
$\rightsquigarrow \mathcal{B}' = \mathbb{C}[u,v]/\langle T_{n/2}(u), T_{n/2}(v)\rangle$

$\dim = \frac{n^2}{4}$

not enough ?

$w = T_1(x) T_1(y) \in \mathcal{B}'$ $\qquad (u+1)(v+1) = 4w^2$

$= xy$




$$\longrightarrow \mathcal{B} = \mathbb{C}[u,v,w] \Big/ \langle T_{n/2}(u), T_{n/2}(v), 4w^2 - (u+1)(v+1) \rangle$$

filter algebra and
signal module
[illegible]

<u>Basis:</u>

Task: express $T_i(x) T_j(y)$, $2 \mid i+j$, in $u, v, w$

Case 1: $i$ and $j$ even, $i = 2i'$, $j = 2j'$

$$\Rightarrow T_i(x) = T_{i'}(u), \quad T_j(y) = T_{j'}(v)$$

$$T_i(x) T_j(x) = T_{i'}(u) T_{j'}(v) \qquad 0 \le i', j' < \tfrac{n}{2} \qquad \tfrac{n^2}{4} \text{ many}$$

Case 2: $i$ and $j$ odd, $i = 2i'+1$, $j = 2j'+1$

[we will use $T_{n+1} + T_n = (x+1) V_n$]

$$4 \underbrace{T_1(x) T_1(y)}_{w} \left(T_{2i'+1}(x) T_{2j'+1}(y)\right) = \left(T_{2i'}(x) + T_{2i'+2}(x)\right)\left(T_{2j'}(y) + T_{2j'+2}(y)\right)$$

$$= \left(T_{i'}(u) + T_{i'+1}(u)\right)\left(T_{j'}(v) + T_{j'+1}(v)\right)$$

$$= (u+1) V_{i'}(u) (v+1) V_{j'}(v)$$

$$= 4w^2 V_{i'}(u) V_{j'}(u)$$

$$\Longrightarrow T_i(x) T_j(x) = w V_{i'}(u) V_{j'}(u)$$

## Signal Model

$$\mathcal{A} = \mathcal{M} = \mathbb{C}[u,v,w] \big/ \langle T_{n/2}(u),\ T_{n/2}(v),\ 4w^2-(u+1)(v+1) \rangle$$

$$\begin{aligned}
b = \{\ & T_0(u)\,T_0(v), \ \ldots,\ T_0(u)\,T_{n/2-1}(v), && \text{1. row}\\
& w\,V_0(u)\,V_0(v), \ \ldots,\ w\,V_0(u)\,V_{n/2-1}(v), && \text{2. row}\\
& \ldots\ldots \\
& T_{n/2-1}(u)\,T_0(v), \ \ldots,\ T_{n/2-1}(u), T_{n/2-1}(v), \\
& w\,V_{n/2-1}(u)\,V_0(v), \ \ldots,\ w\,V_{n/2-1}(u)\,V_{n/2-1}(v)\ \}
\end{aligned}$$

$n$ rows of $n/2$ elements each.

## Spectrum and FT

Task: solve $T_{n/2}(u) = T_{n/2}(v) = 4w^2-(u+1)(v+1) = 0$

$$u_k = \cos\frac{k+\frac{1}{2}}{n/2}\pi, \qquad 0 \le k < \frac{n}{2}$$

$$v_l = \cos\frac{l+\frac{1}{2}}{n/2}\pi, \qquad 0 \le l < \frac{n}{2}$$

$$w_{k,l,\pm} = \pm\frac{1}{2}\sqrt{(u_k+1)(v_l+1)}$$

$\Rightarrow$ zeros: $\{(u_k, v_l, w_{k,l,\pm}) \mid 0 \le k,l < \frac{n}{2},\ \pm \in \{+,-\}\}$

$n^2/2$ many ✓

simplify $w_{k,l,\pm}$:

use $1+\cos(x) = \left(\cos\left(\frac{\pi}{4}-\frac{x}{2}\right) + \sin\left(\frac{\pi}{4}-\frac{x}{2}\right)\right)^2$

$$\cancel{\Rightarrow\ w_{k,l,\pm} = \pm\frac{1}{2}(\cos\gamma_k + \sin\gamma_k)(\cos\gamma_l + \sin\gamma_l)}$$

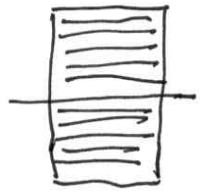
$$\cancel{\gamma_n = \frac{\pi}{4} - \cos\frac{n+\frac{1}{2}}{n}\pi}$$

Better: use $1+\cos(x) = 2\cos^2\left(\frac{x}{2}\right)$

$$\Rightarrow\ w_{k,l,\pm} = \pm\cos\frac{k+\frac{1}{2}}{n}\pi\,\cos\frac{l+\frac{1}{2}}{n}\pi$$

$$\Delta : \mathbb{C}[u,v,w] / \langle \ldots \rangle \longrightarrow \bigoplus_{k,l,\pm} \mathbb{C}[u,v,w] / \langle (u-u_k), (v-v_l), (w-w_{k,l,\pm}) \rangle$$

$$s(u,v,w) \longmapsto \left( s(u_k, v_l, w_{k,l,\pm}) \right)_{k,l,\pm}$$

$\mathcal{F}$: evaluate basis  at zeros

Case 1: even rows $T_i(u)T_j(v) \rightarrow DCT3_{n/2} \otimes DCT3_{n/2}$

Case 2: odd rows $w V_i(u) V_j(v) \rightarrow D_{n^2/4}\left(\overline{DCT4}_{n/2} \otimes \overline{DCT4}_{n/2}\right)$

Result: $\hat{s} =$ 


- scan rowwise
- start top left

zeros: ordered as $k, l, \pm$ ↖ runs fastest

$$DQT_{n \times \frac{n}{2}} =$$

$$= \underset{\text{permutation}}{L^{n^2/2}_{n^2/4}} \begin{pmatrix} DCT3_{n/2} \otimes DCT3_{n/2} & D_{n^2/4}\left(\overline{DCT4}_{n/2} \otimes \overline{DCT4}_{n/2}\right) \\ '' & -D_{n^2/4}\left( \quad '' \quad \right) \end{pmatrix} \underset{\text{permutation}}{(L^n_2 \otimes I_{n/2})}$$

$$= L^{n^2/2}_{n^2/4} \left( DFT_2 \otimes I_{n^2/4} \right) \left( \left( DCT3_{n/2} \otimes DCT3_{n/2} \right) \oplus D_{n^2/4}\left( \overline{DCT4}_{n/2} \otimes \overline{DCT4}_{n/2} \right) \right) \cdot \left( L^n_2 \otimes I_{n/2} \right)$$

<u>Complexity</u>: $n$ a 2-power

- if $T \in \mathbb{C}^{n \times n}$ needs $\pi(n)$ ops, then $T \otimes T$ needs $2n\,\pi(n)$ ops
- $DCT3_n$ can be done using $2n\log_2(n) - n + 1$ ops
- $\overline{DCT4}_n$ " $2n\log_2(n)$ ops

yields:

<u>Lemma</u>: $DCT_{n \times \frac{n}{2}}$ can be done using

$$2n^2 \log_2(n) - n^2 + n \quad \text{ops.}$$

Better is possible:

- row-column for $T \otimes T$ is almost always suboptimal
- there are (slightly) better algorithms for DCT3.