recap: - 2-D finite, shift-invariant, regular signal models

- $\mathbb{C}[x,y]/\langle p(x,y), q(x,y)\rangle$ vs. $\mathbb{C}[x]/p(x)$

- running example $\mathbb{C}[x,y]/\langle x^n-1, y^n-1\rangle \Longleftrightarrow$ 2-D DFT $= \text{DFT} \otimes \text{DFT}$

- tensor product $\leftrightarrow$ separable 2-D SP (more in this lecture)

Tensor product of vector spaces

<u>Definition</u>: Let V, W be vector spaces. Then

$$V \otimes W = \langle (v,w) \mid v \in V, w \in W \rangle_{vs}$$

by obeying the equations

a.) $(v+v', w) = (v,w) + (v',w)$

b.) $(v, w+w') = (v,w) + (v,w')$

c.) $(\alpha v, w) = (v, \alpha w) = \alpha(v,w)$, $\alpha \in \mathbb{C}$

More formally:

$$V \otimes W = \langle V \times W \rangle / \langle (v+v',w)-(v,w)-(v',w), (v,w+w')-(v,w)-(v,w'), (\alpha v,w)-(v,\alpha w)-\alpha(v,w) \rangle_{vs}$$

The equ. class of (v,w) is written as $v \otimes w$. But not all elements are of this form.

For example $0 = \{(v,w) \mid v=0 \text{ or } w=0\}$.

<u>Lemma</u>: Let $b = \{b_0, \dots, b_{n-1}\}$, $c = \{c_0, \dots, c_{m-1}\}$ be bases of V and W, respectively. Then $\{b_i \otimes c_j \mid 0 \le i < n, 0 \le j < m\}$ is a basis of $V \otimes W$. In particular, $\dim(V \otimes W) = \dim V \cdot \dim W$.

<u>Proof</u>:

a.) generating set: generic element in $V \otimes W$:

$$\sum_{i=0}^{N} \alpha_i (v_i \otimes w_i)$$

$$= \sum_{i=0}^{N} \alpha_i \left(\sum_{k=0}^{n-1} \beta_{i,k} b_k \otimes \sum_{l=0}^{m-1} \gamma_{i,l} c_l \right)$$

$$\overset{a.)-c.)}{=} \sum_i \sum_k \sum_l \alpha_i \beta_{i,k} \gamma_{i,l} (b_k \otimes c_l) \quad \checkmark$$

b.) linear independent: omitted.

<u>Notes:</u>

- think of $v \otimes w$ as formal product vw because of a.) – c.)
- not every element in $V \otimes W$ has the form $v \otimes w$.
  E.g. $b_0 \otimes c_0 + b_1 \otimes c_1$ is not.

	$V \oplus W$	$V \otimes W$
definition	$\{(v,w) \mid v \in V, w \in W\}$ + comp.wise operation	<u>generated</u> by $\{(v,w)\}$ + a.) – c.)
basis	$\{(b_i, 0)\} \cup \{(0, c_j)\}$	$\{b_i \otimes c_j\}$
dimension	$\dim V + \dim W$	$\dim V \cdot \dim W$

<u>Example:</u> $\mathbb{C}[x]/_{x^n - 1} \otimes \mathbb{C}[y]/_{y^n - 1} \cong \mathbb{C}[x,y]/_{\langle x^n - 1, y^n - 1 \rangle}$

bases: $\{1, \dots, x^{n-1}\}$ $\quad \{1, \dots, y^{n-1}\}$ $\quad \{x^i y^j \mid 0 \le i,j < n\}$

define: $\varphi : r(x) \otimes s(y) \longrightarrow r(x) s(y)$

and by linear extension to all elements:

$$\varphi\left(\sum \alpha_i (r_i \otimes s_i) \right) = \sum \alpha_i \varphi(r_i \otimes s_i)$$

Claim: φ is bijective linear mapping a.) well-defined:

$$\varphi((r+r')\otimes s - r\otimes s - r'\otimes s) = (r+r')s - rs - r's = 0$$

same for other two relations

b.) linear: by definition

c.) surjective: yes, all basis elements can be obtained

$$\varphi(x^i \otimes y^j) = x^i y^j$$

d.) injective:
$$0 = \varphi\Big(\sum_{0\le i,j<n} \alpha_{i,j}(x^i \otimes y^j)\Big)$$
$$= \sum_{0\le i,j<n} \alpha_{i,j} x^i y^j \quad \Rightarrow \text{ all } \alpha_{i,j} = 0$$

Tensor product of algebras

Algebras are vector spaces, so defined as above.

Multiplication:

$$\sum_i \alpha_i (b_i \otimes c_i) \cdot \sum_j \beta_j (b_j \otimes c_j) = ?$$

- distributivity law
- $(b_i \otimes c_i) \cdot (b_j \otimes c_j) = \underbrace{b_i b_j}_{} \otimes \underbrace{c_i c_j}_{}$ defined in resp. algebras

Tensor product of signal models

Definition? Let $(\mathcal{A}, \mathcal{M}, \Phi)$, $\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/p(x)$,

$\Phi: \hat{s} \mapsto \sum_{k=0}^{n-1} s_k p_k$ be a 1-D signal model.

The associated 2-D separable model is given by

$(\mathcal{A}' = \mathcal{A} \otimes \mathcal{A}, \mathcal{M}' = \mathcal{M} \otimes \mathcal{M}, \Phi' = \Phi \otimes \Phi)$ with

$$\Phi': \mathbb{C}^{n \times n} \longrightarrow \mathcal{M} \otimes \mathcal{M}$$
$$\hat{s} = (s_{i,j}) \longmapsto \sum_{0 \le i,j < n} s_{i,j}\, p_i(x) p_j(y)$$

Basic concepts

a.) shift matrices:

$$\varphi'(x) = I_n \otimes \varphi(x)$$

$$\varphi'(y) = \varphi(y) \otimes I_n$$

note, $\varphi(x) = \varphi(y)$

visualization: $\varphi'(x) + \varphi'(y)$

b in $\mathcal{M} \otimes \mathcal{M}$:

$$\{p_0(x)p_0(y), \dots, p_{n-1}(x)p_0(y),$$
$$\dots$$
$$p_0(x)p_{n-1}(y), \dots, p_{n-1}(x)p_{n-1}(y)\}$$

b.) filter matrices:

$$\varphi'\Big(\sum_{i,j} h_{i,j}\, p_i(x)\, p_j(y)\Big)$$

$$= \sum_{i,j} h_{i,j}\Big(p_i(\varphi(x)) \otimes p_j(\varphi(y))\Big)$$

<u>note</u>: basis in $\mathcal{A}$ not always = basis in $\mathcal{M}$

c.) spectrum and FT: (zeros of $p(x)$: $\alpha = (\alpha_0, \dots, \alpha_{n-1})$)

zeros of $p(x) = p(y) = 0$:

$$(\alpha_k, \alpha_\ell)_{0 \le k, \ell < n}$$

$$\Delta: \mathbb{C}[x,y]/\langle p(x), p(y)\rangle \to \bigoplus_{0 \le k,\ell < n} \mathbb{C}[x,y]/\langle x-\alpha_k, x-\alpha_\ell\rangle$$

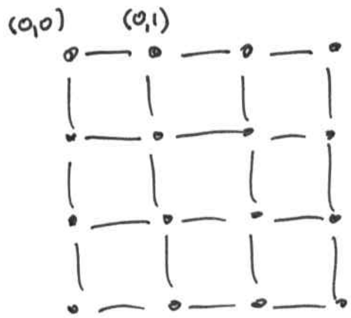
$$s(x,y) \mapsto (s(\alpha_k, \alpha_\ell))_{0 \le k,\ell < n}$$

$$\mathcal{F}' = \big[p_i(\alpha_k)\, p_j(\alpha_\ell)\big]_{(i,j),(k,\ell)} \in \mathbb{C}^{n^2 \times n^2}$$

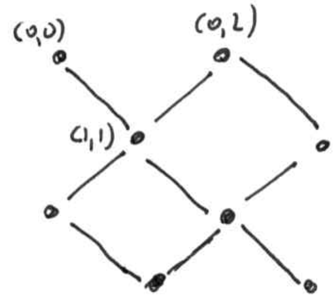
$$= \mathcal{F} \otimes \mathcal{F}$$

Finite Spatial Quincunx Model (2-D, nonseparable)

Signal model for 16 2-D DCTs and DSTs (without b.c.'s)



Associated quincunx lattice (every other point omitted)



e.g., for DCT-3 ⊗ DCT-3

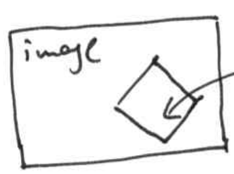
$\mathcal{A} = \mathcal{M} = \mathbb{C}[x,y]/\langle T_n(x), T_n(y)\rangle$

$\Phi: \hat{s} \mapsto \sum s_{i,j} T_i(x) T_j(y)$

signal model? (spectrum, FT, ... ?)

How does quincunx arise?

- downsampling
- or



has quincunx structure

Constructing the signal model

Idea: - start with standard spatial lattice and consider the special case

$$\mathcal{A} = \mathcal{M} = \mathbb{C}[x,y]/\langle T_n(x), T_n(y)\rangle$$

with T-basis: $T_i(x) T_j(y)$

- construct quincunx model by constructing a subalgebra of $\mathcal{A}$.

quincunx lattice points: $T_i(x)\,T_j(y)$, $i+j \equiv 0 \mod 2$

question: $\mathcal{B} = \langle T_i(x)\,T_j(y) \mid i+j \equiv 0 \mod 2 \rangle_{VS}$ an algebra?

$$T_k(x)T_\ell(y) \cdot T_i(x)\,T_j(y)$$

$$= \frac{1}{4}\left(T_{i-k}(x) + T_{i+k}(x)\right)\left(T_{j-\ell}(y) + T_{j+\ell}(y)\right)$$

$$= \frac{1}{4}\left(T_{i-k}(x)\,T_{j-\ell}(y) + \ldots\ldots\right)$$

$i-k+j-\ell = (i+j) - (k+\ell) \equiv 0 \mod 2$ ✓

but what if boundary is exceeded?

b.c.'s: $T_{-k} = T_k$ $\qquad -k \equiv k \mod 2$ ✓

$\qquad T_{n+k} = -T_{n-k}$ $\qquad n+k \equiv n-k \mod 2$ ✓

$\Longrightarrow$ $\mathcal{B}$ an algebra, $\dim \mathcal{B} = \frac{n^2}{2}$.