- 2-D finite, shift-invariant, regular signal models
  \[ \langle x(y), q(x, y) \rangle \quad \text{vs.} \quad \langle r(x), p(x) \rangle \]
- running example \[ \langle x(y_1), y(y_1) \rangle \implies \text{2-D DFT} \]
- tensor product \( \Rightarrow \) separable 2-D I/I
  (more in this lecture)

**Tensor product of vector spaces**

**Definition:** Let \( V, W \) be vector spaces. Then

\[ V \otimes W = \{ (v, w) \mid v \in V, w \in W \} \]

By obeying the equations

a.) \( (v + v', w) = (v, w) + (v', w) \)

b.) \( (v, w + w') = (v, w) + (v, w') \)

c.) \( (\alpha v, w) = (v, \alpha w) = \alpha (v, w), \ \alpha \in \mathbb{C} \)

Prove formally:

\[ V \otimes W = \langle V \times W \rangle \]

\[ \langle (v + v', w) - (v, w) - (v', w), (v, w + w') - (v, w) - (v, w'), \\ (\alpha v, w) - (v, \alpha w) - \alpha (v, w) \rangle \]

The eqn. class of \((v, w)\) is written as \( \mathbb{E}v \otimes w \). But not all elements are of this form.

For example \( \mathbb{O} = \{ (v, w) \mid v = 0 \text{ or } w = 0 \} \).

**Lemma:** Let \( \mathbb{E}v_1, \ldots, v_n, \mathbb{E}w_1, \ldots, w_m \) be bases of \( V \) and \( W \), respectively. Then \( \{ \mathbb{E}v_i \otimes w_j \mid 0 \leq i \leq n, 0 \leq j \leq m \} \) is a basis of \( V \otimes W \). In particular, \( \dim (V \otimes W) = \dim V \cdot \dim W \).

**Proof:**

a.) generating set: generic element in \( V \otimes W \):

\[ \sum_{i=0}^{N} \alpha_i (v_i \otimes w_i) \]
\[ \sum_{i=0}^{n} \alpha_i (\sum_{k=0}^{m} \beta_k \xi_k \otimes \sum_{l=0}^{p} \gamma_l \zeta_l) \]

a), c)
\[ \sum_{i} \sum_{\xi} \alpha_i \beta_i \mu_i \nu_i \zeta_i \epsilon_i (d_i \otimes c_i) \]

6.) Linear independent: omitted.

Notes:
- Think of \( v \otimes w \) as formal product \( vw \)
  because of a)-c.)
- Not every element in \( V \otimes W \) has the form \( v \otimes w \).
  E.g., \( 60 \otimes c_0 + 5 \otimes c_1 \) is not.

<table>
<thead>
<tr>
<th>( V \oplus W )</th>
<th>( V \otimes W )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Definition</strong></td>
<td>( {(v,w)</td>
</tr>
<tr>
<td><strong>Basis</strong></td>
<td>( {b_i \otimes v_i } \cup {0, c_j } )</td>
</tr>
<tr>
<td><strong>Dimension</strong></td>
<td>( \dim V + \dim W )</td>
</tr>
</tbody>
</table>

Example:
\[ \mathbb{C}^2 x \mathbb{C}^2 \oplus \mathbb{C}^2 y \mathbb{C}^2 / y_i^2 - 1 \cong \mathbb{C}^2 x \mathbb{C}^2 / \langle x_1, x_2, y_1 \rangle \]

Bases:
- \( \{1, \ldots, x_i, y_i \} \)
- \( \{1, \ldots, y_i \} \)
- \( \{x_2, \ldots, x_i, y_2, \ldots, y_i \} \)

Define:
\( \Psi: v(x) \otimes sc(y) \rightarrow v(x)sc(y) \)
and by linear extension to all elements:
\( \Psi \left( \sum_{i} (v_i \otimes sc_i) \right) = \sum_{i} \Psi(v_i \otimes sc_i) \)

Claim: \( \Psi \) is bijective linear mapping.
a.) well-defined:
\[ \Psi((v + v') \otimes s - v \otimes s - v' \otimes s) = (v + v')s - vs - v's = 0 \]

same for other two relations

b.) linear: by definition

c.) surjective: yes, all basis elements can be obtained
\[ \Psi(x^i \otimes y^j) = x^i y^j \]
d.) injective: \[ 0 = \Psi(\sum_{0 < i,j} x^i y^j) = x^i y^j \]

Tensor product of algebras

Algebras are vector spaces, so defined as above.

Multiplication:
\[ \sum_i \alpha^i \cdot (b_i \otimes c_i) \cdot \sum_j \beta^j \cdot (b_j \otimes c_j) = ? \]
- distribution law
- \( (b_i \otimes c_i) \cdot (b_j \otimes c_j) = b_i b_j \otimes c_i c_j \)

Tensor product of signal models

Definition? Let \((\varphi, \mu, \Phi)\), \(\varphi = \mu = \text{Cir} \n / \rho(k)\), \(\Phi, \sum_{k=0}^{m-1} \Phi_k \mu_k \) be a 1-D signal model.
The associated 2-D separable model is given by
\((\varphi', \mu', \Phi') = \varphi \otimes \Phi\) with
\[ \Phi' : C^{nr} \rightarrow \mu' \otimes \mu' \]
\[ \hat{S} = (s_{ij}) \rightarrow \sum_{0 \leq i,j < n} s_{ij} p_i(t)p_j(y) \]
Basic concepts

a.) shift matrices:
\[ \mathbf{\psi}'(x) = I_n \otimes \mathbf{\psi}(x) \]
\[ \mathbf{\psi}'(y) = \mathbf{\psi}(y) \otimes I_n \]

Note: \( \mathbf{\psi}(x) = \mathbf{\psi}(y) \)

Visualization: \( \mathbf{\psi}'(x) + \mathbf{\psi}'(y) \)

b.) filter matrices:
\[ \mathbf{\psi}'(\sum_{i,j} h_{ij} \otimes p_i(x) p_j(y)) \]
\[ = \sum_{i,j} h_{ij}(p_i(\mathbf{\psi}(x)) \otimes p_j(\mathbf{\psi}(y))) \]

Note: basis in \( \mathbf{U} \) not always dense in \( \mathbf{U} \)

c.) spectrum and FT:
\( \text{zeros of } \mu(x) = 0 \Rightarrow (a_0, \ldots, a_m) \)
\( \text{zeros of } \mu(y) = 0 \Rightarrow (b_0, \ldots, b_n) \)

\[ \mathbf{D} : \mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}^n \times \mathbb{C}^n \]
\[ \langle \mu(x), \nu(y) \rangle \to \langle x-a_0, x-b_0 \rangle \]
\[ S(x, y) \mapsto (S(x-a_0, x-b_0))_{0 \leq k, l < n} \]

\[ \mathbf{F}': \left[ p_i(\delta(x) \otimes p_j(\delta(y)) \right]_{i,j} \in \mathbb{C}^{n \times n \times n} \]
\[ = \mathbf{F} \otimes \mathbb{F} \]
Finite Spatial Quincuax Model (2-D, nonseparable)

Signal model for 16
2-D DCTs and DSTs
(without b.c.'s)

\( \begin{array}{cc}
(q_0,0) & (q_0,1) \\
(0,q_1) & \cdot & \cdot & \cdot \\
(1,q_1) & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot \\
\end{array} \)

\( \Phi : \sum \sum S_{ij} T_i(x) T_j(y) \)

\( \Phi \) for DCT-3 \& DST-3

\( \mathbf{c} \mathbf{u} = \mathbf{c} \mathbf{U} = \mathbf{C} \mathbf{R} \mathbf{J} / \langle T_i(x), T_j(y) \rangle \)

Associated quincuax lattice (every other point omitted)

Signal model 2
(spectrum, FT, \ldots ?)

How does quincuax arise?
- down sampling
- or

Constructing the signal model

Idea: start with standard spatial lattice and control the special case

\( \mathbf{c} \mathbf{u} = \mathbf{c} \mathbf{U} = \mathbf{C} \mathbf{R} \mathbf{J} / \langle T_i(x), T_j(y) \rangle \)

with \( T \)-basis: \( T_i(x) T_j(y) \)

- construct quincuax model by constructing a subalgebra of \( \mathbf{c} \mathbf{U} \).
question: \( \mathcal{B} \subset \langle T_i(x), T_j(y) \mid i+j \equiv 0 \mod 2 \rangle \) an algebra?

\[
T_k(x)T_e(y) + T_i(x) T_j(y)
= \frac{1}{4} (T_{i-k}(x) + T_{i+k}(x)) (T_{j-e}(y) + T_{j+e}(y))
= \frac{1}{4} (T_{i-k}(x) T_{j-e}(y) + \ldots)
\]

\( i-k+j-e = (i+j) - (k+e) \equiv 0 \mod 2 \) 

but what if boundary is exceeded?

5.c.'s: \( T_{-k} = T_k \)

\[ -k \equiv k \mod 2 \] \( \checkmark \)

\[ T_{-k} = -T_{-k} \]

\[ n \equiv n \mod 2 \] \( \checkmark \)

\[ \Rightarrow \] \( \mathcal{B} \) an algebra, \( \dim \mathcal{B} = \frac{n^2}{2} \)