Construction of a signal model :
(time, space, GNN)

(1) Introduce time/space marks t_n, abstract shift operator q and its operation on t_n ;

(2) Extend to linear combinations of marks $\longrightarrow$ signal module ;
extend to linear combination of shifts $\longrightarrow$ filter algebra .

What is the mathematical nature of step (1) ?

Discrete Markov Chain

System S of mutually exclusive states t_n, $n \in I \subseteq \mathbb{Z}^{4}$.
At time k ($k \in \mathbb{N}_0$) the uncertainty of the state is a random variable X_k. Its distribution is $(a_{k,n})_{n \in I}$, where $a_{k,n} = prob(X_k = t_n)$.

The transition probability $p^k_{m,n} = prob(X_{k+1} = t_m \mid X_k = t_n)$.

The transition probability matrix $Q^k = [p^k_{m,n}]_{m,n \in I}$. $(a_{k+1,n}) = (a_{k,n}) Q^k$

Definition : Given initial distribution $(a_{0,n})$ of X_0 and Q^k, $k \in \mathbb{N}_0$, the sequence $X_0, X_1, \ldots, X_k, \ldots$ is called a Markov chain with discrete state space and of discrete time.

If $p^k_{m,n} = p^l_{m,n}$ for any $k, l \in \mathbb{N}_0$, then the chain has stationary transition probabilities, and it is homogeneous . Then $Q^k = Q$ for all k.

Q^k (or simply Q) is stochastic , i.e. all entries are non-negative, all column sums are equal to 1, no row consists only of zeros.

If $|I|$ is finite, the chain is finite

Relation to signal models

signal model	discrete Markov chains
time/space marks t_n	states t_n
matrix $\varphi(q)$	probability transition matrix Q

Examples:

(a) Infinite discrete time:

$$\begin{pmatrix} \ddots & & & \\ \ddots & 0 & & \\ & 1 & 0 & \\ & & 1 & \ddots \\ & & & \ddots \end{pmatrix}$$

(b) Infinite space:

$$\begin{pmatrix} \ddots & & & & & \\ \ddots & 1/2 & 0 & 1/2 & & \\ & & 1/2 & 0 & 1/2 & \\ & & & 1/2 & 0 & 1/2 & \ddots \\ & & & & \ddots & \ddots & \ddots \end{pmatrix}$$

Matrices, graphs, finite Markov chains, and signal models.

Concepts:

(1) a square matrix $A \in \mathbb{C}^{n \times n}$

(2) a weighted graph with n vertices

(3) a finite Markov chain with n states

(4) a shift-invariant signal model $(\mathcal{A}, \mathcal{M}, \Phi)$ with $\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/p(x)$

$$
\begin{tikzcd}
 & & \text{signal model } (\mathcal{A},\mathcal{M},\Phi) \\[-2em]
 & & \mathcal{A}=\mathcal{M}=\mathbb{C}[x]/p(x) \arrow[dl, "\varphi(x)"] \\
\text{weighted graph} \arrow[r, shift left, "\text{adjacency matrix}"] & \text{matrix } A \arrow[l, shift left] \arrow[ur, dashed, bend left, "m_A(x)=p_A(x)"] \arrow[dr, dashed, bend left, "A \geq 0"] & \\
 & & \text{Markov chain} \arrow[ul, "\text{probability transition matrix}"]
\end{tikzcd}
$$

Equivalence of concepts:

(a) Matrix $\rightleftarrows$ weighted graph

(b) Matrix $\rightleftarrows$ Markov chain

→ Note the condition $A \geq 0$, as any matrix $A \in \mathbb{C}^{n \times n}$ can have its rows scaled, so that each has a sum of 1.

(c) Matrix $\rightleftarrows$ signal model

Regular model: $\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/p(x)$

← $\mathcal{M}$ uniquely determines $A = \varphi(x)$.

→

Lemma: $A \in \mathbb{C}^{n \times n}$. There exists $p(x) \in \mathbb{C}[x]$, $\deg p(x) = n$, and a basis of $\mathbb{C}[x]/p(x)$, such that $\varphi(x) = A$ iff $m_A(x) = p_A(x)$. Then $p(x) = p_A(x)$.

(Reminder: $p_A(x) = \det(xI - A)$ is called a characteristic polynomial of A. $m_A(x)$, a monic polynomial of minimal degree, such that $m_A(A) = 0$, is called a minimal polynomial of A)

Non-regular model: $\mathcal{A} \neq \mathcal{M}$

What if $\mathcal{A}$ is a subalgebra of $\mathbb{C}[x]/p(x)$?

Lemma: $p(x) = \prod_{i=0}^{n-1}(x-\alpha_i)$, $\alpha_i \neq \alpha_j$ $(i \neq j)$; $\mathcal{A} = \mathbb{C}[x]/p(x)$, and $g(x) \in \mathcal{A}$.

$$\langle g(x) \rangle_{Alg.} = \mathcal{A} \iff g(\alpha_i) \neq g(\alpha_j) \quad (i \neq j).$$

Theorem (not in the paper):

Same conditions on $p(x)$, $g(x)$.

$$\dim \langle g(x) \rangle_{Alg.} = \left| g(\alpha_0), \dots, g(\alpha_{n-1}) \right|.$$

Example: $p(x) = x^4 - 1$, $g(x) = \frac{x + x^{-1}}{2} = \frac{x + x^{4-1}}{2} = \frac{x + x^3}{2}$.

Lemma: $A \in \mathbb{C}^{n \times n}$:
$$\varphi : \mathbb{C}[x]/m_A(x) \to \mathbb{C}^{n \times n}$$
$$x \mapsto A$$

φ is a representation of $\mathcal{A} = \mathbb{C}[x]/m_A(x)$, i.e. a homomorphism of algebras.

Lemma: $A \in \mathbb{C}^{n \times n}$. There exists a module $\mathcal{M} = \mathbb{C}[x]/p(x)$ (p is separable) with the regular representation φ (w.r.t. suitable basis b), such that $A = \varphi(g(x))$ iff A is diagonalizable.

In this case $\mathcal{A} \cong \mathbb{C}[x]/m_A(x)$.