<u>Recap:</u>
- 16 finite space models $\longleftrightarrow$ 16 DTTs
- polynomial DTTs: $\overline{\text{DTT}}$
- relationships between DTTs
- unitary/orthogonal matrices
- $MAM^{-1} =$ diagonal

  $\uparrow$ columns are Eigenvectors of A
- A symmetric, real $\Rightarrow$ A is diagonalized by an orthogonal matrix M

Orthogonal DTTs

DTTs are "almost" orthogonal since they diagonalize symmetric or "almost" symmetric matrices, e.g. $\varphi(x)$.

<u>Theorem:</u> For every DTT there are diagonal matrices D, E such that

$$D \cdot \text{DTT} \cdot E$$

is orthogonal.

The proof is constructive and uses Christoffel-Darboux formula (see ASP paper).

<u>Example: DCT-3</u>

signal model: $\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/T_n(x)$, Φ finite T-transform

$$\varphi(x) = \frac{1}{2}\begin{pmatrix} 0 & 1 & & & \\ 2 & 0 & \ddots & & \\ 0 & 1 & \ddots & & \\ \vdots & & \ddots & \ddots & 1 \\ 0 & & & 1 & 0 \end{pmatrix} \qquad \text{"almost symmetric"}$$

visualization: 

Set $E = \mathrm{diag}(\sqrt{2}, 1, \dots, 1)$:

$$E \cdot \varphi(x) \cdot E^{-1} = \frac{1}{2}\begin{pmatrix} 0 & \sqrt{2} & & & \\ \sqrt{2} & & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & & 1 \\ & & & 1 & 0 \end{pmatrix} \quad \text{is symmetric}$$

$\Rightarrow$ DCT-3 $\cdot E^{-1}$ diagonalizes a symmetric matrix with pairwise distinct Eigenvalues

$\Rightarrow$ there is diagonal D:

$$\underset{\substack{\uparrow \\ \text{base change} \\ \text{in spectrum}}}{D} \cdot \text{DCT-3} \cdot \underset{\substack{\uparrow \\ \text{base change} \\ \text{in } \mathcal{M}}}{E^{-1}} \quad \text{is orthogonal.}$$

base change in $\mathcal{M}$ $\Rightarrow$ changes signal model

Signal model for orthogonal DCT-3:

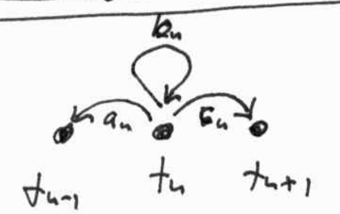
$$\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/T_n(x), \quad \Phi: \hat{s} \mapsto s_0 \tfrac{1}{\sqrt{2}} T_0 + \sum_{1 \le k < n} s_k T_k$$

visualization: $\bullet \overset{\sqrt{2}}{—} \bullet — \bullet \;\dots\; \bullet — \bullet$

We call such a signal model symmetric, i.e., if $\varphi(x)$ and hence all $\varphi(h(x))$ are symmetric.

Generic next-neighbour (GNN) ~~shift~~ model

GNN shift:



space variant (weights depend on n) but model will still be shift invariant

realization:

$$P_{n+1} = \frac{x - b_n}{c_n} P_n - \frac{a_n}{c_n} P_{n-1} \qquad c_n \neq 0$$

$$P_0 = 1, \; P_1 = \alpha x + \beta$$

note:
a.) again left boundary, but monomial signal extension in general not possible
b.) polynomials that satisfy a three-term recurrence are orthogonal polynomials:

- exists interval $[a,b]$, weight function $\omega(x)$:

$$\int_a^b P_n(x)\,P_m(x)\,\omega(x)\,dx = \sigma_n \cdot \delta_{n,m} \qquad \text{orthogonality}$$

$$\delta_{n,m} = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases} \quad \text{Kronecker delta}$$

- P_n have pairwise distinct zeros, all the zeros are in $[a,b]$
- example: $P = T$

$[a,b] = [-1,1]$

$\omega(x) = \frac{1}{\sqrt{1-x^2}}$

$$\int_{-1}^{1} T_n(x)\,T_m(x)\,\frac{1}{\sqrt{1-x^2}}\,dx = \frac{\pi}{2}\cdot \delta_{n,m} \qquad (\pi \text{ for } n=0)$$

zeros of T_n: $\cos \frac{k+\frac{1}{2}}{n}\pi$, $0 \le k < n$

(in general no closed form for zeros)

c.) ~~no~~ I did not find a suitable notion of k-fold shift

<u>infinite GNN model:</u>

$\mathcal{A} = \{ \sum_{k \geq 0} h_k x^k \mid \hat{h} \in \ell^1(\mathbb{N}) \}$

$\mathcal{M} = \{ \sum_{n \geq 0} s_n P_n \mid \hat{s} \in \ell^1(\mathbb{N}) \}$ (ℓ^2 makes trouble in general)

$\Phi: \hat{s} \longmapsto \sum_{n \geq 0} s_n P_n$

<u>finite GNN model:</u>

- again no monomial right signal extension possible
- choose s.c. $P_n = 0$ so spectrum consists of one-dim components (irred. submodules)

$\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/P_n(x)$, $\Phi: \hat{s} \mapsto \sum_{k \geq 0} s_k P_k$

$$\ell(x) = \begin{pmatrix} -\beta/\alpha & a_1 & & & \\ 1/\alpha & b_1 & \ddots & & \\ 0 & c_1 & \ddots & \ddots & \\ \vdots & & \ddots & \ddots & a_{n-1} \\ \vdots & & & c_{n-2} & b_{n-1} \end{pmatrix} \quad \text{tridiagonal}$$

(note: bug in ASP paper)

Notes:

- again there are diagonal matrices D, E such that
  $$D \cdot \mathcal{P}_{b,\alpha} \cdot E \quad \text{is orthogonal}$$
  (same reason as for the DTTs)
- $\mathcal{P}_{b,\alpha} = [P_\ell(\alpha_k)]_{k,\ell}$ possesses an $O(n \log^2(n))$ algorithm.
- strong connection to Gaussian quadrature:
  (quadrature = numerical integration)
  $$\int_a^b f(x)\, \underset{\substack{\uparrow \\ \text{weight function}}}{w(x)}\, dx \approx \sum_{i=1}^n w_i f(x_i), \quad x_i \in [a,b]$$
  questions: - best nodes x_i? (zeros of orth. polys)
  - error bound?
- strong connection to approximation theory

Examples:

- generalized Chebyshev: $\left(C_n^{(d)}(x) = \sqrt{d}^{\,n} C_n\left(\frac{x}{\sqrt{d}}\right)\right)$
  $$C_0^{(d)} = 1, \quad C_1^{(d)} = ?, \quad C_{n+1}^{(d)} = 2x\, C_n - \underset{\substack{\uparrow \\ \text{constant}}}{d}\, C_{n-1}$$

  $d = 1 \to$ Chebyshev, space model

  $d = 0 \to (2x)^n$, essentially time model

  continuum of models in between?

- Legendre polynomials:
  $$L_0 = 1, \quad L_1 = x, \quad (n+1) L_{n+1} = (2n+1)\, x L_n - n L_{n-1}$$
  $$\int_{-1}^{1} L_n L_m \, dx = \frac{2}{2n+1} \delta_{n,m}$$

- Hermite polynomials:

  $H_0 = 1$, $H_1 = 2x$, $H_{n+1} = 2x H_n - 2n H_{n-1}$

  $$\int_{-\infty}^{\infty} H_n H_m \cancel{dx}\, e^{-x^2} dx = n! 2^n \sqrt{\pi}\, \delta_{n,m}$$

- dozens of more well-known orth. polys
  each series provides valid signal model

= lot of things to play around with

+ connect facts of orth. polys to signal processing