Recap finite space models "space shift" 16 choices with monomial signal extension

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<th>( s_0 = s_{n-2} )</th>
<th>( s_2 = 0 )</th>
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Transposition:
- Types 1, 4, 5, 8 are symmetric
- \((s_1 \text{ type } 2)^T = \text{ type } 3\) \(, (s_1 \text{ type } 6)^T = \text{ type } 2\)
Example 2: DCT-2

Signal model: \( u = U = \frac{C(x)}{V_x-V_{x-1}} = \frac{C(x)}{(x-1)U_{n-1}} \) \( V \)-basis

Shift matrix:
\[
\Phi(x) = \begin{pmatrix}
\frac{V_x}{V_x} & \frac{V_x}{V_x} & \cdots & 0 & \frac{V_x}{V_x} \\
\frac{V_x}{V_x} & 0 & \cdots & \frac{V_x}{V_x} & \frac{V_x}{V_x} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \frac{V_x}{V_x} & \cdots & \frac{V_x}{V_x} & \frac{V_x}{V_x} \\
\frac{V_x}{V_x} & \frac{V_x}{V_x} & \cdots & 0 & \frac{V_x}{V_x}
\end{pmatrix}
\]

Visualization:
\[
\begin{array}{ccccccc}
\circ & - & \circ & \cdots & \circ & - & \circ \\
\end{array}
\] (scaled by \( \frac{V_x}{V_x} \))

Signal extension:
- Left: \( V_x = V_{x-1} \) (H.S)
- Right: \( V_{x+1} = V_{x-1} + V_x \) (H.S)

Spectrum and FT: zeros of \( U_n \):
\[
U_n(x) = \frac{\sin((n+1)\cos x)}{\sin(\cos x)} \implies \text{zeros: } \cos \frac{k(x+1)}{n}, 0 \leq k \leq n
\]

Zeros of \( (x-1)U_{n-1} = \cos \frac{k(x+1)}{n}, 0 \leq k \leq n \)

\( F = \frac{C(x)}{(x-1)U_{n-1}} \rightarrow \frac{C(x)}{x-cos \frac{k(x+1)}{n}} \)

as matrix (use \( V_n(x) = \frac{\cos(k(x+1)\cos x)}{\cos(k\cos x)} \))

\[
\mathbb{P}_{b,a} = [V_n(x)] = \\
= \left[ \begin{array}{c}
\cos \frac{k(x+1)}{n} \\
\cos \frac{k(x+1)}{n} \\
\vdots \\
\cos \frac{k(x+1)}{n}
\end{array} \right] = \text{DCT-2}_n
\]

\( \text{polynomial DCT-2}^n \)

\[
= \text{diag} \left( \frac{1}{\cos \frac{k(x+1)}{n}} \right) \left[ \cos \frac{k(x+1)}{n} \right]
\]

\( \text{DCT-2} \)

\[ \implies \text{DCT-2} = \text{diag} \left( \frac{1}{\cos \frac{k(x+1)}{n}} \right) \cdot \text{DCT-2} \]

scaled polynomial transform
Definition: The polynomial transform associated with a DTT (i.e., they are FT's for the same signal model) is called "polynomial DTT," written as \( \overline{DTT} \).

- \( \overline{DTT} = DTT \) (i.e., now in the 4x4 table).
- Clearly: \( \overline{DTT} = D \cdot DTT \), \( D \) diagonal.

Relationship between DTTs

Duality:

Definition: We call 2 DTTs "dual" if they have mirrored d.c.'s, i.e., they are at mirrored positions in the 4x4 table. Necessarily, they have the same \( D \).

Lemma: If \( DTT, DTT' \) are dual, then

\[
\text{diag}(c^{-1}) \cdot DTT_u = DTT'_u \cdot J_u
\]

\( J_u = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix} \)

Base change:

Lemma: DTTs in the same group ("T-group" etc.) have (almost) the same \( dT \). As a consequence, they can be translated into each other using \( O(d) \) operators.

Example:

- DCT-3: \( u = U = \cos \theta / T_u \), T-basis
- DCT-4: \( u = U = \cos \pi / T_u \), V-basis

use \( T_u = (V_u + V_{u-1}) / 2 \)
T-basis $\rightarrow$ V-basis:

$$S_u = \frac{1}{2} \begin{pmatrix} 1 & \cdots & 1 \\ 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

So:

$$\begin{array}{cccc}
    & \mathbb{C}^3 & \xrightarrow{S_u} & \mathbb{C}^3 \\
\mathbb{C}^3 & \text{(T-basis)} & \text{(V-basis)} & \text{(V-basis)} \\
\end{array}$$

$$\begin{array}{cccc}
    \oplus \mathbb{C}^3/\mathbb{T}_u & \xrightarrow{F_u} & \oplus \mathbb{C}^3/\mathbb{T}_u \\
\mathbb{C}^3 & \text{(T-basis)} & \text{(V-basis)} & \text{(V-basis)} \\
\end{array}$$

$$= \begin{array}{ccc}
    \text{DCT-3}_u & \xrightarrow{\text{DCT-4}_u} & S_u \\
\end{array}$$

$$= \begin{array}{ccc}
    \text{DCT-3}_u & \xrightarrow{\text{DCT-4}_u \cdot S_u} & \text{DCT-4}_u \\
\end{array}$$

$$D_u = \text{diag}(\cos \frac{\pi u \lambda}{4})$$

These identities can be inverted/transpose to get other identities.

**Orthogonality**

**Definition:** Let $M \in \mathbb{C}^{m \times n}$. Then $$M^H = (M^T)^*$$ (transpose-conjugate) is sometimes called "Hermitian adjoint."

**Definition:** $M \in \mathbb{C}^{m \times n}$ is called "unitary" if $$MM^H = M^HM = I.$$ If it is unitary and real, i.e., $M^T = M$, it is called "orthogonal."

We write $\langle x, y \rangle = \sum x_k \overline{y_k}$ for the standard scalar product. $$\langle x, x \rangle = \|x\|^2$$ is the "energy" of $x \in \mathbb{C}$. 
Lemmas: The following are equivalent:

a) \( \Pi \) is unitary
b) the rows of \( \Pi \) form an orthonormal basis
c) the columns of \( \Pi \)
d) \( \langle \Pi x, \Pi y \rangle = \langle x, y \rangle \) for \( x, y \in \mathbb{C}^n \)
e) \( \| \Pi x \| = \| x \| \) for \( x \in \mathbb{C}^n \)

("energy preserving")

Lemma: If \( A \in \mathbb{R}^{n \times n} \) is symmetric, then \( A \) has an orthonormal basis of Eigenvectors, i.e., there is \( \Pi \in \mathbb{R}^{n \times n} \) orthogonal:

\[ \Pi A \Pi^{-1} = \text{diagonal} \]

Lemma: If the Eigenvalues of any \( A \) all have multiplicity 1, i.e., \( A \) has \( n \) distinct Eigenvalues, then

- there is \( \Pi \): \( \Pi A \Pi^{-1} = \text{diagonal} \)
- all \( \Pi \) that diagonalize \( A \) are given by

\[ \text{Eigenvectors of } \Pi, \text{E diagonal, invertible} \]

Intuition: Let \( \Pi \) diagonalize \( A \):

\[ A = \Pi^{-1} \Pi, D = \text{diag}(\lambda_i) \]

\[ A \cdot (\lambda_1, \ldots) = (\lambda_1 \lambda_1 \lambda_2 \ldots) \]

- if each Eigenspace has dimension 1, then choosing any other Eigenspace yields

\[ \Pi^{-1}, F \neq \Pi \text{ diagonal} \]

(since \( \Pi^{-1} \Pi \text{ diagonalizes } A \))

- if \( A = \Xi(\xi) \) is a filter matrix and \( M = \Xi \)

\( \xi \) columns of \( \Xi \) are the pure frequencies (we knew they already)
Application to DTTs:

$\psi(x)$ "almost" symmetric (because it represents symmetric space shift)

$\implies$ DTTs are "almost" orthogonal.