Recap Finite space models $\overset{1/2}{\curvearrowleft}\bullet\overset{1/2}{\curvearrowright}$ "space shift"

16 choices with monomial signal extension

	left b.c. \ right b.c.	$s_n = s_{n-2}$	$s_n = 0$	$s_n = s_{n-1}$	$s_n = -s_{n-1}$
T	$s_{-1} = s_1$	DCT-1	DCT-3	types 5-8	
U	$s_{-1} = 0$	DST-3	DST-1		
V	$s_{-1} = s_0$	types 5-8		DCT-2	DCT-4
W	$s_{-1} = -s_0$			DST-4	DST-2
- transform					

groups:

DTTs, type 3,4: $\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/T_n$ but different bases in $\mathcal{M}$

"T-group DTTs"

DTTs, type 1,2: $\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/(x^2-1)U_{n-2},\ \mathbb{C}[x]/U_n,\ \mathbb{C}[x]/(x-1)U_{n-1},$

$\mathbb{C}[x]/(x+1)U_{n-1},$

for DCT-1, DST-1, DCT-2, DST-2

"U-group DTTs"

DTTs, type 5-8: 4 "V-group DTTs", 4 "W-group DTTs"

transposition:

- type 1, 4, 5, 8 are symmetric
- $(\text{type } 2)^T = \text{type } 3$, $(\text{type } 6)^T = \text{type } 7$

Example 2: DCT-2

Signal model: $\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/V_n - V_{n-1} = \mathbb{C}[x]/(x-1)U_{n-1}$, V-basis

shift matrix:

$$\varphi(x) = \begin{pmatrix} \frac12 & \frac12 & & & \\ \frac12 & 0 & \frac12 & & \\ & \frac12 & \ddots & \ddots & \\ & & \ddots & 0 & \frac12 \\ & & & \frac12 & \frac12 \end{pmatrix}$$

visualization:  (scaled by $\frac12$)

signal extension:

left: $V_{-\ell} = V_{\ell-1}$ (HS)

right: $V_{n+\ell} = V_{n-1-\ell}$ (HS)

spectrum and FT: zeros of U_n?

$$U_n(x) = \frac{\sin((n+1)\operatorname{acos} x)}{\sin(\operatorname{acos} x)} \Rightarrow \text{zeros: } \cos\frac{k+1}{n+1}\pi,\ 0 \le k < n$$

zeros of $(x-1)U_{n-1}$: $\cos\frac{k}{n}\pi$, $0 \le k < n$

$$\mathcal{F}: \mathbb{C}[x]/(x-1)U_{n-1} \longrightarrow \bigoplus_{k=1}^{n-1} \mathbb{C}[x]/x - \cos\frac{k}{n}\pi$$

as matrix (use $V_n(x) = \dfrac{\cos((n+\frac12)\operatorname{acos} x)}{\cos(\frac12 \operatorname{acos} x)}$)

$$\mathcal{P}_{b,\alpha} = [V_\ell(\alpha_k)] = \left[\frac{\cos\frac{k(\ell+\frac12)}{n}\pi}{\cos\frac{k}{2n}\pi}\right] = \overline{\text{DCT-2}}_n$$

"polynomial DCT-2"

$$= \operatorname{diag}_k\left(\frac{1}{\cos\frac{k}{2n}\pi}\right) \underbrace{\left[\cos\frac{k(\ell+\frac12)}{n}\pi\right]}_{\text{DCT-2}}$$

$$\Rightarrow \text{DCT-2} = \operatorname{diag}_k\left(\cos\frac{k}{2n}\pi\right) \cdot \overline{\text{DCT-2}}$$

scaled polynomial transform

Definition: The polynomial transform associated with a DTT (i.e., they are FT's for the same signal model) is called "polynomial DTT", written as $\overline{DTT}$.

- $DTT = \overline{DTT} \iff$ first ~~col~~ row in the 4x4 table.
- clearly: $DTT = D \cdot \overline{DTT}$, D diagonal

Relationship between DTTs

Duality:

Definition: We call 2 DTTs "dual" if they have mirrored b.c.'s, i.e., they are at mirrored positions in the 4x4 table. Necessarily, they have the same $\mathcal{A} = \mathcal{M}$.

Lemma: If DTT, DTT' are dual, then

$$\operatorname*{diag}_{0 \le k < n}\left((-1)^k\right) \cdot DTT_n = DTT'_n \cdot J_n.$$

$$J_n = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$$

Base change:

Lemma: DTTs in the same group ("T-group" etc.) have (almost) the same $\mathcal{A} = \mathcal{M}$. As a consequence, they can be translated into each other using $O(n)$ operations.

Example:

DCT-3: $\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/T_n$, T-basis

DCT-4: $\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/T_n$, V-basis

use $T_e = (V_e + V_{e-1})/2$

T-basis $\longrightarrow$ V-basis: $$S_n = \frac{1}{2}\begin{pmatrix} 2 & 1 & & 0 \\ & 1 & \ddots & \\ & & \ddots & 1 \\ 0 & & & 1 \end{pmatrix}$$

so:

$$\begin{array}{ccc}
\begin{array}{c}\mathbb{C}[x]/T_n \\ (\text{T-basis})\end{array} & \xrightarrow{\;S_n\;} & \begin{array}{c}\mathbb{C}[x]/T_n \\ (\text{V-basis})\end{array} \\
\Big\downarrow{\scriptstyle \mathrm{DCT\text{-}3}_n} & & \Big\downarrow{\scriptstyle \overline{\mathrm{DCT\text{-}4}_n}} \\
\bigoplus_k \mathbb{C}[x]/x-\alpha_k & \xrightarrow{\;I_n\;} & \bigoplus \mathbb{C}[x]/x-\alpha_k
\end{array}$$

$\Rightarrow$ $\mathrm{DCT\text{-}3}_n =$ [illegible] $\cdot\, \overline{\mathrm{DCT\text{-}4}_n} \cdot S_n$

$\Rightarrow$ $D_n \cdot \mathrm{DCT\text{-}3}_n = \mathrm{DCT\text{-}4}_n \cdot S_n$, $\quad D_n = \mathrm{diag}_k\left(\cos\frac{(2k+1)\pi}{4n}\right)$

These identity can be inverted/transposed to get other identities.

Orthogonality

Definition: Let $M \in \mathbb{C}^{n\times n}$. Then
$$M^H = (M^T)^* \quad \text{(transpose-conjugate)}$$
is sometimes called "Hermitian adjoint."

Definition: $M \in \mathbb{C}^{n\times n}$ is called "unitary" if
$$MM^H = M^H M = I.$$
if M is unitary and real, i.e.,
$$MM^T = M^T M = I$$
it is called "orthogonal."

We write $\langle x, y\rangle = \sum x_k y_k^*$ for the standard scalar product.
$\langle x, x\rangle = \|x\|_2^2$ is the "energy." of $x \in \mathbb{C}^n$.

Lemma: The following are equivalent:

a.) M is unitary

b.) the rows of M form an orthonormal basis

c.) the columns of M "

d.) $\langle Mx, My \rangle = \langle x, y \rangle$ for $x, y \in \mathbb{C}^n$

e.) $\|Mx\|_2 = \|x\|_2$ for $x \in \mathbb{C}^n$

("energy preserving")

Lemma: If $A \in \mathbb{R}^{n\times n}$ is symmetric, then A has an orthonormal basis of real Eigenvectors, i.e., there is $M \in \mathbb{R}^{n\times n}$, orthogonal, :

$$MAM^{-1} = \text{diagonal}$$

Lemma: If the Eigenvalues of any $A \in \mathbb{C}^{n\times n}$ all have multiplicity 1, i.e., A has n distinct Eigenvalues, then

- there is M: $MAM^{-1} = D$ diagonal
- all M that diagonalize A are given by

$$EM, \quad E \text{ diagonal, invertible}$$

Intuition: Let M diagonalize A:

$$A \cdot M^{-1} = M^{-1} \cdot D, \quad D = \text{diag}(\lambda_n)$$

$$A \cdot \underbrace{\left(| \, | \, | \cdots \right)}_{\text{colums = Eigenvectors of } A} = \left(\lambda_1 \cdot \,|\, \lambda_2 \cdot \,|\, \lambda_3 \cdot \,|\, \cdots \right)$$

- if each Eigenspace has dimension 1, then choosing any other Eigenbasis yields

$$M^{-1} \cdot F, \quad F \text{ diagonal}$$

(so $F^{-1}M$ diagonalizes A)

- if $A = \varphi(h)$ is a filter matrix and $M = \hat{\mathcal{F}}$

$\Rightarrow$ colums of $\hat{\mathcal{F}}$ are the pure frequencies (we knew that already)

Application to DTTs:

$\varphi(x)$ "almost" symmetric (because it represents symmetric space shift)

$\Longrightarrow$ DTTs are "almost" orthogonal.