recap:
- signal model $(\mathcal{A}, \mathcal{M}, \Phi)$
  - regular ($\mathcal{A} = \mathcal{M}$)
  - shift-invariant ($\mathcal{A}$ commutative)
  - finite ($\dim(\mathcal{M}) < \infty$)
  - 1-D ($\mathcal{A} = \langle x \rangle$ alg)

$\boxed{\text{signal model}}$ $\longrightarrow$ signals, filters, filtering
$\phantom{\boxed{\text{signal model}}}$ $\longrightarrow$ spectrum, FT, frequ. response

- why polynomial algebras $\mathbb{C}[x]/p(x)$
- visualization

Finite, shift-invariant, 1-D signal models

Necessarily $\mathcal{A} = \mathbb{C}[x]/p(x)$. Assume $\mathcal{M} = \mathcal{A}$, $b = \{p_0, \dots, p_{n-1}\}$ basis of $\mathcal{M}$.

Then $(\mathcal{A}, \mathcal{M}, \Phi)$ is a signal model ~~with~~ for $V = \mathbb{C}^n$ with

$$\Phi : \mathbb{C}^n \longrightarrow \mathcal{M}$$
$$\hat{s} \longmapsto s = s(x) = \sum_{\ell=0}^{n-1} s_\ell \, p_\ell(x)$$

(running) example: $\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/x^n - 1$, $b = \{1, x, \dots, x^{n-1}\}$
Φ is called "finite z-transform"

Filtering: $s(x) \in \mathcal{M}$, $h(x) \in \mathcal{A} \longrightarrow h(x) s(x) \in \mathcal{M}$
(multiplication of polynomials modulo $p(x)$; the polynomials are expressed in the basis b)

example: $\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/x^{n} - 1$

filtering: $h(x) \cdot s(x) \mod (x^n - 1)$

$$\Longleftrightarrow \hat{h} \circledast \hat{s}$$

circular convolution

Filtering in coordinates

Let φ be the representation (of $\mathcal{A}$) afforded by $\mathcal{M}$ with basis b.

$$\varphi: \mathcal{A} \longmapsto \mathbb{C}^{n \times n}$$
$$h \longmapsto \varphi(h)$$

then: $\quad h(x)s(x) \bmod p(x) \longleftrightarrow \varphi(h)\cdot \hat{s}$

example: $\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/x^n - 1$, $\quad b = \{1, x, \dots, x^{n-1}\}$

$$\varphi(x) = \begin{pmatrix} 0 & & & 1 \\ 1 & \ddots & & \\ 0 & \ddots & \ddots & \\ \vdots & & 1 & \end{pmatrix}, \quad \varphi(x^2) = \begin{pmatrix} 0 & & 1 & \\ 0 & & & 1 \\ 1 & \ddots & & \\ & & 1 & \end{pmatrix} \quad \dots\dots$$

$$\Longrightarrow \varphi\Big(\sum_{k=0}^{n-1} h_k x^k\Big) \underset{\substack{\uparrow \\ \varphi \text{ hom. of algebras}}}{=} \sum_{k=0}^{n-1} h_k \varphi(x^k) = \underbrace{\begin{pmatrix} h_0 & \ddots & & h_2 & h_1 \\ h_1 & \ddots & & & h_2 \\ h_2 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & h_2 & h_1 & h_0 \end{pmatrix}}_{\text{circulant matrix}}$$

$$h(x)s(x) \bmod (x^n - 1) \longleftrightarrow \begin{pmatrix} h_0 & & \cdots & h_1 \\ h_1 & \ddots & & \\ & \ddots & \ddots & \\ \cdots & & h_1 & h_0 \end{pmatrix} \cdot \hat{s}$$

Visualization: Graph that has $\varphi(x)$ as adjacency matrix $=$ let x operate on b

$$\begin{tikzcd} x \text{ operates:} & \bullet \arrow[r] & \bullet \arrow[r] & \cdots \arrow[r] & \bullet \arrow[lll, bend right] \\ b: & x^0 & x^1 & & x^{n-1} \end{tikzcd}$$

structure
$=$ a circle
$=$ periodic signal extension

Spectrum and FT

$\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/p(x)$, assume $p(x) = \prod_{k=0}^{n-1}(x - \alpha_k)$,

$\alpha = (\alpha_0, \dots, \alpha_{n-1})$ pairwise distinct

FT (coordinate-free):

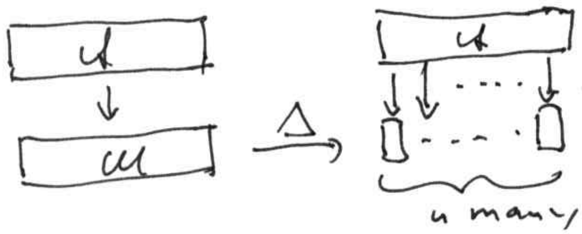
(one-dim, hence irreducible, $\mathcal{A}$-modules)

$$\Delta: \mathcal{M} \longrightarrow \bigoplus_{k=0}^{n-1} \mathbb{C}[x]/(x-\alpha_k)$$

$$s = s(x) \longmapsto (s(x) \bmod (x-\alpha_0), \dots, s(x) \bmod (x-\alpha_{n-1}))$$
$$= (s(\alpha_0), \dots, s(\alpha_{n-1}))$$

"spectrum of s w.r.t. the signal model"

pure frequencies: $f_i = \Delta^{-1}(e_i)$, $i = 0 \dots n-1$



FT in coordinates:

basis in $\mathcal{M}$: b (fixed by signal model)

basis in each $\mathbb{C}[x]/_{x-\alpha_k}$: 1

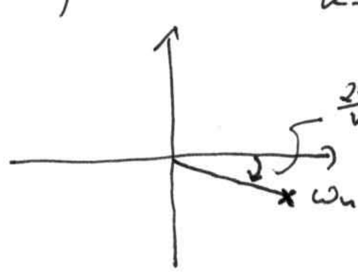
$$\Rightarrow \mathcal{F} = \begin{pmatrix} p_0(\alpha_0) & p_1(\alpha_0) & \\ \vdots & \vdots & \cdots \\ p_0(\alpha_{n-1}) & p_1(\alpha_{n-1}) & \end{pmatrix} = [p_\ell(\alpha_k)]_{0 \le k, \ell < n}$$

$$= \mathcal{P}_{b,\alpha}$$ "polynomial transform"

other basis in spectrum: $\tilde{\mathcal{F}} = \underbrace{D}_{\text{diagonal, invertible}} \cdot \mathcal{P}_{b,\alpha}$

$$\Delta(s) \longleftrightarrow \tilde{\mathcal{F}} \cdot \hat{s}$$

example: $\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/_{x^n-1}$, $x^n - 1 = \prod_{k=0}^{n-1} (x - \omega_n^k)$, $\omega_n = e^{-\frac{2\pi i}{n}}$



$$\Delta: \mathbb{C}[x]/x^n-1 \longrightarrow \bigoplus \mathbb{C}[x]/x-\omega_n^i$$

$$s = s(x) \longmapsto \left(s(\omega_n^0), \dots, s(\omega_n^{n-1})\right)$$

$$\mathcal{F} = \mathcal{P}_{b,\alpha} = [p_\ell(\alpha_k)] = [(\omega_n^k)^\ell] = [\omega_n^{k\ell}] = \underbrace{DFT_n}_{\text{DFT matrix}}$$

$$\Delta(s) \longleftrightarrow DFT_n \cdot \hat{s}$$

pure frequencies:

- $\hat{f}_i = DFT_n^{-1} e_i$, $i = 0 \dots n-1$, or
- $f_i = \dfrac{\prod_{j \neq i}(x - \omega_n^j)}{\prod_{j \neq i}(\omega_n^i - \omega_n^j)}$

<u>Frequency response</u>

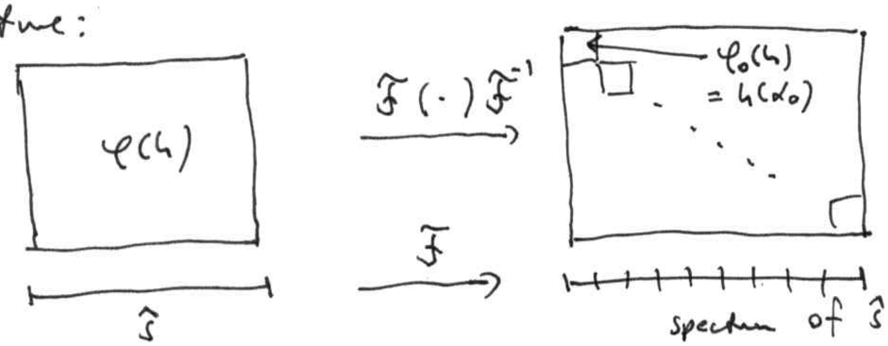
spectral component $\mathbb{C}[x]/x-\alpha_k$ w. basis (1)

irreducible representation:

$$\varphi_k: \mathbb{C}[x]/p(x) \longrightarrow \mathbb{C}^{1\times 1} = \mathbb{C}$$

$$h(x) \longmapsto \varphi_k(h(x)) = h(\alpha_k)$$

picture:



$$= \underbrace{\mathrm{diag}(h(\alpha_0), \dots, h(\alpha_{n-1}))}_{\text{frequ. response of } h}$$

$$\Longrightarrow \mathcal{F}\varphi(h)\mathcal{F}^{-1} = \mathrm{diag}(h(\alpha_0), \dots, h(\alpha_{n-1})) \quad \text{for } \underline{\underline{\text{all}}}\ h \in \mathcal{A}$$

as commutative diagram: (basis chosen everywhere)

$$\begin{tikzcd}
\mathbb{C}[x]/p(x) \arrow[r, "\tilde{\mathcal{F}}"] \arrow[d, "\varphi(h)"'] \arrow[dr, phantom, "///"] & \bigoplus \mathbb{C}[x]/x-\alpha_k \arrow[d, "\operatorname{diag}(h(\alpha_0), \ldots, h(\alpha_{n-1}))"] \\
\mathbb{C}[x]/p(x) \arrow[r, "\tilde{\mathcal{F}}"] & \bigoplus \mathbb{C}[x]/x-\alpha_k
\end{tikzcd}$$

example: $\mathcal{A} = \mathcal{M} = \mathbb{C}[x]/x^n-1$, $h = h(x) = \sum h_k x^k \in \mathcal{A}$

$$\mathrm{DFT}_n \cdot \begin{pmatrix} h_0 & & \cdots & h_1 \\ h_1 & \ddots & & \\ & & \ddots & \\ \vdots & & h_1 & h_0 \end{pmatrix} \cdot \mathrm{DFT}_n^{-1} = \operatorname{diag}(h(\omega_n^0), \ldots, h(\omega_n^{n-1}))$$

$[\Rightarrow$ char poly of $\downarrow$ $= \prod (x - h(\omega_n^k))]$

filtering in signal domain:

$$\varphi(h) \cdot \hat{s} = \begin{pmatrix} h_0 & & h_1 \\ h_1 & \ddots & \\ & \ddots & \\ & h_1 & h_0 \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix}$$

filtering in frequ. domain:

$$\begin{pmatrix} h(\omega_n^0) & & 0 \\ & \ddots & \\ 0 & & h(\omega_n^{n-1}) \end{pmatrix} \begin{pmatrix} s(\omega_n^0) \\ \vdots \\ s(\omega_n^{n-1}) \end{pmatrix}$$

$$= (h(\omega_n^0) s(\omega_n^0), \ldots, h(\omega_n^{n-1}) s(\omega_n^{n-1}))^T.$$