Short history of linear algebra

200–100 B.C.: first matrix methods for solving a system of linear equations (China)

~1700: theory of determinants to solve lin. equ. (Leibniz)

1750: Cramer's rule for the solution of systems of lin. equ.

~1800: Gauss elimination (~~[illegible]~~ Gauss)

1843: term "vector" (Hamilton)

1848: term "matrix" = latin for womb (Sylvester)

~1850: matrix theory (matrix mult. etc.) (Cayley)

1848/62: abstract theory of algebras (Grassmann) incl. lin. dependence

1888: abstract theory of vector spaces (Peano) dimension, existence of basis, ...

Recap: Coordinatization in vector spaces (vectors and linear mappings)

Algebraic Signal Processing Theory (ASP)

(linear) SP concepts:	discrete: infinite time	discrete: finite time	generic theory = ASP
basic			
filter			
signal			
filtering (conv.)	linear conv.	circular conv.	
z-transform	z-transform	?	
spectrum			
Fourier trafo	DTFT	DFT	
frequ. response			
advanced			
sampling			
filterbanks			
uncertainty			
multiresolution	all developed	parts missing	
⋮			

goal: - better understanding
- go beyond time SP

SP is algebraic

- show
- filter space = an algebra $\mathcal{A}$
- signal space = an $\mathcal{A}$-module $\mathcal{M}$.

Definition: Let $\mathcal{A}$ be an algebra. An $\mathcal{A}$-module $\mathcal{M}$ is a vector space (same field as $\mathcal{A}$) with an operation:

$$\cdot : \mathcal{A} \times \mathcal{M} \to \mathcal{M}$$

such that

- for $a, b \in \mathcal{A}$, $m, n \in \mathcal{M}$:

$$(a+b)m = am + bm$$
$$a(m+n) = am + an$$
$$a(bm) = (ab)m$$
$$1 \cdot m = m$$

Norms:

Definition: Let V be a real or complex vector space.
A norm on V is a mapping

$$\|\cdot\| : V \longrightarrow \mathbb{R}$$
$$v \longmapsto \|v\|$$

such that

a.) $\|v\| \geq 0$ for $v \in V$ and $\|v\| = 0 \iff v = 0$

b.) $\|\alpha v\| = |\alpha| \cdot \|v\|$

c.) $\|v + w\| \leq \|v\| \, \|w\|$

examples: p-norms, $p \geq 1$

$$\text{discrete:} \begin{cases} \|(\alpha_0, \dots, \alpha_{n-1})^T\|_p = \left(\sum_i |\alpha_i|^p\right)^{1/p} & p = \infty: \max_i |\alpha_i| \\ \|(\dots \alpha_{-1}, \alpha_0, \alpha_1, \dots)\|_p = \quad " & \quad " \end{cases}$$

most important: $p = 1$, $p = 2$ (Euclidean norm, energy),
$p = \infty$ (maximum norm)

ℓ^p-spaces:

Definition: $\ell^p(\mathbb{Z}) = \{ v = (\dots \alpha_{-1}, \alpha_0, \alpha_1, \dots) \in \mathbb{C}^{\mathbb{Z}} \mid \|v\|_p < \infty \} \leq \mathbb{C}^{\mathbb{Z}}$

$p = 1$: absolute summable
$p = 2$: finite energy
$p = \infty$: bounded

$$\ell^1(\mathbb{Z}) \leq \ell^2(\mathbb{Z}) \leq \ell^\infty(\mathbb{Z}) \leq \mathbb{C}^{\mathbb{Z}}$$

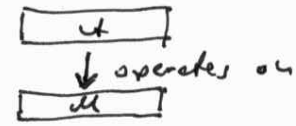
Back to ASP

Given filter algebra $\mathcal{A}$ and signal module $\mathcal{M}$, all basic SP concepts are automatically defined.
Running example: infinite discrete time SP.

<u>Algebra = filter space</u>

- elements: h, algebra: $\mathcal{A}$
- discrete time: $\mathcal{A} = \{ h = H(z) = \sum_{n \in \mathbb{Z}} h_n z^{-n} \mid \hat{h} \in \ell_1(\mathbb{Z}) \}$

<u>$\mathcal{A}$-Module = signal space</u> picture:



- elements: s, module: $\mathcal{M}$
- discrete time: $\mathcal{M} = \{ s = S(z) = \sum_{n \in \mathbb{Z}} s_n z^{-n} \mid \hat{s} \in \ell_2(\mathbb{Z}) \}$

<u>Regular module</u>

If $\mathcal{A}$ is an algebra, then $\mathcal{A}$ can be viewed as $\mathcal{A}$-module w.r.t. the multiplication in $\mathcal{A}$. $\mathcal{A}$ is called "regular module."

- discrete time: module is <u>not</u> regular

<u>Representations of algebras = filters in coordinates</u>

Let $\mathcal{A}, \mathcal{M}$ be given. For any $h \in \mathcal{A}$,

$$\begin{aligned} \mathcal{M} &\to \mathcal{M} \\ s &\mapsto hs \end{aligned}$$

is a linear mapping.

Assume $\underbrace{\dim(\mathcal{M})}_{=n} < \infty$ and choose a basis b in $\mathcal{M}$.

Then, in coordinates:

$s \in \mathcal{M}$ becomes $\hat{s} \in \mathbb{C}^n$ so $\underbrace{h \cdot s}_{\text{filtering}}$ becomes $\underbrace{M_h \cdot \hat{s}}_{\text{in coord.}}$

$h \in \mathcal{A}$ " $M_h \in \mathbb{C}^{n \times n}$

<u>Lemma and definition:</u>

$$\begin{aligned} \varphi: \mathcal{A} &\to \mathbb{C}^{n \times n} \\ h &\mapsto M_h \end{aligned}$$

is a homomorphism of algebras, called "(matrix) representation of $\mathcal{A}$ afforded by $\mathcal{M}$ with basis b.

- discrete time: $s = (\ldots z^{-1}, 1, z^{1}, \ldots)$

$$M_{z^{-1}} = \begin{pmatrix} \ddots & & & \\ 1 & \ddots & & \\ & 1 & \ddots & \\ & & 1 & \ddots \end{pmatrix} \quad \text{(infinite matrix)}$$

$h = \sum_{n=0}^{N} h_n z^{-n}$, (FIR filter)

$$M_h = \begin{pmatrix} \ddots & & & \\ h_N \cdots h_0 & & & \\ & h_N \cdots h_0 & & \\ & & h_N \cdots h_0 & \ddots \end{pmatrix}$$

$h \cdot s$
$\longleftrightarrow (\sum h_n z^{-n})(\sum s_n z^{-n})$
$\longleftrightarrow M_h \cdot \hat{s}$

Irreducible submodule = spectral component

__Definition:__ Let $\mathcal{M}$ be an $\mathcal{A}$-module. $\mathcal{M}' \leq \mathcal{M}$ (subvector space) is called "submodule" if it is again an $\mathcal{A}$-module, i.e., $\mathcal{A}\mathcal{M}' \subseteq \mathcal{M}'$, i.e., closed under the operation of $\mathcal{A}$ (= filtering).

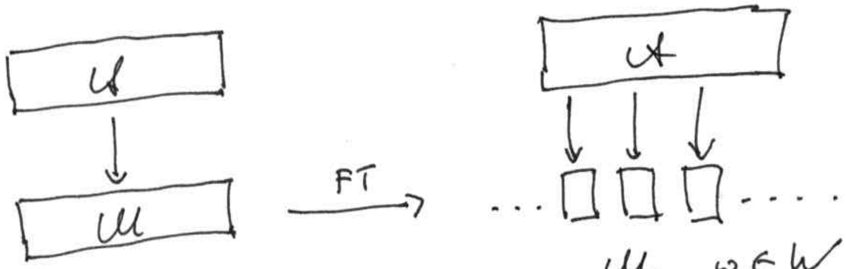
$\mathcal{M}' \leq \mathcal{M}$ is called irreducible if ~~the~~ it has no nontrivial submodules. (Smallest possible: $\dim(\mathcal{M}') = 1$).

- discrete time: $\mathcal{M} = \{ \sum s_n z^{-n} \mid \hat{s} \in \ell^2(\mathbb{Z}) \}$
  - $\langle 1, z^{-1}, \ldots, z^{-N} \rangle_{VS}$ not a submodule
  - $\langle \ldots z^{-2}, z^{0}, z^{2}, z^{4}, \ldots \rangle_{VS}$ not a submodule
  - $\langle \underbrace{\sum e^{j\omega n} z^{-n}}_{\notin \ell^2 \text{ but in } \ell^\infty} \rangle$, $\omega \in [-\pi, \pi)$ module + irreducible since $\dim = 1$
  - $\langle \sum \cos(\omega n) z^{-n}, \sum \sin(\omega n) z^{-n} \rangle$ module but not irreducible

If $\mathcal{M} = \mathcal{A}$ is regular, then $\mathcal{A}$-submodules = (left) ideals of $\mathcal{A}$.

Fourier transform = decomposition into irreducible modules
(= spectral components)

picture:



$\mathcal{M}_\omega$, $\omega \in W$, irreducible

FT $\Delta: \mathcal{M} \longrightarrow \bigoplus_{\omega \in W} \mathcal{M}_\omega$

$\quad s \longmapsto (s_\omega)_{\omega \in W}$ $\qquad$ so FT is a <u>base change</u>

- discrete time:

basis in $\mathcal{M}$: $\{z^{-n}\}_{n \in \mathbb{Z}}$ $\qquad$ basis in $\bigoplus \mathcal{M}_\omega$: $\{\sum e^{j\omega n} z^{-n}\}_{\omega \in [-\pi, \pi)}$

coordinate-free:

$$\Delta: \ s = S(z) \longmapsto \left(S(e^{j\omega}) \cdot \sum e^{j\omega n} z^{-n}\right)_{\omega \in [-\pi,\pi)}$$

in coordinates:

$$\mathcal{F}: \ \hat{s} \longmapsto \left(S(e^{j\omega})\right)_{\omega \in [-\pi,\pi)} = \omega \mapsto S(e^{j\omega})$$

(function on $[-\pi,\pi)$)

Δ is an $\mathcal{A}$-module homomorphism.

Irreducible representations = frequency response

Each irred. $\mathcal{M}_\omega$ with chosen basis b_ω affords an "irreducible" representation φ_ω. The collection $(\varphi_\omega(h))_{\omega \in W}$ for $h \in \mathcal{A}$ is the frequency response of h. Clearly, $\varphi_\omega(h)$ is a $\dim(\mathcal{M}_\omega) \times \dim(\mathcal{M}_\omega)$ matrix for all ω.

- discrete time: $\mathcal{M}_\omega = \langle \sum e^{j\omega n} z^{-n} \rangle$, $\dim = 1$

$$h = H(z): \quad H(z) \cdot \left(\sum e^{j\omega n} z^{-n}\right) = \underbrace{H(e^{j\omega})}_{\varphi_\omega(h)} \cdot \sum e^{j\omega n} z^{-n}$$

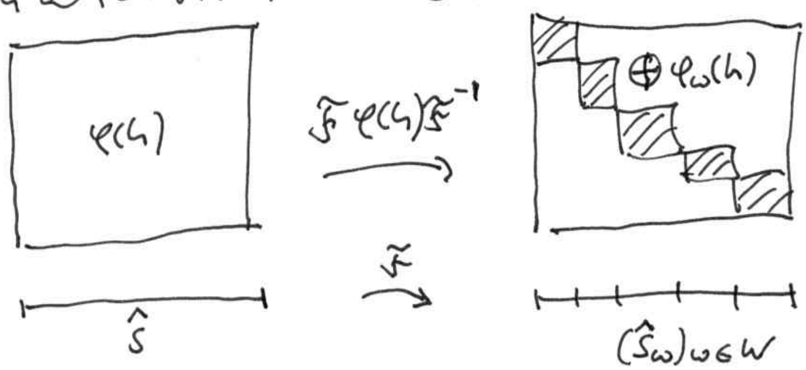
frequ. response: $\left(H(e^{j\omega})\right)_{\omega \in [-\pi,\pi)} = \omega \longmapsto H(e^{j\omega})$

so: coord. free $\qquad$ in coordinates

$$h \cdot s_\omega \longleftrightarrow \varphi_\omega(h) \cdot \hat{s}_\omega$$

$\in \mathcal{A} \quad \in \mathcal{M}_\omega$

diagonalization: assume $\dim(\mathcal{M}) < \infty$



$\tilde{\mathcal{F}}$ (block) diagonalizes all $\varphi(h)$.

<u>Summary</u>: Table: SP concepts, algebraic concepts (coord. free), in coordinates

<u>Signal model</u>

- motivation: signals $\hat{s}$ usually $\in \mathbb{C}^I$
- definition
- regular/finite etc. signal model
- usual form: $\hat{s} \mapsto s = \sum s_n b_n$
- discrete time: z-transform

<u>Shift</u>

- special filter $x \in \mathcal{A}$
  (discrete time: $x = z^{-1}$)
- filters: polynomials or series in x

shift(s) = chosen generators of $\mathcal{A}$

1 shift $\Leftrightarrow$ 1-D SP, 2 shifts $\Leftrightarrow$ 2-D SP, ...

- Note: determine $\langle x\rangle_{grp}$, $\langle x\rangle_{vs}$, $\langle x\rangle_{ring}$, $\langle x\rangle_{algebra}$ and $\langle x,y\rangle_{\dots}$

Shift invariance

- formally: $x \cdot h = h \cdot x$ for all $h \in \mathcal{A}$ and shifts x

shift-invariant signal model
$\iff$ $\mathcal{A}$ is commutative

- which $\mathcal{A}$ are commutative? (+ finite number of shifts)

  $\dim(\mathcal{A}) = \infty$: spaces of series

  $\dim(\mathcal{A}) < \infty$: polynomial algebras $\mathbb{C}[x_1, \dots, x_n]/I$

  + 1-D = one shift : $\mathbb{C}[x]/p(x)$.

Visualization

Operation of the shift on the chosen basis of $\mathcal{M}$, visualized as graph.

- discrete time: $b = \{z^{-n}\}_{n \in \mathbb{Z}}$, $x = z^{-1}$

$$\dots \underset{z^{-1}}{\bullet} \longrightarrow \underset{z^{0}}{\bullet} \longrightarrow \underset{z^{1}}{\bullet} \longrightarrow \underset{z^{2}}{\bullet} \dots$$

= structure imposed on signal