recap: Cartesian products and CRT

HW 2

Coordinates

coordinate vectors

Let $\dim V = n$ be an $\mathbb{F}$-vector space, choose a basis $b = \{b_0, \dots, b_{n-1}\}$. So $V = \langle b_0, \dots, b_{n-1} \rangle_{VS} = \mathbb{F} b_0 + \dots + \mathbb{F} b_{n-1}$.

Then every $v \in V$ has a unique representation $v = \sum_{k=0}^{n-1} \alpha_k b_k, \alpha_k \in \mathbb{F}$

$$\begin{aligned} \varphi : V &\to \mathbb{F}^n \\ v &\mapsto \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} = \hat{v} = \varphi(v) \end{aligned}$$

is a bijective linear mapping (VS isom.).

We call $\hat{v}$ the "<u>coordinate vector</u> of v w.r.t. b."

change of basis

assume a different basis $c = \{c_0, \dots, c_{n-1}\}$ of V

given $\hat{v}$ (coord. vector w.r.t. b) we want to compute $\hat{\hat{v}}$ (coord. vector w.r.t. c)

- express each $b_k \in b$ in the basis c.

$$b_k = \sum_{\ell=0}^{n-1} \mu_{k\ell} c_\ell, \quad \text{i.e.} \quad \hat{\hat{b}}_k = \begin{pmatrix} \mu_{k0} \\ \vdots \\ \mu_{k,n-1} \end{pmatrix}$$

- insert in b-expansion of v

$$v = \sum_k \alpha_k b_k = \sum_k \alpha_k \left(\sum_\ell \mu_{k\ell} c_\ell \right) = \sum_\ell \underbrace{\left(\sum_k \alpha_k \mu_{k\ell} \right)}_{\text{coordinates w.r.t. } c} c_\ell$$

$$\Rightarrow \hat{\hat{v}} = M \cdot \hat{v}, \quad M = M_{b \to c} = [\mu_{k\ell}]_{0 \le k, \ell < n} \quad (k \text{ is row index})$$

as commutative diagram:

$$\begin{tikzcd} & \mathbb{F}^n \arrow[dd, "M"] \\ V \arrow[ur, "\hat{(\cdot)}"] \arrow[dr, "\hat{\hat{(\cdot)}}"'] \arrow[r, phantom, "///"] & \\ & \mathbb{F}^n \end{tikzcd}$$

$\boxed{\text{M has to be invertible}}$

easy to see:

$$M_{c \to b} = M_{b \to c}^{-1}$$

$$M_{c \to d} M_{b \to c} = M_{b \to d}$$

$\boxed{\text{base change: express old basis in new basis and put the resulting column vectors into the columns of } M.}$

example: $V = \mathbb{R}_1[x] = \{\alpha_0 + \alpha_1 x \mid \alpha_0, \alpha_1 \in \mathbb{R}\}$, $b = \{1, x\}$

$v = 1 + 2x$, $\hat{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

new basis: $c = \{1+x, 1-x\}$

express b in c:

$$\left.\begin{aligned} 1 &= \tfrac{1}{2}(1+x+1-x) \\ x &= \tfrac{1}{2}(1+x-(1-x)) \end{aligned}\right\} \Rightarrow M_{b\to c} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix}$$

[easier: compute $M_{c\to b} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and invert.]

$$\text{so } \hat{v} = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 3/2 \\ -1/2 \end{pmatrix}$$

check: $\frac{3}{2}(1+x) - \frac{1}{2}(1-x) = 1 + 2x = v$ ✓

coordinatization of linear mappings

Let $\varphi : V \to W$ be a linear mapping, $\dim V = n$, $\dim W = m$.
Choose bases $b = \{b_0, \dots, b_{n-1}\}$ of V and $c = \{c_0, \dots, c_{m-1}\}$ of W.
We want to express φ on the coordinate vectors:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\hat{(\cdot)}"'] & W \arrow[d, "\hat{(\cdot)}"] \\ \mathbb{F}_n \arrow[r, "?"] & \mathbb{F}_m \end{tikzcd}$$

i.e.
$\varphi(v) = w \iff ?(\hat{v}) = \hat{w}$

- compute $\widehat{\varphi(b_k)}$:
$$\varphi(b_k) = \sum_{l=0}^{m-1} \beta_{kl} c_l$$
- $$\varphi(v) = \varphi\left(\sum_{k=0}^{n-1} \alpha_k b_k\right) = \sum_{k=0}^{n-1} \alpha_k \varphi(b_k)$$
$$= \sum_k \alpha_k \left(\sum_l \beta_{kl} c_l\right)$$
$$= \sum_l \underbrace{\left(\sum_k \beta_{kl} \alpha_l\right)}_{\text{coordinates of } \varphi(v)} c_l$$

$\Rightarrow \quad \widehat{\varphi(v)} = B\,\hat{v}, \quad B = [\beta_{k,l}]_{\substack{0\le k<m \\ 0\le l<n}} \in \mathbb{F}^{m\times n}$

B is the matrix representation of φ w.r.t. the bases b of V and c of W

$\boxed{\text{matrix for } \varphi\text{: map basis of } V \text{ and express in basis of } W. \text{ The resulting column vectors go into the columns of } B.}$

comm. diagram:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "\hat{(\cdot)}"'] \arrow[dr, phantom, "///"] & W \arrow[d, "\hat{(\cdot)}"] \\ \mathbb{F}_n \arrow[r, "B"] & \mathbb{F}_m \end{tikzcd}$$

(if $V = W$ then usually the same basis is chosen)

example:

$V = \mathbb{R}_2[x] = \{\alpha_0 + \alpha_1 x + \alpha_2 x^2 \mid \alpha_i \in \mathbb{R}\}$, $\varphi =$ derivative

$b = \{1, x, x^2\}$

– map basis:

$\varphi: \quad 1 \mapsto 0, \quad \hat{0} = \begin{pmatrix}0\\0\\0\end{pmatrix}$

$x \mapsto 1, \quad \hat{1} = \begin{pmatrix}1\\0\\0\end{pmatrix}$

$x^2 \mapsto 2x, \quad \widehat{2x} = \begin{pmatrix}0\\2\\0\end{pmatrix}$

$\Rightarrow \quad B = \begin{pmatrix}0&1&0\\0&0&2\\0&0&0\end{pmatrix}$

change of basis $M_{b\to c}$ can be expressed as

$$\begin{tikzcd} V \arrow[r, "\text{id. map.}"] \arrow[d, "\hat{(\cdot)}"'] \arrow[dr, phantom, "///"] & V \arrow[d, "\hat{\hat{(\cdot)}}"] \\ \mathbb{F}_n \arrow[r, "M_{b\to c}"'] & \mathbb{F}_n \end{tikzcd}$$

id. map: $v \to v$

easy to see: a.) if $\varphi \longleftrightarrow B$ (w.r.t. bases b, c) and φ bijective

$\Rightarrow \varphi^{-1} \longleftrightarrow B^{-1}$

~~$\varphi \longleftrightarrow B$ (w.r.t. b,c) $\psi: W \to X$ (w.r.t. c,d) $\Rightarrow \psi\circ\varphi \longleftrightarrow$~~

diagram: (φ bijective)

$$
\begin{tikzcd}
V \arrow[r, "\varphi"] \arrow[d, "\hat{(\cdot)}"'] & W \arrow[d, "\hat{(\cdot)}"] \\
\mathbb{F}^n \arrow[r, "B"'] & \mathbb{F}^n
\end{tikzcd}
\quad \Longrightarrow \quad
\begin{tikzcd}
V \arrow[d, "\hat{(\cdot)}"'] & W \arrow[l, "\varphi^{-1}"'] \arrow[d, "\hat{(\cdot)}"] \\
\mathbb{F}^n & \mathbb{F}^m \arrow[l, "B^{-1}"]
\end{tikzcd}
$$

b.) $\underset{\text{basis } b}{V} \xrightarrow{\varphi} \underset{\text{basis } c}{W}$, $\psi : \underset{\text{basis } c}{W} \longrightarrow \underset{\text{basis } d}{U}$

and $\varphi \longleftrightarrow B$, $\psi \longleftrightarrow C$,

$\Longrightarrow$ $\psi \circ \varphi : \underset{\text{basis } b}{V} \longrightarrow \underset{\text{basis } d}{U} \longleftrightarrow CB$

diagram:

$$
\begin{tikzcd}
V \arrow[r, "\varphi"] \arrow[d] \arrow[rr, bend left, "\psi \circ \varphi"] & W \arrow[r, "\psi"] \arrow[d] & U \arrow[d] \\
\mathbb{F}^n \arrow[r, "B"] \arrow[rr, bend right, "CB"'] & \mathbb{F}^m \arrow[r, "C"] & \mathbb{F}^k
\end{tikzcd}
$$

Change of basis for linear mappings

Consider only $\varphi : V \to V$. Let b be a basis of V and $\varphi \longleftrightarrow B$ w.r.t. this basis. ($B \in \mathbb{F}^{n \times n}$)

We want to change the basis $b \to c$ and see what happens to B.

$$
\begin{tikzcd}
V \arrow[d, "\hat{\hat{(\cdot)}}"'] & V \arrow[l, "\text{id.}"'] \arrow[r, "\varphi"] \arrow[d, "\hat{(\cdot)}"'] & V \arrow[r, "\text{id.}"] \arrow[d, "\hat{(\cdot)}"] & V \arrow[d, "\hat{\hat{(\cdot)}}"] \\
\mathbb{F}_n \arrow[rrr, bend right, "M_{b\to c} B M_{b\to c}^{-1}"'] & \mathbb{F}_n \arrow[l, "M_{b\to c}"] \arrow[r, "B"'] & \mathbb{F}_n \arrow[r, "M_{b\to c}"'] & \mathbb{F}_n
\end{tikzcd}
$$

result:

$$C = M_{b\to c} \, B \, M_{b\to c}^{-1}$$
(φ w.r.t. c)
$$= M_{c\to b}^{-1} \, B \, M_{c\to b}$$
(φ w.r.t. b)

(sometimes called: B "conjugated" with $M_{c\to b}$)

Motivates: relation on $\mathbb{F}^{n\times n}$

$B \sim C \iff B, C$ "belong to" the same φ

$\iff C = M^{-1} B M$ for some $M \in GL_n(\mathbb{F})$

- this is an equivalence relation
- equ. classes are sometimes called "conjugacy classes"

<u>On Eigenvectors and Eigenvalues</u>

Let $\varphi : V \to V$, b a basis of V $\Rightarrow B = \left(\widehat{\varphi(b_0)} \; \widehat{\varphi(b_1)} \; \dots \; \widehat{\varphi(b_{n-1})} \right)$

e.g. $\varphi(b_0) = \sum_{k=0}^{n-1} \alpha_k b_k$, $\widehat{\varphi(b_0)} = \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{n-1} \end{pmatrix}$

Goal: find a basis b s.t. B is "easiest"

e.g. $\varphi(b_0) = \underset{\substack{\uparrow \\ \text{Eigen} \\ \text{value} \\ \text{of } \varphi}}{\alpha_0} \; \underset{\substack{\nwarrow \\ \text{corresponding} \\ \text{Eigenvector}}}{b_0}$, $\widehat{\varphi(b_0)} = \begin{pmatrix} \alpha_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow B = \begin{pmatrix} \alpha_0 & \\ 0 & \\ \vdots & * \\ 0 & \end{pmatrix}$

- if b is a basis of Eigenvectors then B is diagonal, diagonal elements = Eigenvalues
  Such a basis does not always exist.
- Compute Eigenvalues:

$$p_A(\lambda) = \det(A - \lambda \underset{\text{identity matrix}}{I})$$ "characteristic polynomial"

  Eigenvalues = zeros of p

- Compute Eigenvector for Eigenvalue λ:

  solve $(A - \lambda I)x = 0$

- $A = M^{-1} B M \Rightarrow p_A = p_B$

- Theorem of Cayley-Hamilton:

$$p_A(A) = 0$$

- Smallest polynomial (by degree) that annihilates A is called "minimal polynomial": m_A

  Necessarily:
  - $m_A \mid p_A$
  - α a zero of $p_A \Rightarrow \alpha$ a zero of m_A
  - p_A pairwise distinct zeros (i.e. no multiple zeros) $\Rightarrow m_A = p_A$

  Easy to check: $\{ p(x) \in \mathbb{F}[x] \mid p(A) = 0 \} \trianglelefteq \mathbb{F}[x]$

$$\{ p(x) \in \mathbb{F}[x] \mid p(A) = 0 \} = m_A[x] \cdot \mathbb{F}[x] \quad \text{(principal ideal)}$$

coordinate-free (basis independent)	coordinates (chosen basis)
$v \in V$	$\hat{v} \in \mathbb{F}^n$
$\varphi : V \to W$	$B \in \mathbb{F}^{m \times n}$
	base change: $M^{-1} B N$
$\varphi(V)$	$\langle \text{columns of } B \rangle_{vs}$
$\ker \varphi$	solution of $B\hat{v} = 0$ (= Eigenspace for Eigenvalue 0)
$\dim \varphi(V)$	$\operatorname{rank} B$
$\dim \ker \varphi$	nullity or defect of B ($\operatorname{null} B$ or $\operatorname{def} B$)
Eigenvector/value $\varphi(v) = \lambda v$	$B\hat{v} = \lambda \hat{v} \Leftrightarrow (B - \lambda I)\hat{v} = 0$