

recap: Cartesian products and CRT

HW 2

Coordinates coordinate vectors

Let $\dim V = n$ be an \mathbb{F} -vector space, choose a basis $b = \{b_0, \dots, b_{n-1}\}$. So $V = \langle b_0, \dots, b_{n-1} \rangle_{\mathbb{F}} = \mathbb{F}b_0 + \dots + \mathbb{F}b_{n-1}$.

Then every $v \in V$ has a unique representation $v = \sum_{k=0}^{n-1} \alpha_k b_k, \alpha_k \in \mathbb{F}$

$$\varphi: V \rightarrow \mathbb{F}^n$$
$$v \mapsto \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{n-1} \end{pmatrix} = \hat{v} = \varphi(v)$$

is a bijective linear mapping (VS isom.).

We call \hat{v} the "coordinate vector of v w.r.t. b ."

change of basis

assume a different basis $c = \{c_0, \dots, c_{n-1}\}$ of V
given \hat{v} (coord. vector w.r.t. b) we want to compute $\hat{\hat{v}}$ (coord. vector w.r.t. c)

- express each $b_k \in b$ in the basis c .

$$b_k = \sum_{\ell=0}^{n-1} \mu_{k\ell} c_\ell, \text{ i.e. } \hat{b}_k = \begin{pmatrix} \mu_{k0} \\ \vdots \\ \mu_{k,n-1} \end{pmatrix}$$

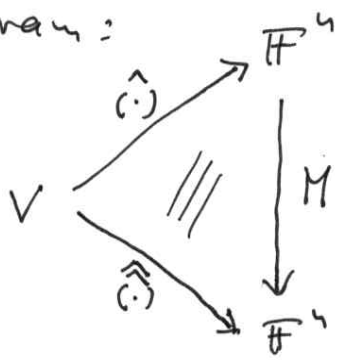
- insert in b -expansion of v

$$v = \sum_k \alpha_k b_k = \sum_k \alpha_k \left(\sum_\ell \mu_{k\ell} c_\ell \right)$$
$$= \sum_\ell \left(\sum_k \alpha_k \mu_{k\ell} \right) c_\ell$$

coordinates w.r.t. c

$$\Rightarrow \hat{\hat{v}} = M \cdot \hat{v}, \quad M = M_{b \rightarrow c} = [\mu_{k\ell}]_{0 \leq k, \ell < n} \quad (k \text{ is row index})$$

as commutative diagram:



It has to be invertible

easy to see:

$$M_{c \rightarrow b} = M_{b \rightarrow c}^{-1}$$

$$M_{c \rightarrow d} M_{b \rightarrow c} = M_{b \rightarrow d}$$

base change: express old basis in new basis and put the resulting column vectors into the columns of Π .

example: $V = \mathbb{R}_1[x] = \{ \alpha_0 + \alpha_1 x \mid \alpha_0, \alpha_1 \in \mathbb{R} \}$, $S = \{1, x\}$

$$v = 1 + 2x, \hat{v} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\text{new basis: } c = \{1+x, 1-x\}$$

express S in c :

$$1 = \frac{1}{2}(1+x + 1-x)$$

$$x = \frac{1}{2}(1+x - (1-x))$$

$$\Rightarrow \Pi_{S \rightarrow c} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

[easier: compute $\Pi_{c \rightarrow S} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ and invert.]

$$\text{so } \hat{v} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} \\ -\frac{1}{2} \end{pmatrix}$$

$$\text{check: } \frac{3}{2}(1+x) - \frac{1}{2}(1-x) = 1+2x = v$$

Coordinateization of linear mappings

Let $\varphi: V \rightarrow W$ be a linear mapping, $\dim V = n$, $\dim W = m$.
Choose bases $S = \{s_0, \dots, s_{n-1}\}$ of V and $c = \{c_0, \dots, c_{m-1}\}$ of W .

We want to express φ on the coordinate vectors:

$$\begin{array}{ccc} V & \xrightarrow{\varphi} & W \\ \hat{(\cdot)} \downarrow & & \downarrow \hat{(\cdot)} \\ \mathbb{F}_n & \xrightarrow{\hat{\varphi}} & \mathbb{F}_m \end{array}$$

i.e.

$$\varphi(v) = w \Leftrightarrow \hat{\varphi}(\hat{v}) = \hat{w}$$

- compute $\hat{\varphi}(s_k)$:

$$\varphi(s_k) = \sum_{e=0}^{m-1} \beta_{ke} c_e$$

$$\varphi(v) = \varphi\left(\sum_{k=0}^{n-1} \alpha_k s_k\right) = \sum_{k=0}^{n-1} \alpha_k \varphi(s_k)$$

$$= \sum_k \alpha_k \left(\sum_e \beta_{ke} c_e \right)$$

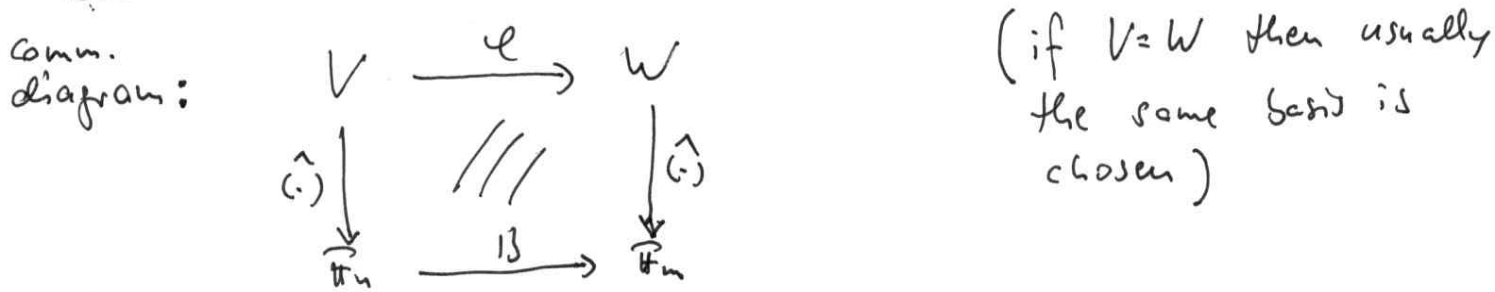
$$= \sum_e \left(\sum_k \beta_{ke} \alpha_k \right) c_e$$

coordinates of $\varphi(v)$

$$\Rightarrow \widehat{\varphi}(v) = B \widehat{v}, \quad B = [\beta_{ij}]_{\substack{0 \leq i < m \\ 0 \leq j < n}} \in \mathbb{F}^{m \times n}$$

B is the matrix representation of φ w.r.t. the bases b of V and c of W

matrix for φ : map basis of V and express in basis of W .
The resulting column vectors go into the columns of B .

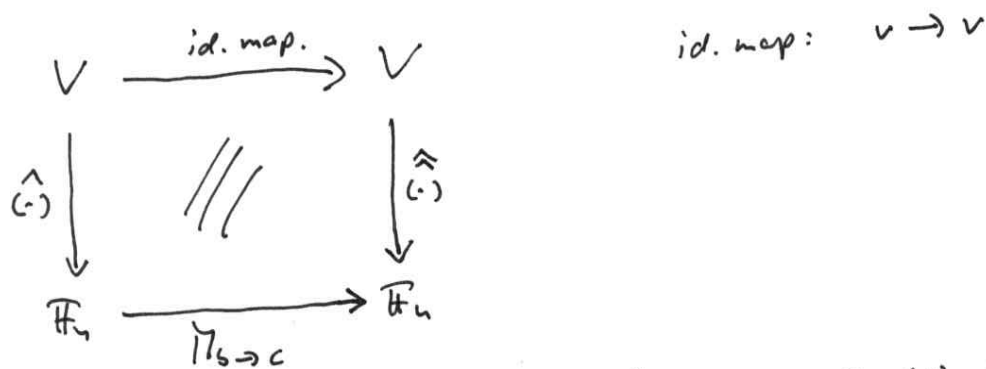


example:
 $V = \mathbb{R}_2[x] = \{ \alpha_0 + \alpha_1 x + \alpha_2 x^2 \mid \alpha_i \in \mathbb{R} \}$, $\varphi = \text{derivative}$
 $b = \{1, x, x^2\}$

- map basis:

$$\begin{aligned} \varphi: 1 &\mapsto 0, & \widehat{0} &= \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\ x &\mapsto 1, & \widehat{1} &= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ x^2 &\mapsto 2x, & \widehat{2x} &= \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \end{aligned} \Rightarrow B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

change of basis $M_{b \rightarrow c}$ can be expressed as

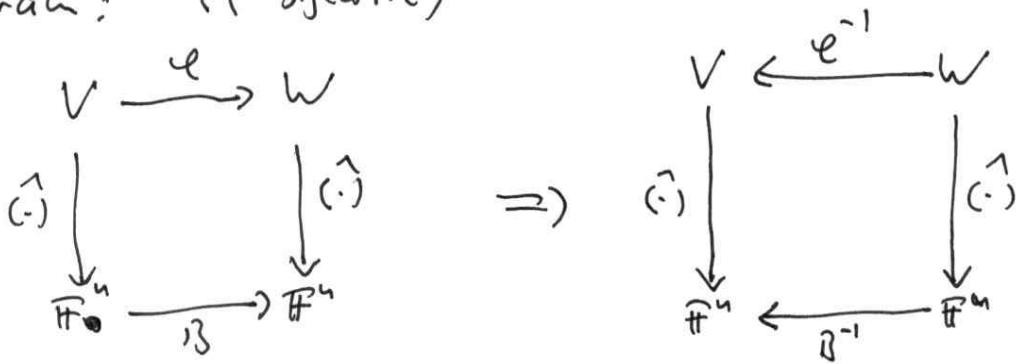


easy to see: a.) if $\varphi \leftrightarrow B$ (w.r.t. bases b, c) and φ bijective

$$\Rightarrow \varphi^{-1} \leftrightarrow B^{-1}$$

~~$\varphi \leftrightarrow B$ (w.r.t. b, c) $\varphi: W \rightarrow X$ (w.r.t. c, d)~~

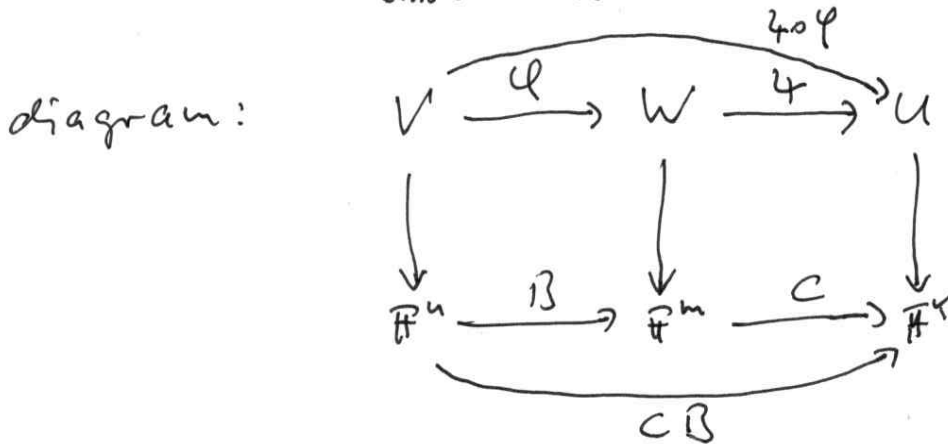
diagram: (φ bijective)



b.) $\varphi: V \rightarrow W$, $\psi: W \rightarrow U$
 basis b basis c basis c basis d

and $\varphi \leftrightarrow \beta$, $\psi \leftrightarrow \gamma$

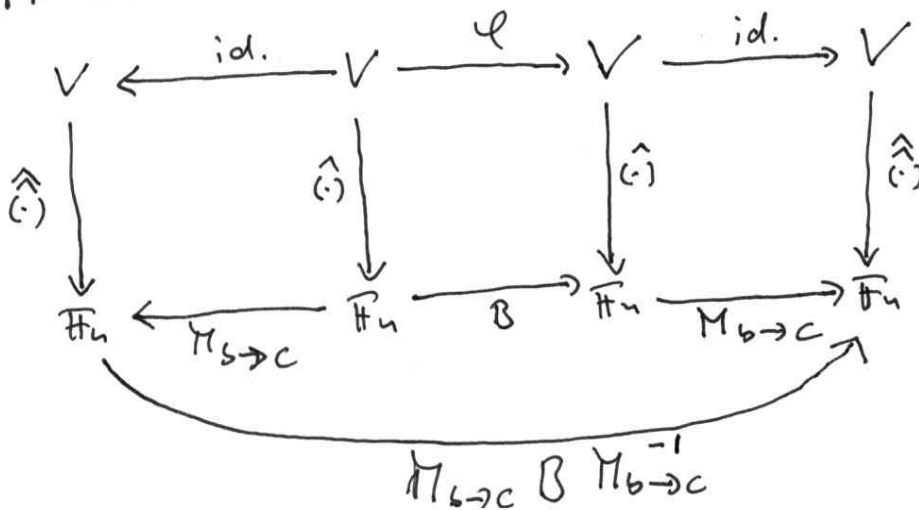
$\Rightarrow \psi \circ \varphi: V \rightarrow U \leftrightarrow \gamma \beta$
 basis b basis d



Change of basis for linear mappings

Consider only $\varphi: V \rightarrow V$, let b be a basis of V
 and $\varphi \leftrightarrow \beta$ w.r.t. this basis, ($\beta \in \mathbb{F}^{n \times n}$)

We want to change the basis $b \rightarrow c$ and see what happens to β .



result:

$$\begin{aligned}
 (\varphi \text{ w.r.t. } \hat{c}) \quad C &= M_{b \rightarrow c} \cdot B \cdot M_{b \rightarrow c}^{-1} \\
 &= M_{c \rightarrow b}^{-1} \cdot B \cdot M_{c \rightarrow b} \quad (\varphi \text{ w.r.t. } b)
 \end{aligned}$$

(sometimes called: B
"conjugated" with $M_{c \rightarrow b}$)

Motivates: relation on $\mathbb{F}^{n \times n}$

$B \sim C \Leftrightarrow B, C$ "belong to" the same φ

$$\Leftrightarrow C = \Pi^{-1} B \Pi \quad \text{for some } \Pi \in GL_n(\mathbb{F})$$

- this is an equivalence relation

- equ. classes are sometimes called "conjugacy classes"

On Eigenvectors and Eigenvalues

Let $\varphi: V \rightarrow V$, b a basis of $V \Rightarrow B = (\varphi(\hat{b}_0) \varphi(\hat{b}_1) \dots \varphi(\hat{b}_{n-1}))$

$$\text{e.g. } \varphi(b_0) = \sum_{k=0}^{n-1} \alpha_k b_k, \quad \varphi(\hat{b}_0) = \begin{pmatrix} \alpha_0 \\ \vdots \\ \alpha_{n-1} \end{pmatrix}$$

Goal: find a basis b s.t. B is "easiest"

$$\text{e.g. } \varphi(b_0) = \lambda_0 b_0, \quad \varphi(\hat{b}_0) = \begin{pmatrix} \lambda_0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow B = \begin{pmatrix} \lambda_0 & & \\ & \ddots & \\ & & 0 \end{pmatrix} *$$

Eigenvalue of φ correspondingly Eigenvector

- if b is a basis of Eigenvectors then B is diagonal, diagonal elements = Eigenvalues
Such a basis does not always exist.

- Compute Eigenvalues: \leftarrow identity matrix

$$p_A(\lambda) = \det(A - \lambda I) \quad \text{"characteristic polynomial"}$$

Eigenvalues = zeros of p

- Compute Eigenvector for Eigenvalue λ :

$$\text{solve } (A - \lambda I)x = 0$$

$$- A = \Pi^{-1} B \Pi \Rightarrow P_A = P_B$$

- Theorem of Cayley-Hamilton:

$$p_A(A) = 0$$

- Smallest polynomial (by degree) that annihilates A is called "minimal polynomial": m_A

Necessarily: $m_A \mid p_A$

- α a zero of $p_A \Rightarrow \alpha$ a zero of m_A
- p_A pairwise distinct zeros (i.e. no multiple zeros) $\Rightarrow m_A = p_A$

Easy to check: $\{p(x) \in \mathbb{F}[x] \mid p(A) = 0\} \cong \mathbb{F}[x]$

$m_A \cdot \mathbb{F}[x]$ (principal ideal)

coordinate-free
(basis independent)

coordinates
(chosen basis)

$$v \in V$$

$$\varphi: V \rightarrow W$$

$$\varphi(v)$$

$$\ker \varphi$$

$$\dim \varphi(V)$$

$$\dim \ker \varphi$$

$$\text{Eigenvector/value } \varphi(v) = \lambda v$$

$$\hat{v} \in \mathbb{F}^n$$

$$B \in \mathbb{F}^{n \times n}$$

$$\text{base change: } B^{-1}BN$$

< columns of $B \rangle_{vs}$

solution of $B\hat{v} = 0$
(= Eigenspace for Eigenvalue 0)

$$\text{rank } B$$

nullity or defect of B
(null B or def B)

$$B\hat{v} = \lambda\hat{v} \Leftrightarrow (B - \lambda I)\hat{v} = 0$$