Bits of HW 1

Recap: Vector spaces, generating sets (basis, linear dependence), dimension
subspaces, factor spaces

Theorem: $U, V \leq W$ vector spaces.

a.) $U+V$ is again a VS

b.) $U \cap V$ is "

c.) $\dim(U+V) = \dim U + \dim V - \dim(U \cap V)$

visualization (careful):

proof: a.) to show: $x, y \in U+V$, $\alpha, \beta \in \mathbb{F} \Rightarrow \alpha x + \beta y \in U+V$

$x = u+v$, $y = u'+v' \Rightarrow \alpha x + \beta y = \underbrace{(\alpha u + \beta u')}_{\in U} + \underbrace{(\alpha v + \beta v')}_{\in V} \in U+V$ ✓

c.) idea: start with a basis of $U \cap V$, extend to basis of U, and to basis of V.
($\dim W < \infty$) and count.

Definition: If $U \cap V = \{0\}$ we write $U + V = U \oplus V$
and call it the (inner) direct sum of U and V.

$\dim(U \oplus V) = \dim U + \dim V$.

Lemma: $W = U \oplus V \Leftrightarrow$ every $x \in W$ has a unique decomposition
$x = u + v$, $u \in U$, $v \in V$

Examples:

a.) $V = \mathbb{R}^3 = \langle e_1 \rangle \oplus \langle e_2 \rangle \oplus \langle e_3 \rangle$

b.) $V = \mathbb{R}[x] = \langle 1, x \rangle_{VS} \oplus x^2 \mathbb{R}[x]$

c.) $V = \mathbb{R}^3$, $U = \langle e_1, e_2 \rangle$ $\quad \dim U = 2$

$U' = \langle \begin{pmatrix}1\\0\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\end{pmatrix} \rangle$ $\quad \dim U' = 2$

- $V = U + U'$ since $e_3 = \begin{pmatrix}1\\0\\1\end{pmatrix} - e_1$, $e_2, e_1 \in V$

$\Rightarrow \dim(U \cap U') = \dim U + \dim U' - \dim V = 1$

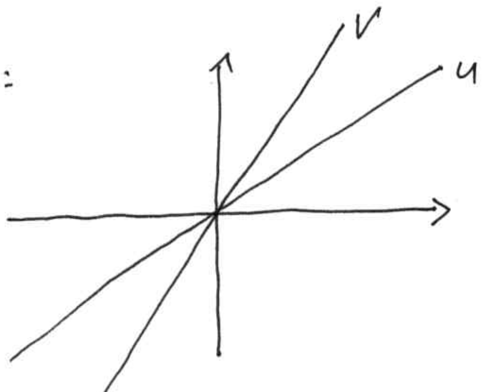
so ~~that~~ the sum is not direct.

indeed: $\begin{pmatrix}1\\-1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix} - \begin{pmatrix}0\\1\\1\end{pmatrix} = e_1 - e_2 \in U \cap V$.

d.) $\mathbb{F}((x)) = \{ \sum_{n \geq N} a_n x^n \mid a_n \in \mathbb{F},\ N \in \mathbb{Z} \}$ "truncated Laurent series"

$= \mathbb{F}[-x] \oplus x\mathbb{F}[[x]]$

e.) $\mathbb{R}^2$:



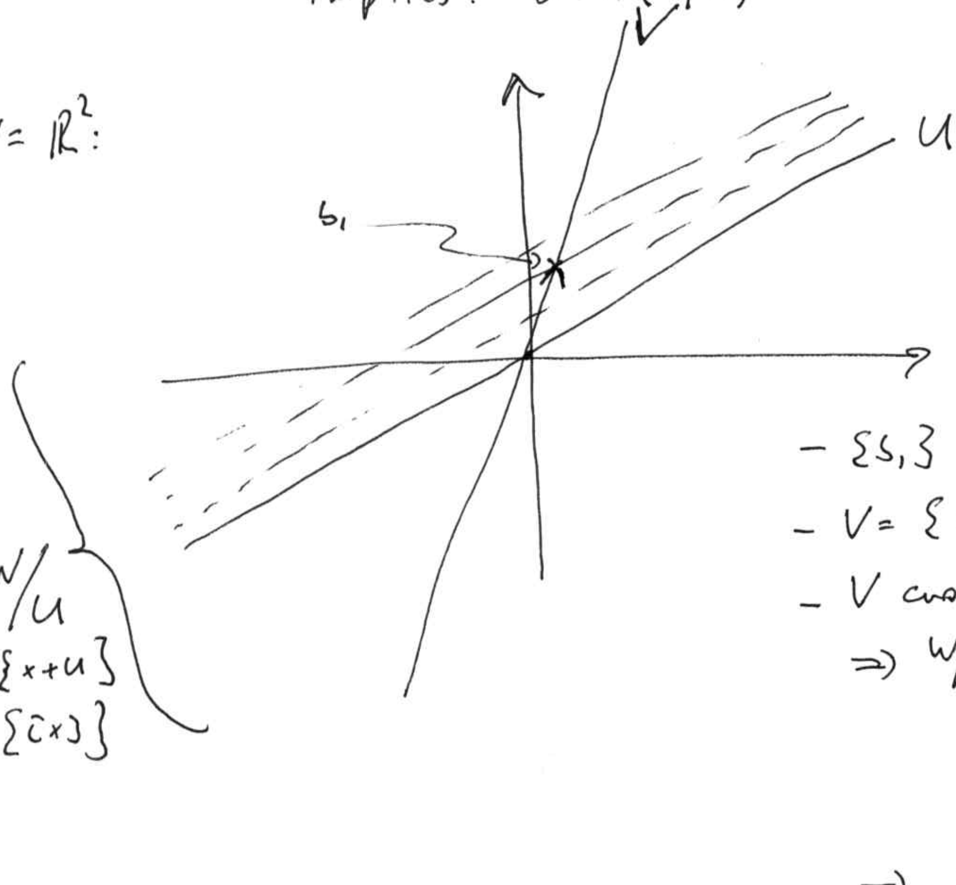
$\mathbb{R}^2 = U \oplus V$

f.) $\mathbb{R}^3$, $U, V \leq \mathbb{R}^3$ planes $\Rightarrow$ never a direct sum

assume $\dim(W) < \infty$

- $U \leq W$: we can always find V s.t. $W = U \oplus V$
- <u>Lemma</u>: ~~bla basis~~ if $b = \{b_1, \dots, b_n\}$ basis of V then $\{[b_1], \dots, [b_n]\}$ is a basis of W/U, which implies: $\dim(W/U) = \dim W - \dim U = \dim V$

$W = \mathbb{R}^2$:



W/U
$= \{x + U\}$
$= \{[x]\}$

- $\{b_1\}$ is a basis of V
- $V = \{\alpha b_1 \mid \alpha \in \mathbb{R}\}$
- V crosses all equ. classes $x + U$

$\Rightarrow W/U = \{\alpha b_1 + U \mid \alpha \in \mathbb{F}\}$

$= \{[\alpha b_1] \mid \alpha \in \mathbb{F}\}$

$= \langle [b_1] \rangle_{VS}$

$\Rightarrow \{[b_1]\}$ is a basis of W/U

proof: Let $b = \{b_1, \dots, b_n\}$ be a basis of V.

- $\{[b_1], \dots, [b_n]\}$ lin. indep.:

  let $\alpha_1 [b_1] + \dots + \alpha_n [b_n] = [0]$

  $\Rightarrow [\sum \alpha_i b_i] = [0]$

  $\Rightarrow \sum \alpha_i b_i \in U$ but also $\in V$

  $\Rightarrow \sum \alpha_i b_i = 0$ (since $W = U \oplus V$)

  $\Rightarrow$ all $\alpha_i = 0$. ✓

- $\{[b_1], \dots, [b_n]\}$ generates W/U

  let $[x] \in W/U$ i.e. x is any element $\in W$

  $\Rightarrow x = u + v$, $u \in U$, $v \in V$

  $\Rightarrow x = u + \sum_{i=1}^{n} \alpha_i b_i$

  $\Rightarrow [x] = [\quad " \quad] = [\sum_{i=1}^{n} \alpha_i b_i] = \sum_{i=1}^{n} \alpha_i [b_i]$ ✓

$\boxed{\text{Homomorphisms}}$:

Definition: V, W $\mathbb{F}$-vector spaces. $\varphi: V \to W$ is called a "vector space hom." or "linear mapping" if

$$\varphi(\alpha x + \beta y) = \alpha \varphi(x) + \beta \varphi(y) \quad \text{for } \alpha, \beta \in \mathbb{F}, \; x, y \in V$$

Further, $\ker \varphi = \{x \in V \mid \varphi(x) = 0\}$ is the "kernel of φ".

If φ is bijective it is called VS isomorphism, V, W are then called isomorphic: $V \cong W$.

Lemma:

a.) $\ker \varphi \leq V$

b.) φ injective $\Leftrightarrow \ker \varphi = \{0\} \Leftrightarrow \dim(\ker \varphi) = 0$

c.) $\varphi(V) \leq W$ d.) φ isom. $\Rightarrow \dim V = \dim W$

Theorem (hom. theorem)

$$V/_{\ker \varphi} \cong \varphi(V)$$

In particular: $\dim V = \dim(\ker \varphi) + \dim(\varphi(V))$

Examples:

a.) $\varphi : V \to V,\ x \mapsto x$ (identity mapping)

$\ker\varphi = \{0\},\ \varphi(V) = V$

b.) $\varphi : \mathbb{F}[x] \to \{(\alpha_0, \alpha_1, \alpha_2, \dots) \mid \alpha_i \in \mathbb{F}\} = \mathbb{F}^{\mathbb{N}_0}$

$\sum_{i=0}^{n} \alpha_i x^i \mapsto (\alpha_0, \dots, \alpha_n, 0, 0, \dots)$

$\ker\varphi = \{0\},\ \varphi(\mathbb{F}[x]) =$ all series that are eventually 0

c.) $\varphi : \mathbb{F}^{n\times n} \to \mathbb{F}^{n^2}$

$A = [a_{ke}] \mapsto \varphi(A) = (a_{00}, a_{01}, a_{02}, \dots)^T$

(putting matrix into a vector in row-major order)

is isom.

d.) $\dim(V) = n$, $b = \{b_1, \dots, b_n\}$ a basis

$\varphi : V \to \mathbb{F}^n$

$\sum \alpha_i b_i \mapsto (\alpha_1, \dots, \alpha_n)^T$

is isomorphism

$$\boxed{\dim V = n \implies V \cong \mathbb{F}^n}$$

$$\boxed{\dim V = \dim W \implies V \cong W}$$