18-799 Algorithms and Computation in Signal Processing
Spring 2005
Assignment 3 - Solution

1. Determine the arithmetic cost of a radix-2 Cooley-Tukey FFT for a DFT of 2-power size $n = 2^k$. The cost measure is $(A(n), M(n))$, where $A(n)$ is the number of complex adds/subs, $M(n)$ is the number of mults by a complex constant not equal to 1 or -1. Is this a good cost measure when considering implementation?

   Solution:

   $\mathrm{DFT}_{2^k} = (\mathrm{DFT}_2 \otimes \mathrm{I}_{2^{k-1}})\, \mathrm{T}^{2^k}_{2^{k-1}} (\mathrm{I}_2 \otimes \mathrm{DFT}_{2^{k-1}}) L^{2^k}_2$

   Cost of additions: Note that the permutation and the twiddle introduce no additions.

$$\begin{aligned} A(n) &= T_k = \mathrm{Cost}(\mathrm{DFT}_{2^k}) \\ &= 2^{k-1}\,\mathrm{Cost}(\mathrm{DFT}_2) + 2 * \mathrm{Cost}(\mathrm{DFT}_{2^{k-1}}) \\ &= 2^{k-1} * 2 + 2 * \mathrm{Cost}(\mathrm{DFT}_{2^{k-1}}) \\ &= 2^k + 2 * \mathrm{Cost}(\mathrm{DFT}_{2^{k-1}}) \\ T_k &= 2^k + 2 * T_{k-1}, \quad T_0 = 0 \\ T_k &= \sum_{i=0}^{k-1} 2^i * 2^{k-i} \\ T_k &= k * 2^k \end{aligned}$$

   Cost of multiplications: Note that the permutation does not introduce any mults. The twiddle contains a total of 2^k entries, of which $2^{k-1}+1$ entries are equal to 1. So there are $2^{k-1}-1$ entries not equal to one, resulting in that many multiplications.

$$\begin{aligned} M(n) &= M_k = 2^{k-1} * \mathrm{Cost}(\mathrm{DFT}_2) + (2^{k-1}-1) + 2 * \mathrm{Cost}(\mathrm{DFT}_{2^{k-1}}) \\ M_k &= 2^{k-1} - 1 + 2 * \mathrm{Cost}(\mathrm{DFT}_{2^{k-1}}) \\ M_k &= 2^{k-1} - 1 + 2 * M_{k-1}, \quad M_0 = 0 \\ &= \sum_{i=0}^{k-1} 2^i * (2^{k-i-1} - 1) \\ &= \sum_{i=0}^{k-1} 2^{k-1} - \sum_{i=0}^{k-1} 2^i \\ &= 2^{k-1}(k-2) + 1 \end{aligned}$$

   Therefore, the cost $(A(n), M(n))$ is: $(k * 2^k, 2^{k-1}(k-2)+1)$

   This cost measure counts complex operations. On current computers instruction sets and execution units only cover real operations. For this reason, a cost measure counting real operations would be more appropriate. Note that complex multiplications may translate into real operations differently. For example, a generic multiplication by a complex constant involves 6 real operations (see Lecture 13), but a multiplication by $i = \sqrt{-1}$ no real operation at all.

2. Show that the arithmetic cost (as defined and measured as in exercise 4.) of the Cooley-Tukey FFT for a DFT of 2-power size $n = 2^k$ is independent of the chosen recursion strategy. (Hint: use induction).

   Solution:

   Adds: $n = 2^k$

$$
\begin{aligned}
A(k) &= k * 2^k \\
M(k) &= (k-2) * 2^{k-1} + 1 \\
&= (1/2)k * 2^k - 2^k + 1
\end{aligned}
$$

Proof by induction:

- $k = 1$: Clear, since there is only one decomposition.
- $k-1$ to k: Assume the assumption holds for $1..(k-1)$. Show that all $k-1$ splits for s^k lead to the same cost.

  Arbitrary split: $2^k = 2^i * 2^{k-1}, i \in 1..k-1$.

  $\mathrm{DFT}_{2^k} = (\mathrm{DFT}_{2^i} \otimes \mathrm{I}_{2^{k-1}})\, \mathrm{T}^n_m (\mathrm{I}^i_2 \otimes \mathrm{DFT}_{2^{k-1}}) L^{2^k}_{2^i}$

  Adds:

$$
\begin{aligned}
A(k) &= 2^{k-i} * A(i) + 2^i * A(k-i) \\
&= 2^{k-i} * i * 2^i + 2^i (k-i) 2^{k-i} \\
&= i * 2^k + (k-i) 2^k \\
&= k * 2^k
\end{aligned}
$$

  as defined.

  Mults:

  Note: $w(k,i)$ = number of elements $\neq 0$ in T^n_m, which, as seen by inspecting the definition, is $2^k - 2^i - 2^{k-i} + 1$.

$$
\begin{aligned}
M(k) &= 2^{k-i} * M(i) + 2^i M(k-i) + w(k,i) \\
&= 2^{k-i}((i-2)2^{i-1} + 1) + 2^i((k-i-2) * 2^{k-i-1} + 1) + 2^k - 2^i - 2^{k-i} + 1 \\
&= 2^{k-i}(i-2) + 2^{k-i} + 2^{k-1}(k-i-2) + 2^i + 2^k - 2^i - 2^{k-i} + 1 \\
&= 2^{k-1}(k-4) + 2^k + 1 \\
&= 2^{k-1}(k-2) + 1
\end{aligned}
$$

  as defined.