The Finite-Dimensional Witsenhausen Counterexample

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UC Berkeley

Joint work with Prof. Anant Sahai, Se Yong Park

There are handouts for this talk, please take one!

Outline

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- Witsenhausen’s counterexample

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- Infinite-length vector extension

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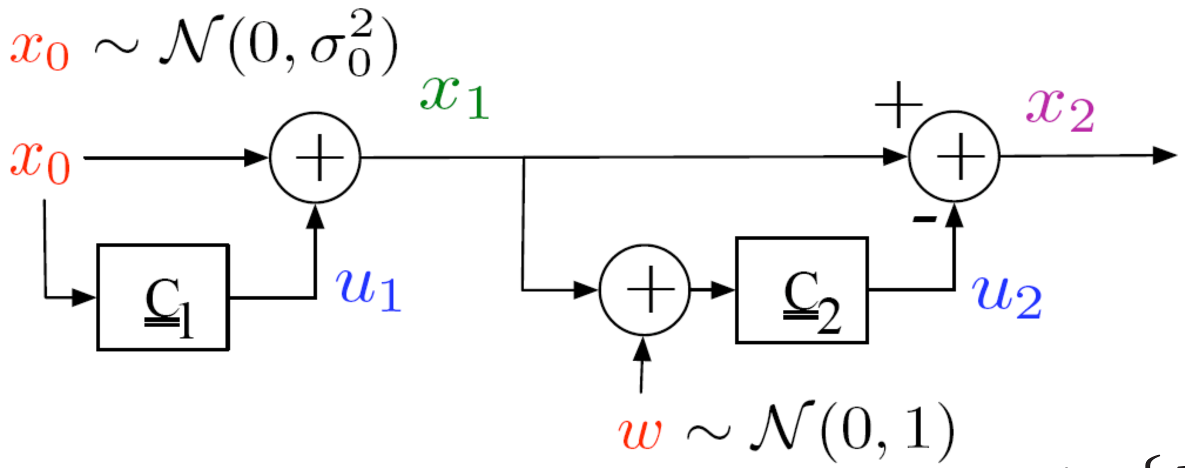
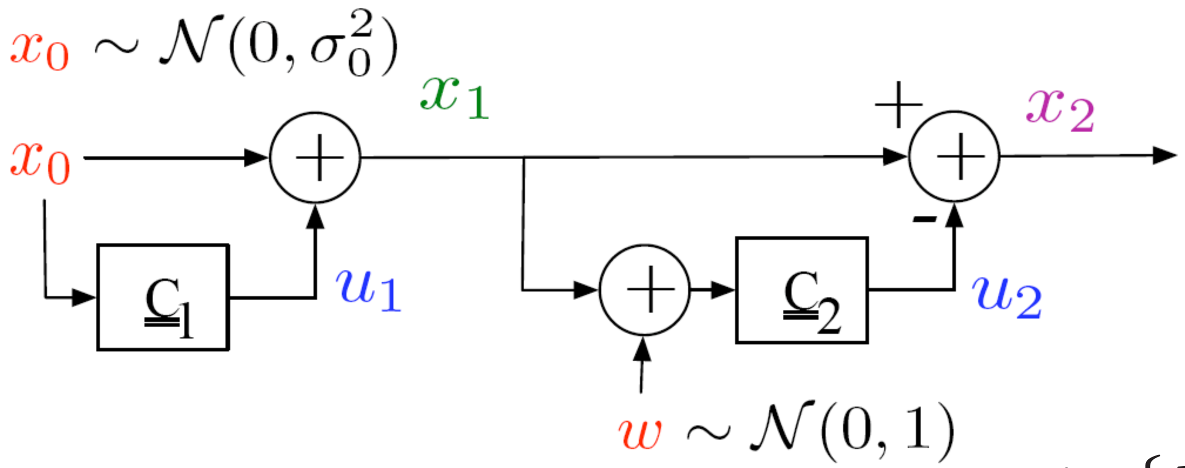
It is easier to approximate

Witsenhausen's counterexample



$$\min \left\{ k^2 \mathbb{E}\left[{u_1}^2\right] + \mathbb{E}\left[{x_2}^2\right] \right\}$$

Witsenhausen's counterexample



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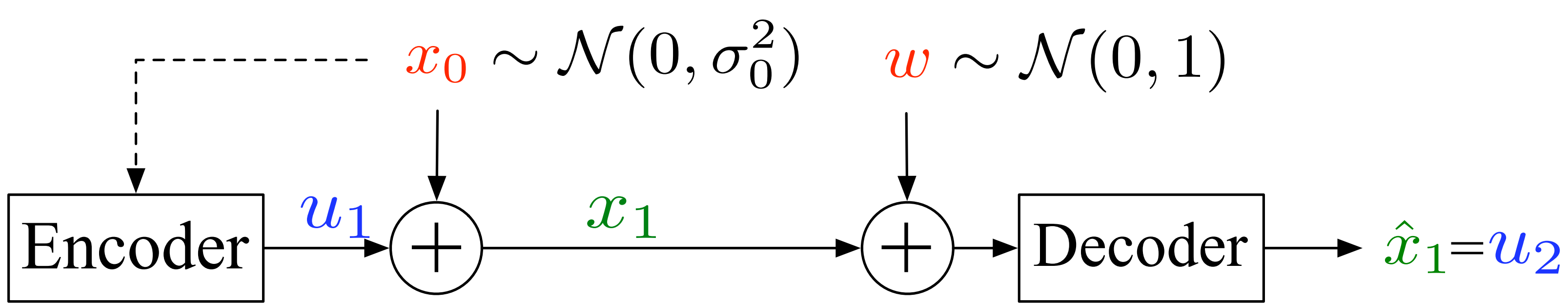
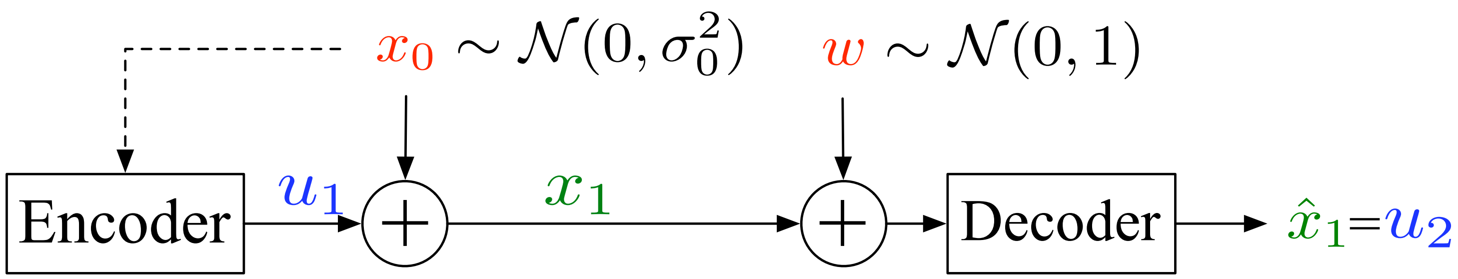
"Implicit channel"



$\mathbb{E}\left[{u_1}^2\right] \leq P$ $\quad$ $\min \mathbb{E}\left[(x_1 - \hat{x}_1)^2\right]$

Implicit channel based signaling strategies

"Implicit channel"



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[Mitter and Sahai, '99]

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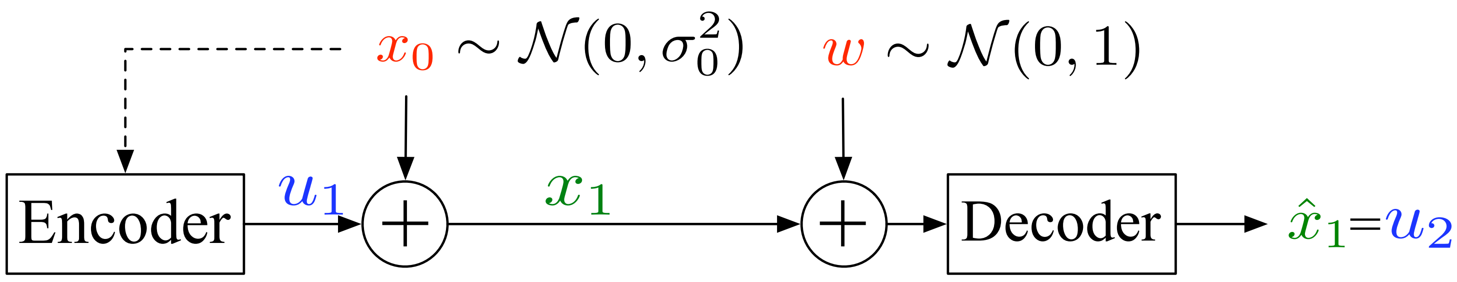

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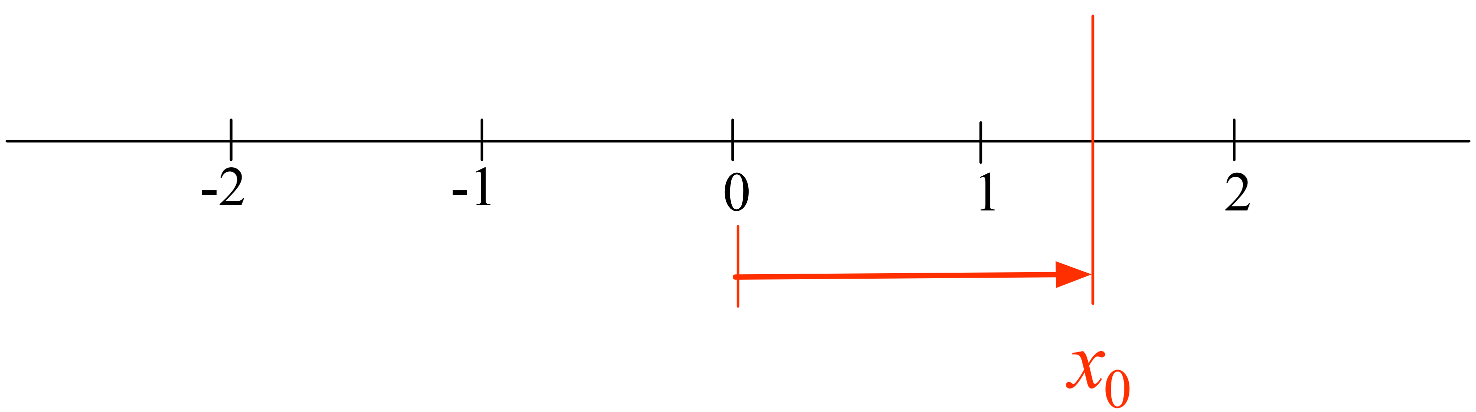

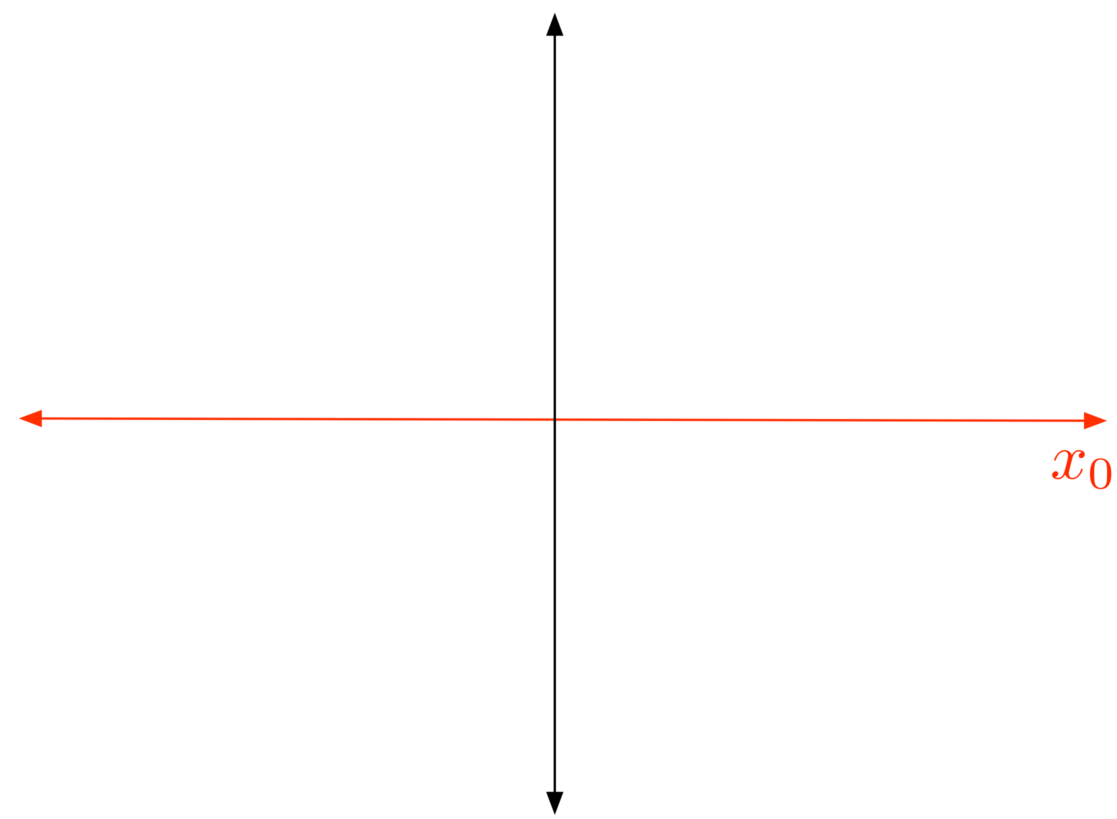

[Mitter and Sahai, '99]

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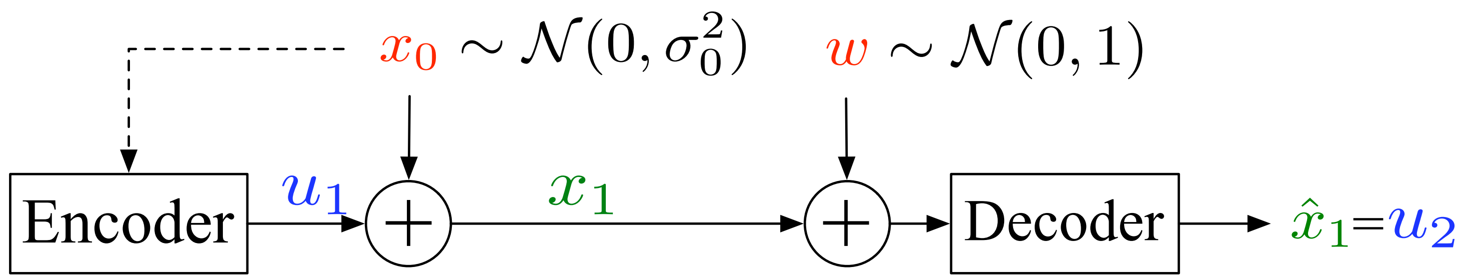

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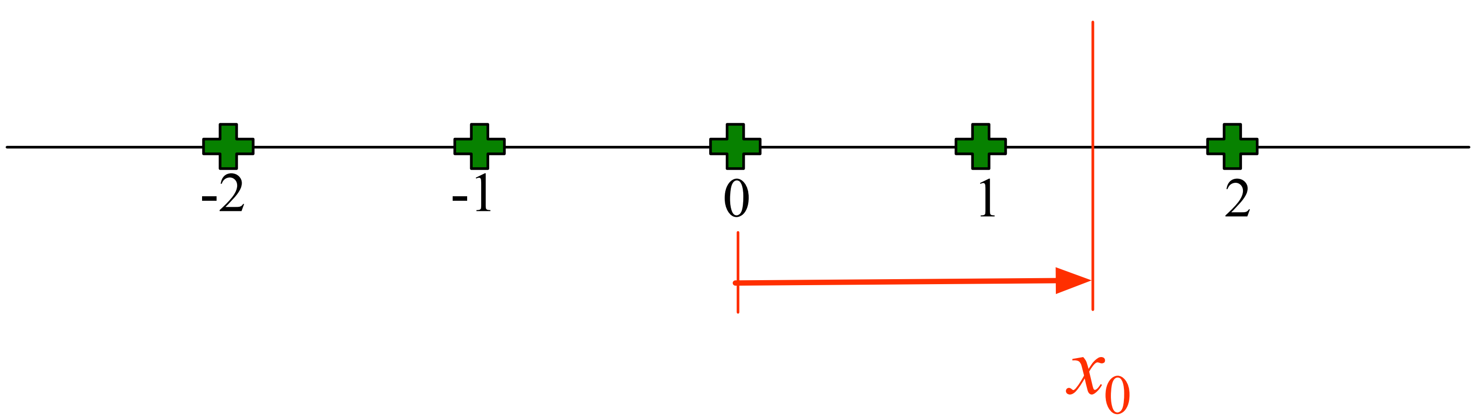

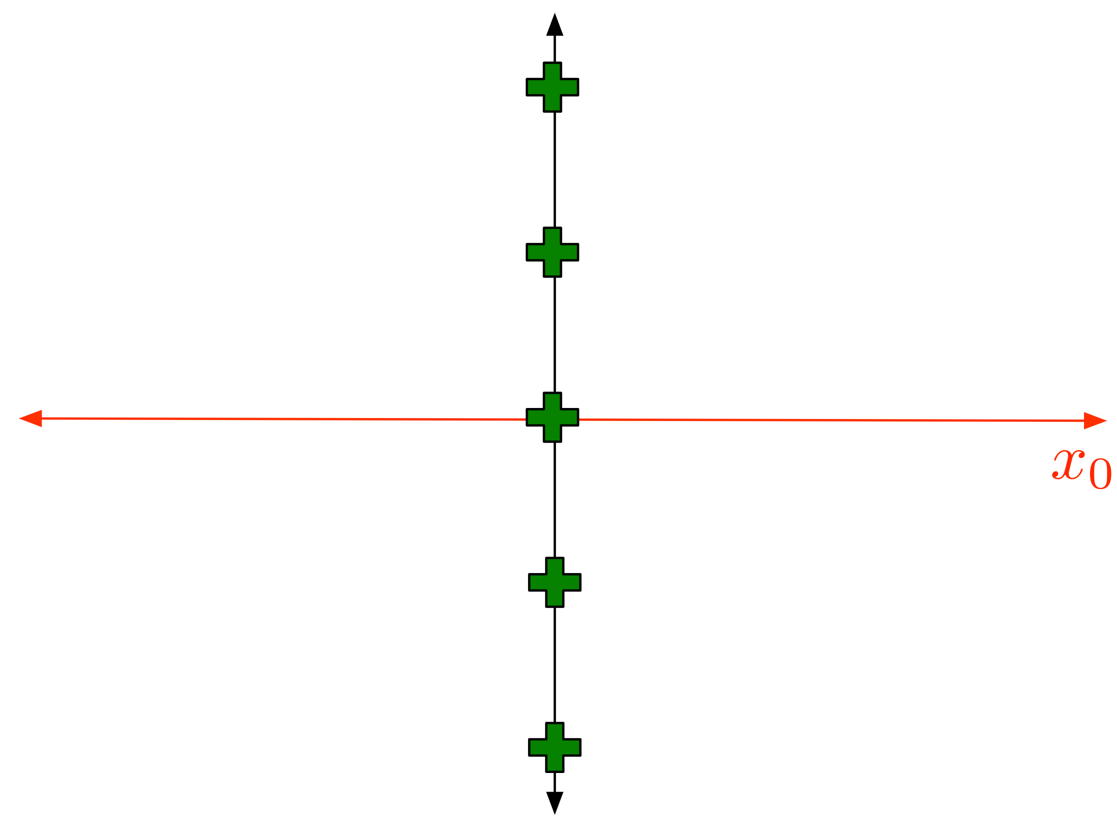

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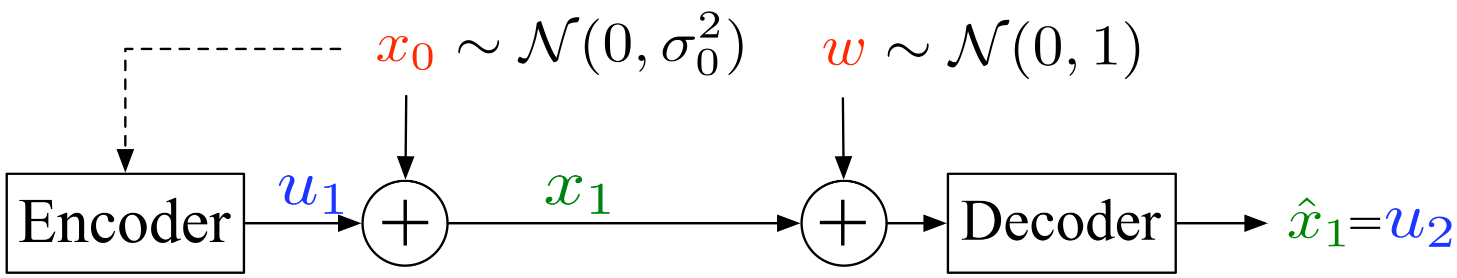

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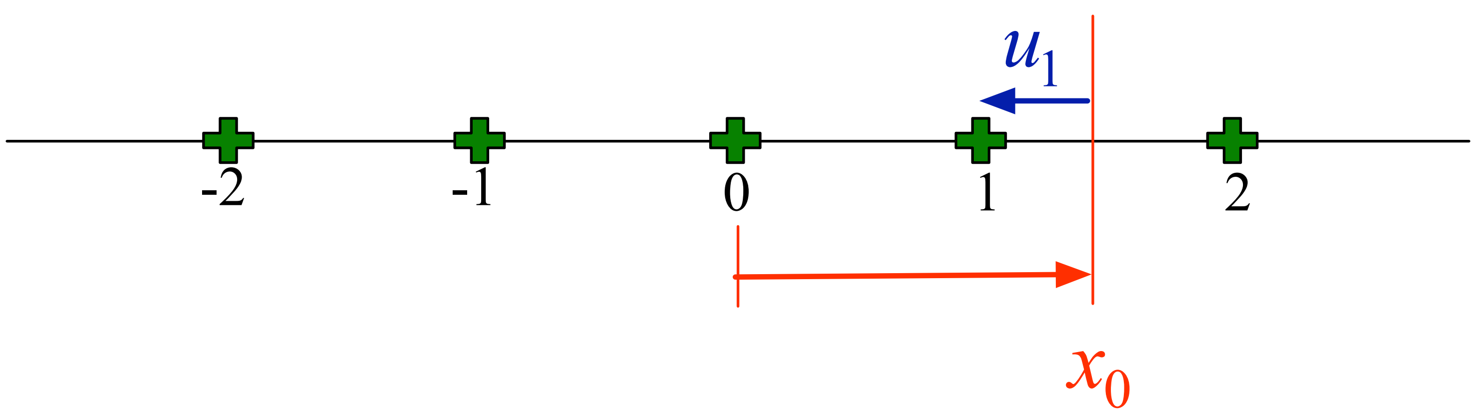

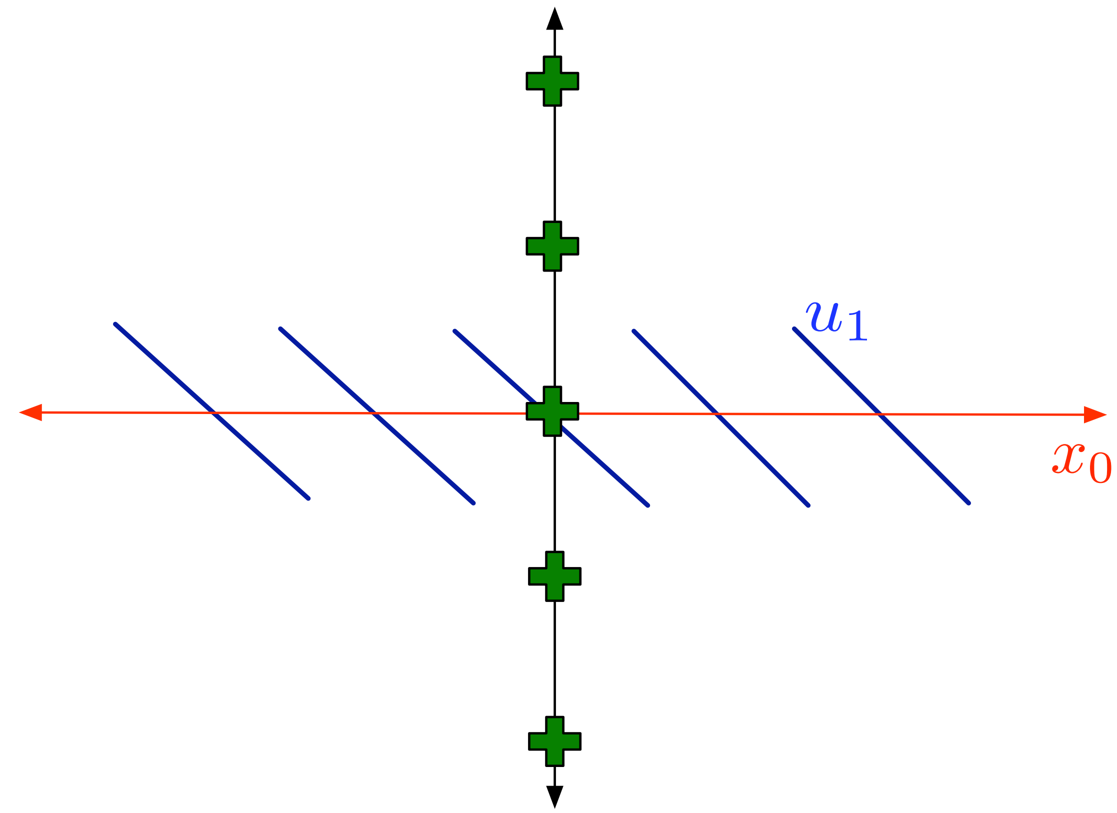

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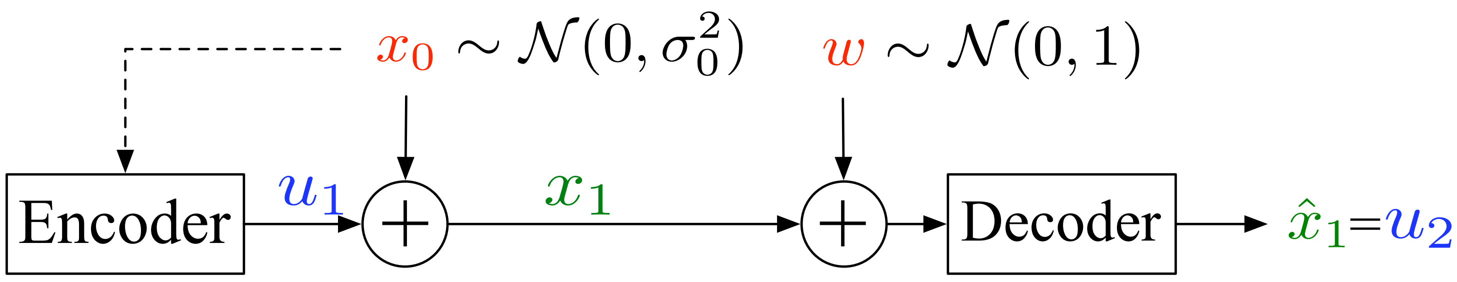

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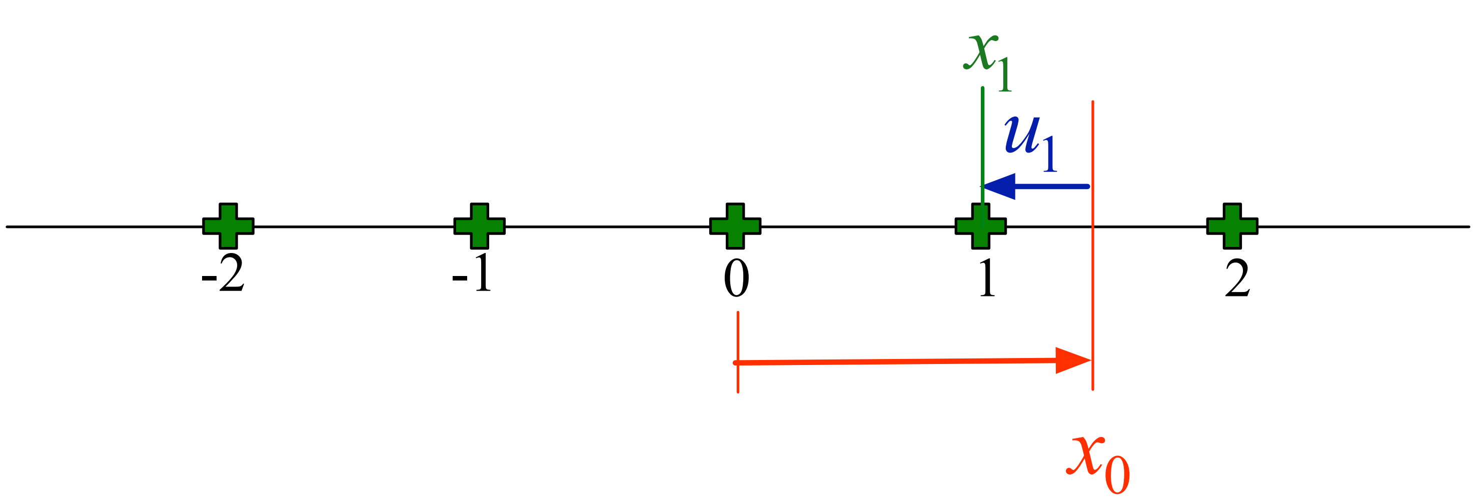

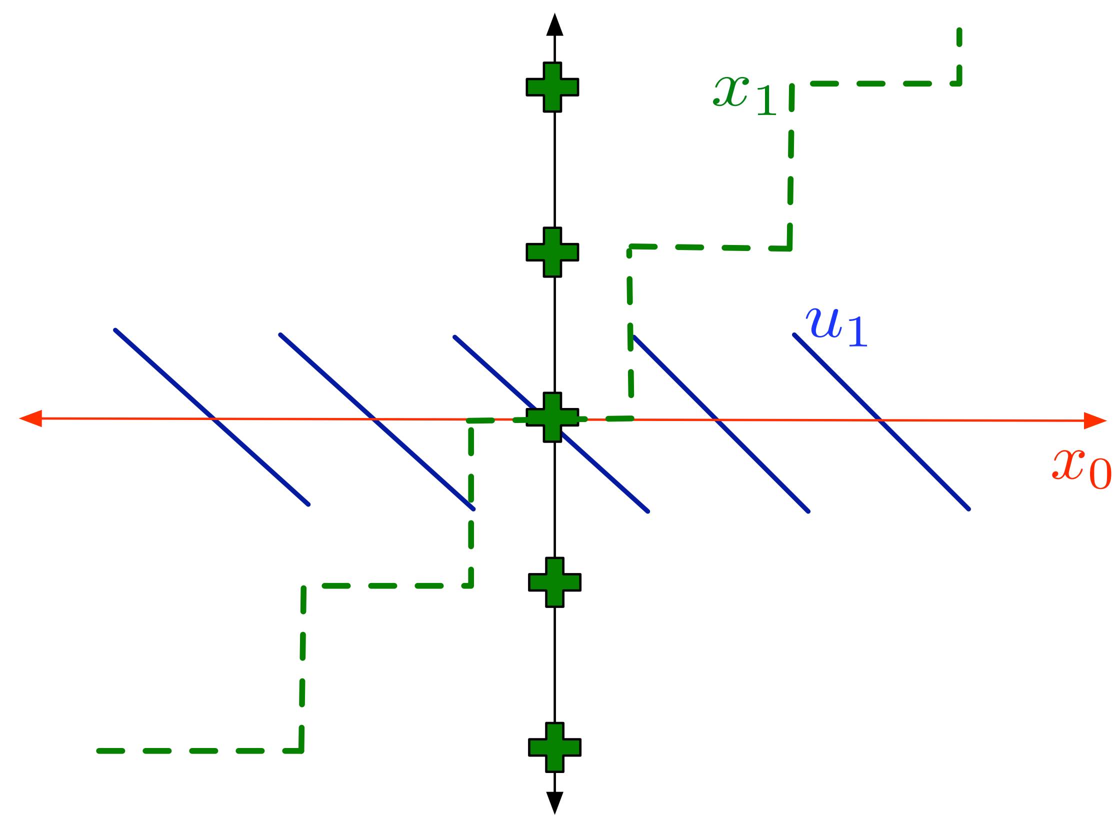

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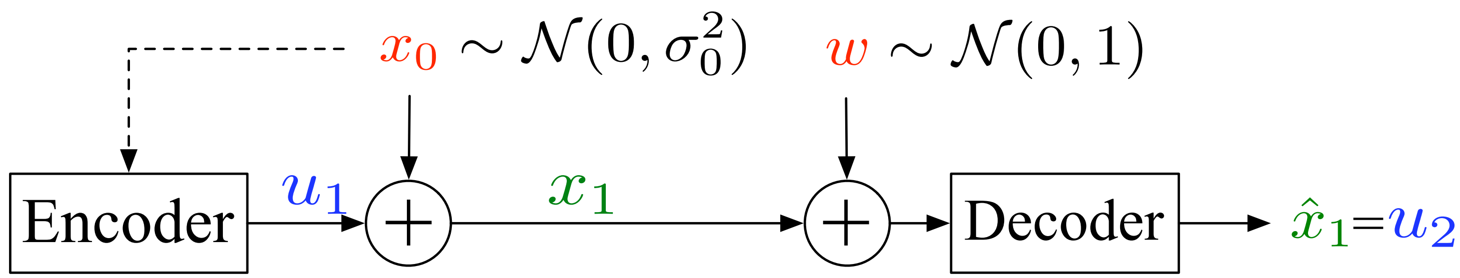

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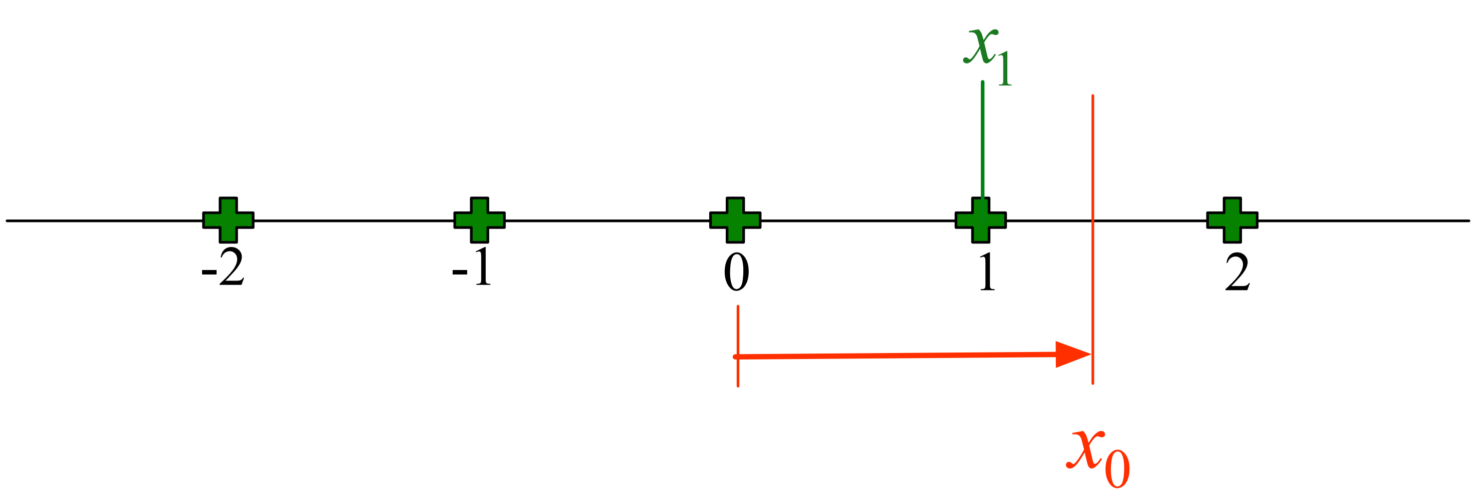

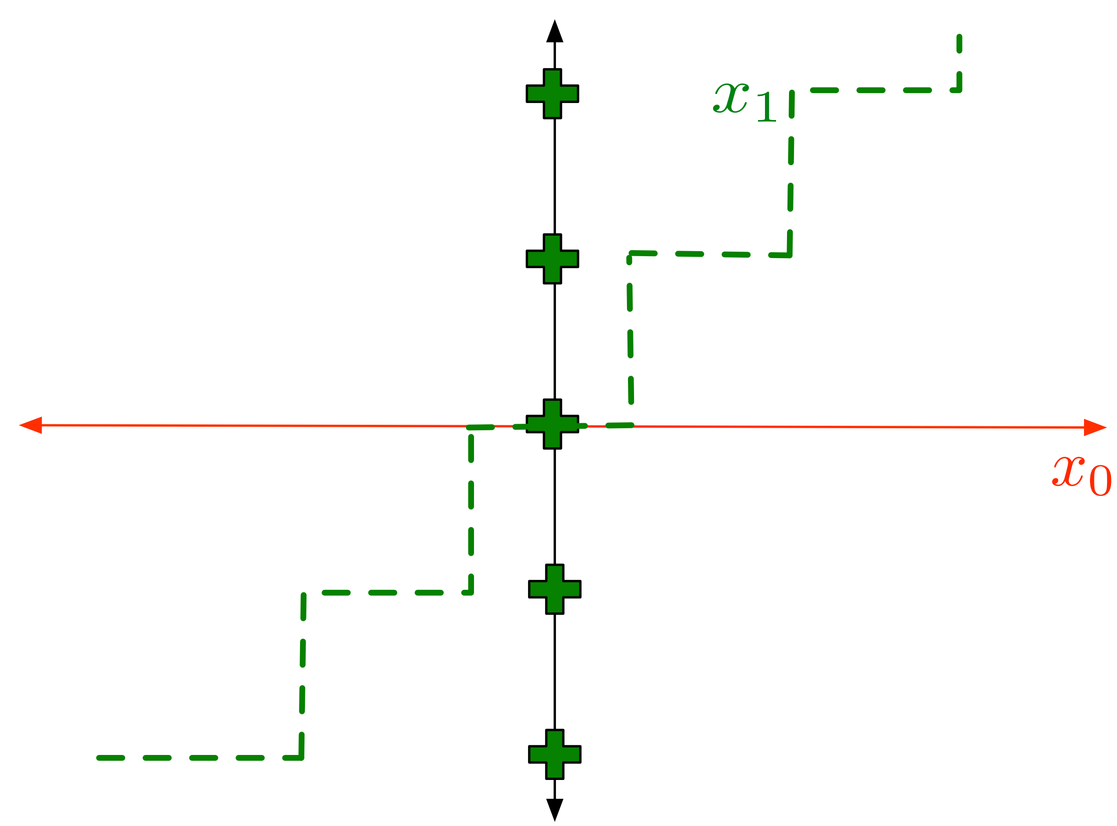

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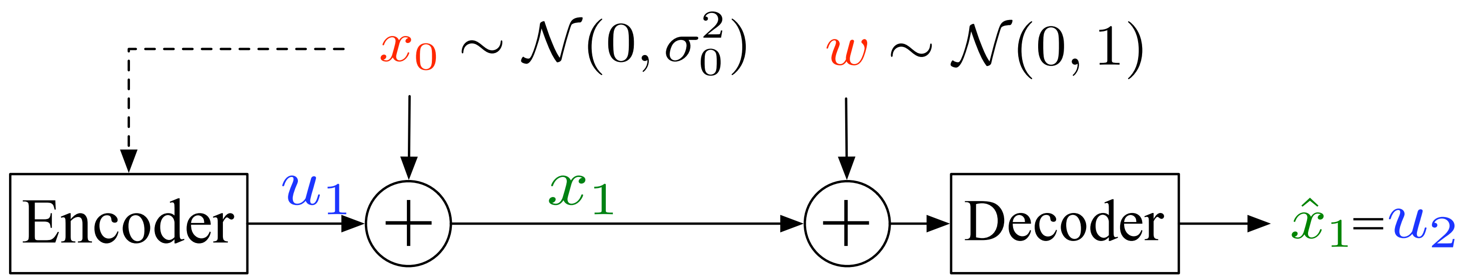

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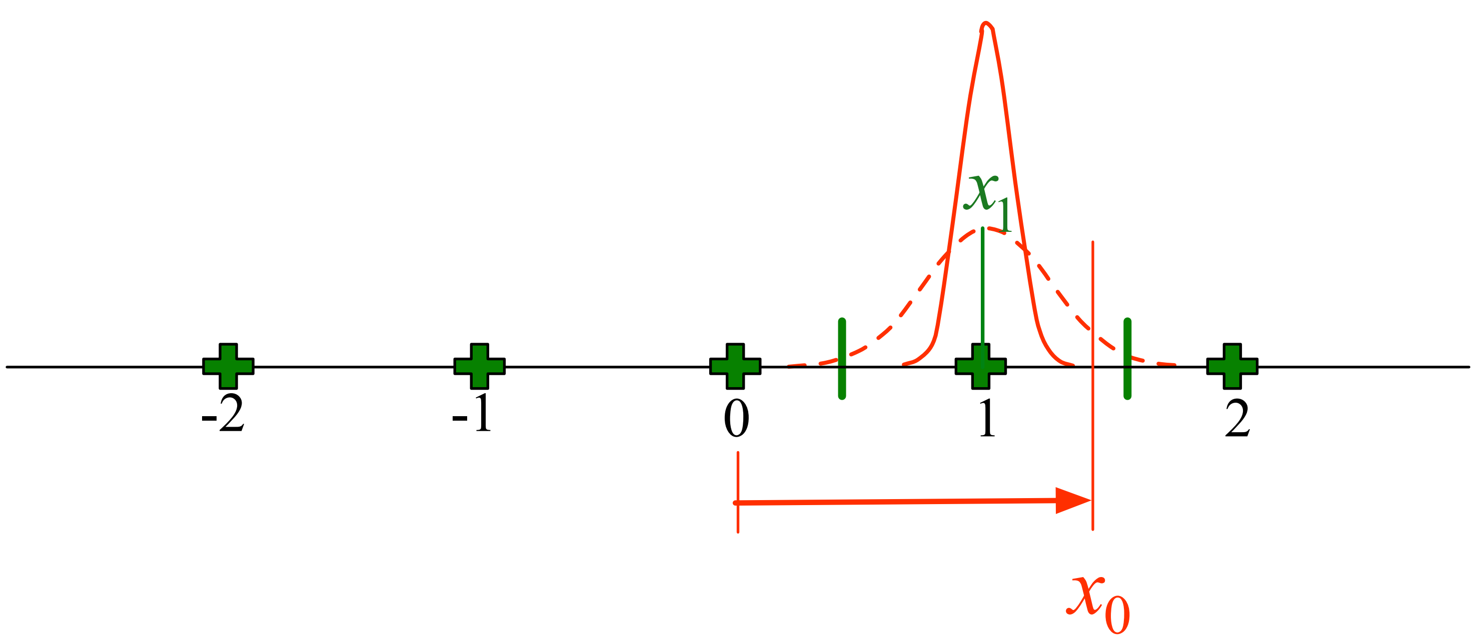
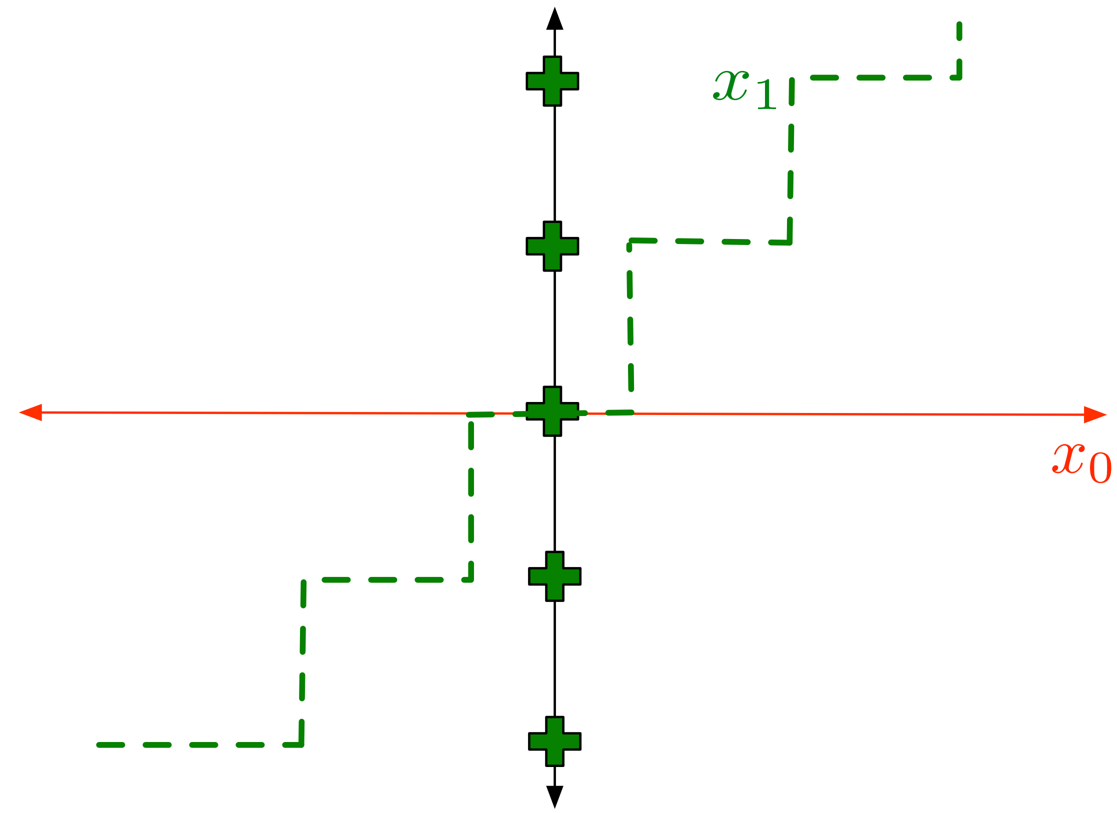
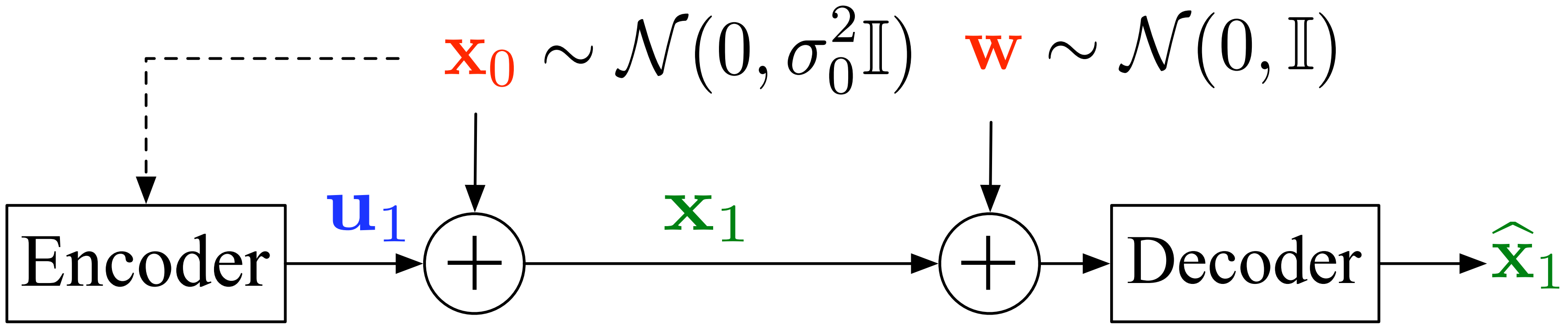




[Mitter and Sahai, '99]

Infinite-length counterexample

A simplification

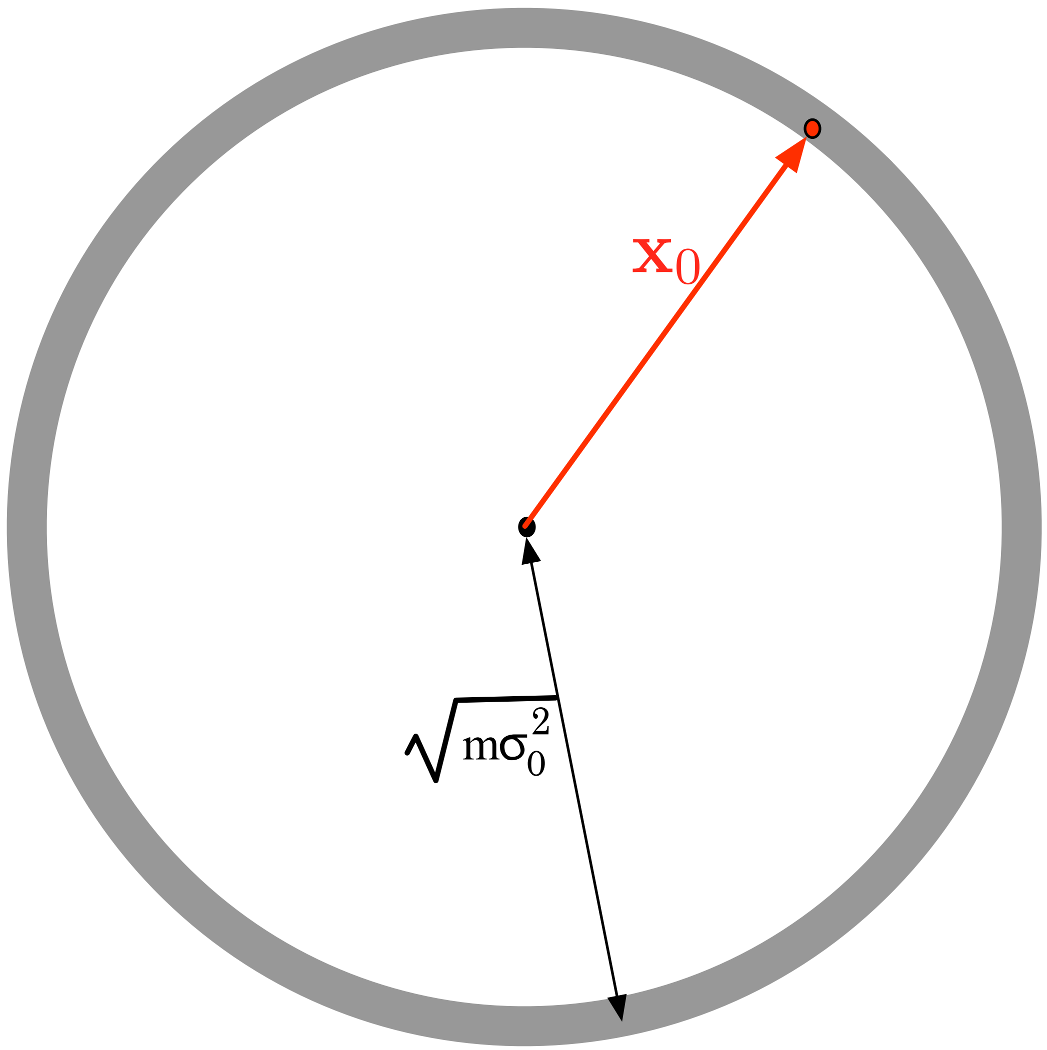


$$\frac{1}{m}\mathbb{E}\left[\|\mathbf{u}_1\|^2\right] \leq P \qquad \min \frac{1}{m}\mathbb{E}\left[\|\mathbf{x}_1 - \widehat{\mathbf{x}}_1\|^2\right]$$

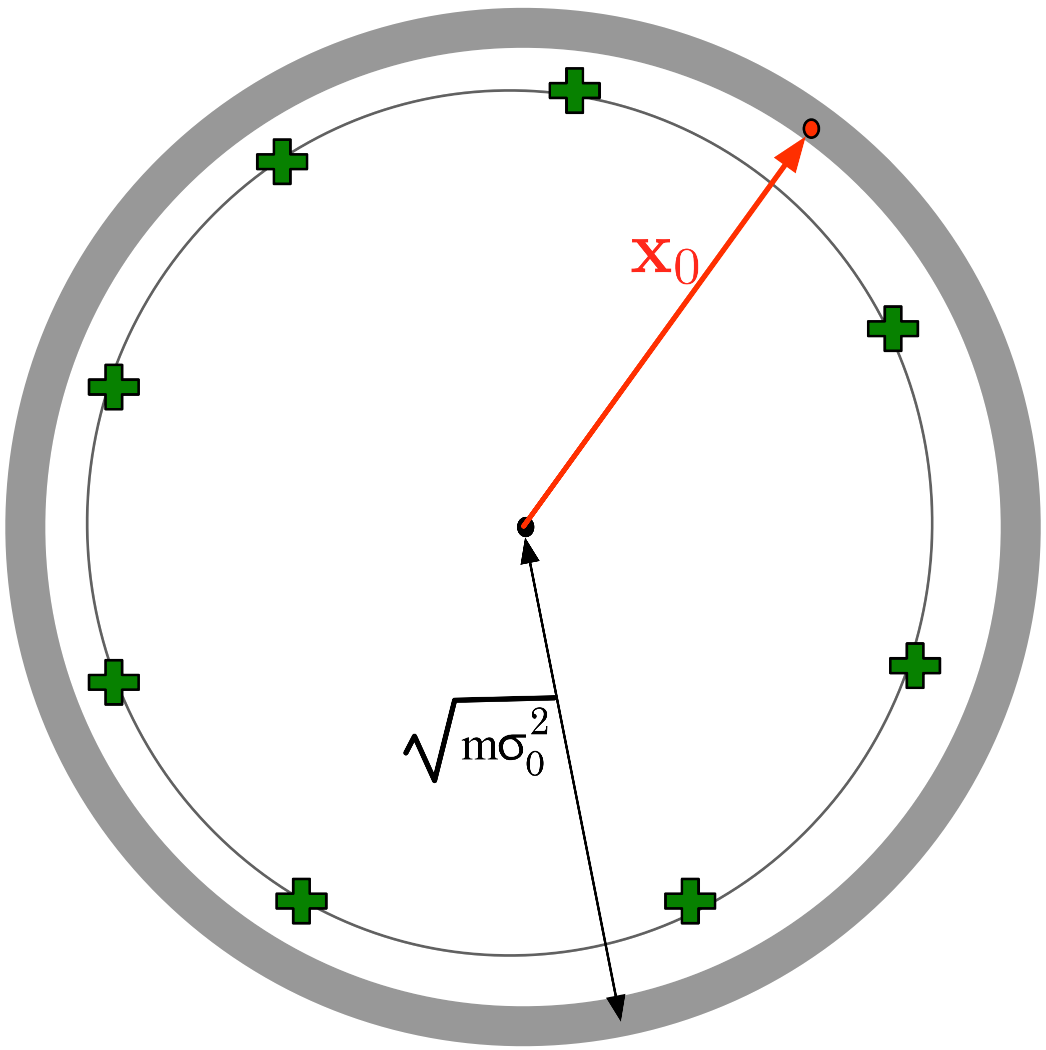
$$\mathbb{E}[\mathcal{C}] = k^2 P + \frac{1}{m}\mathbb{E}\left[\|\mathbf{x}_1 - \widehat{\mathbf{x}}_1\|^2\right]$$

[Ho, Kastner, Wong '78]

Vector quantization



Vector quantization



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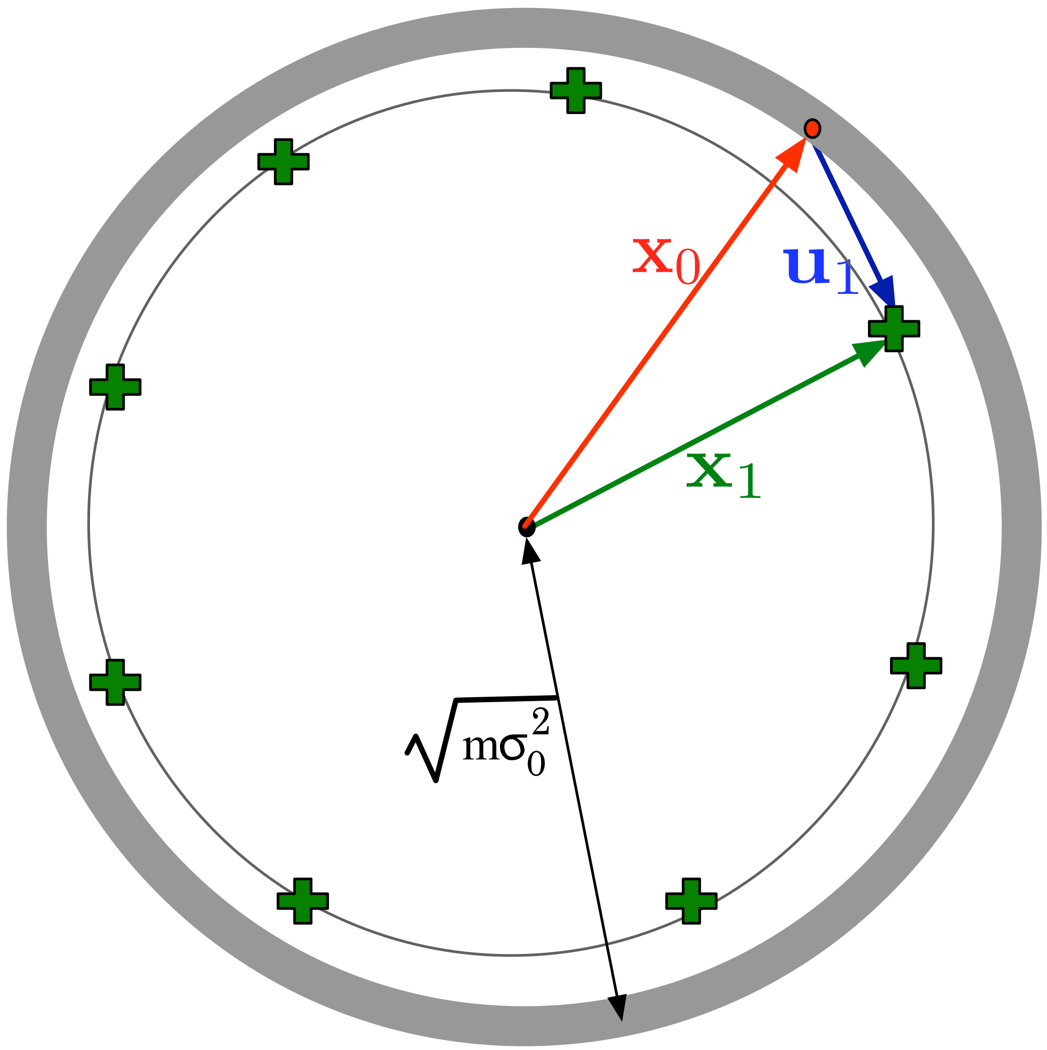

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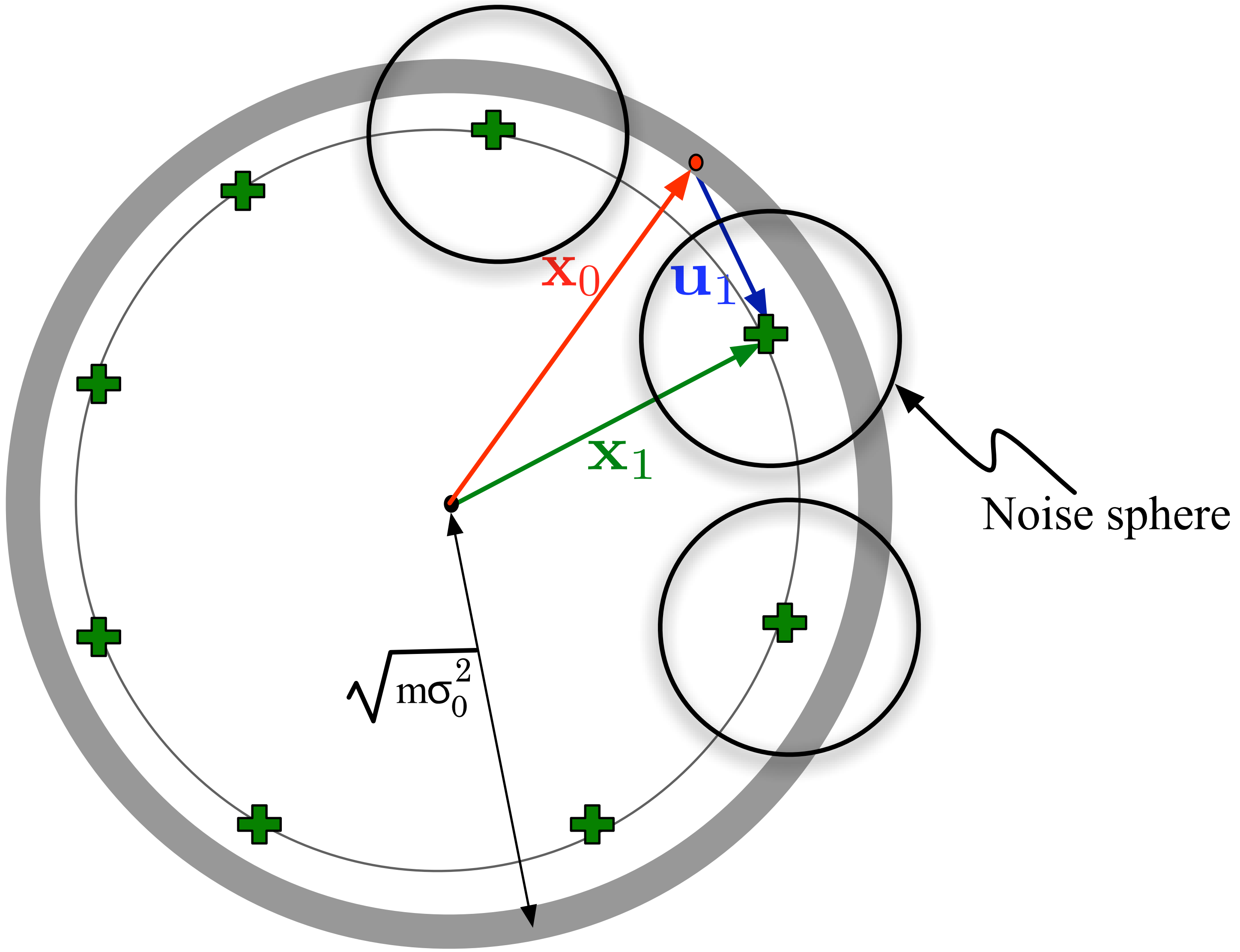

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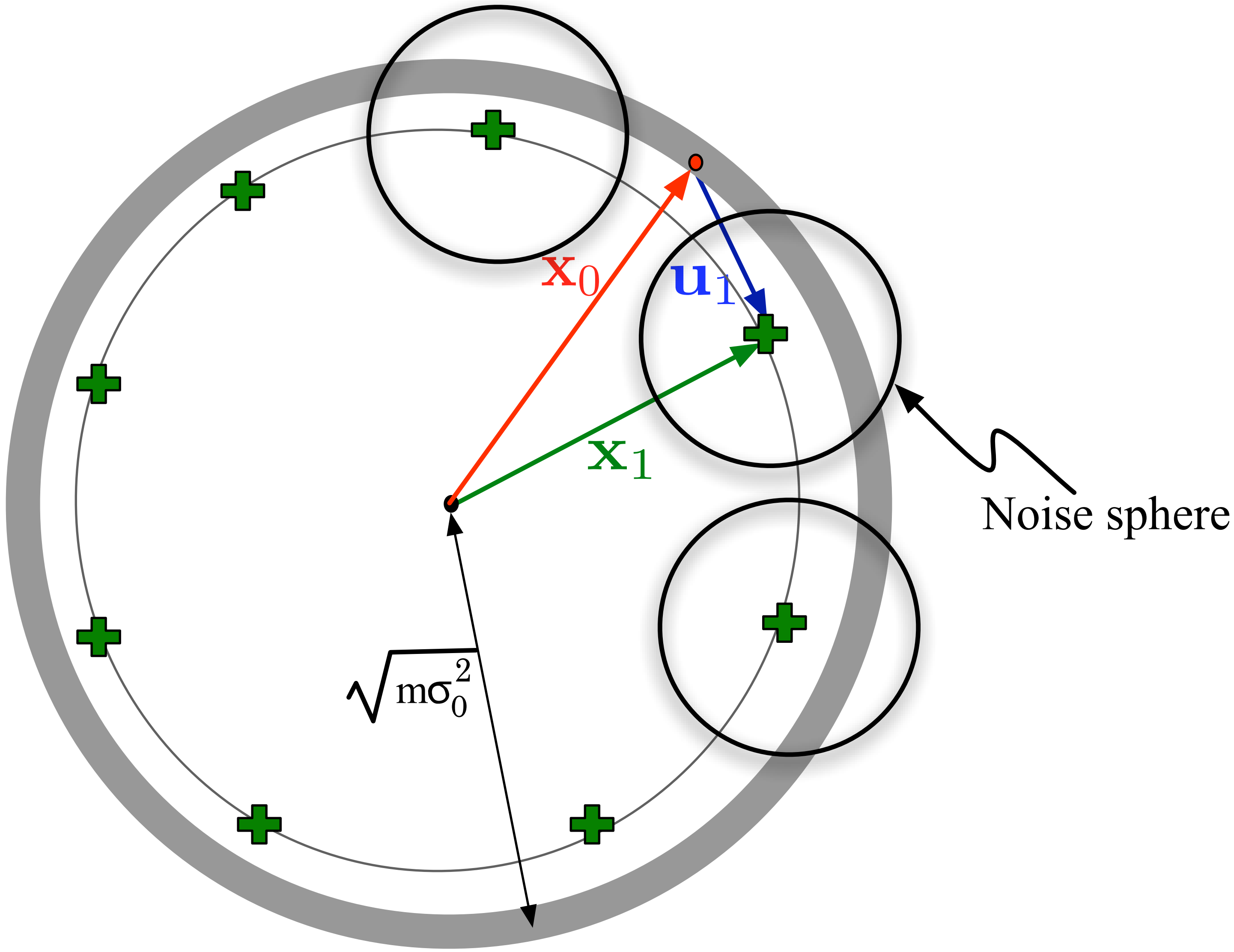


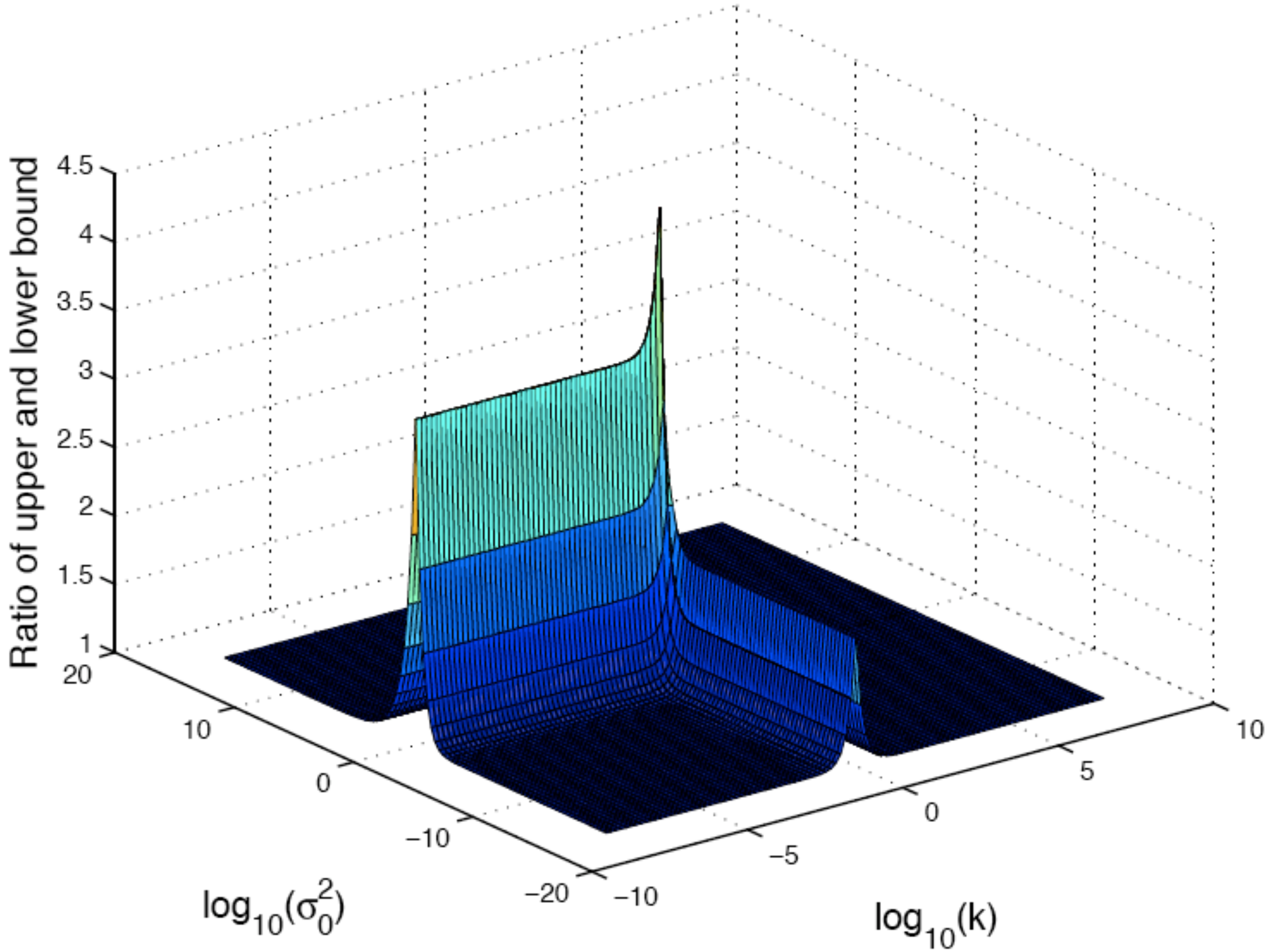
$$\bar{\mathcal{C}} = k^2 + 0$$

A lower bound for the vector extension

$$\bar{\mathcal{C}}_{\min} \geq \inf_{P \geq 0} k^2 P + \left(\left(\sqrt{\kappa(P)} - \sqrt{P} \right)^+ \right)^2$$

$$\kappa(P) = \frac{\sigma_0^2}{(\sigma_0 + \sqrt{P})^2 + 1}$$

Optimal costs within a factor of 4.45



A Dirty Paper Coding-based strategy

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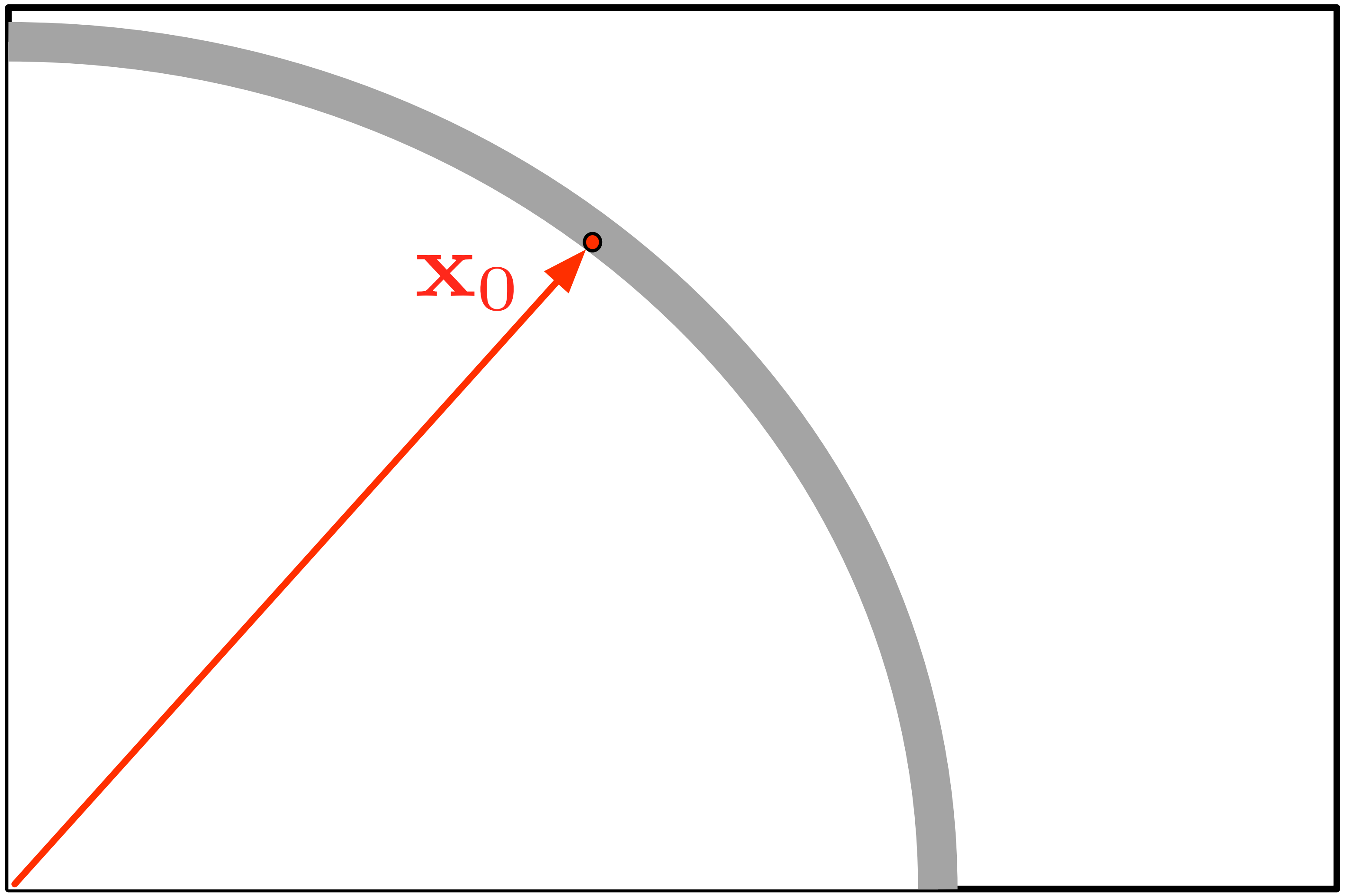

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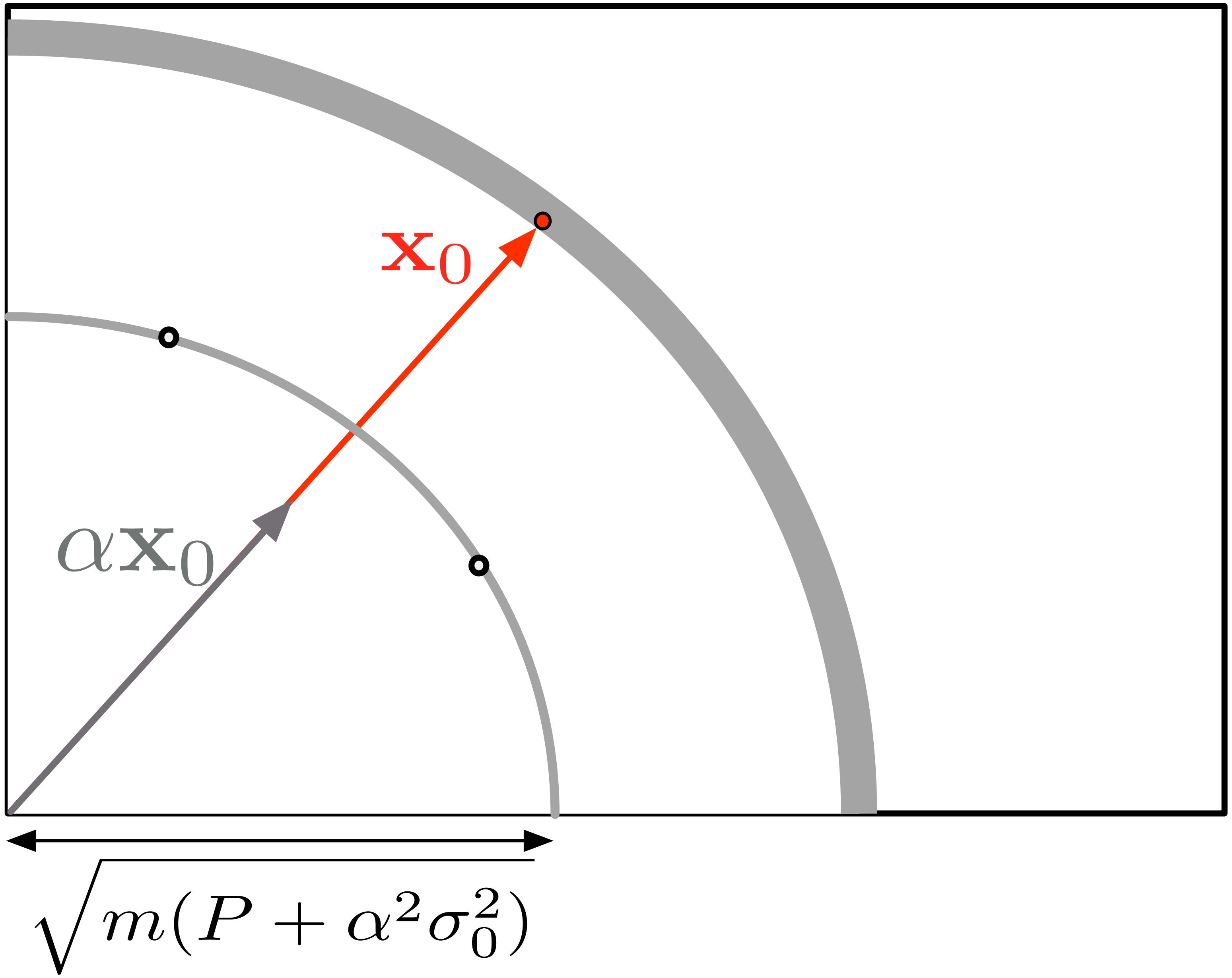

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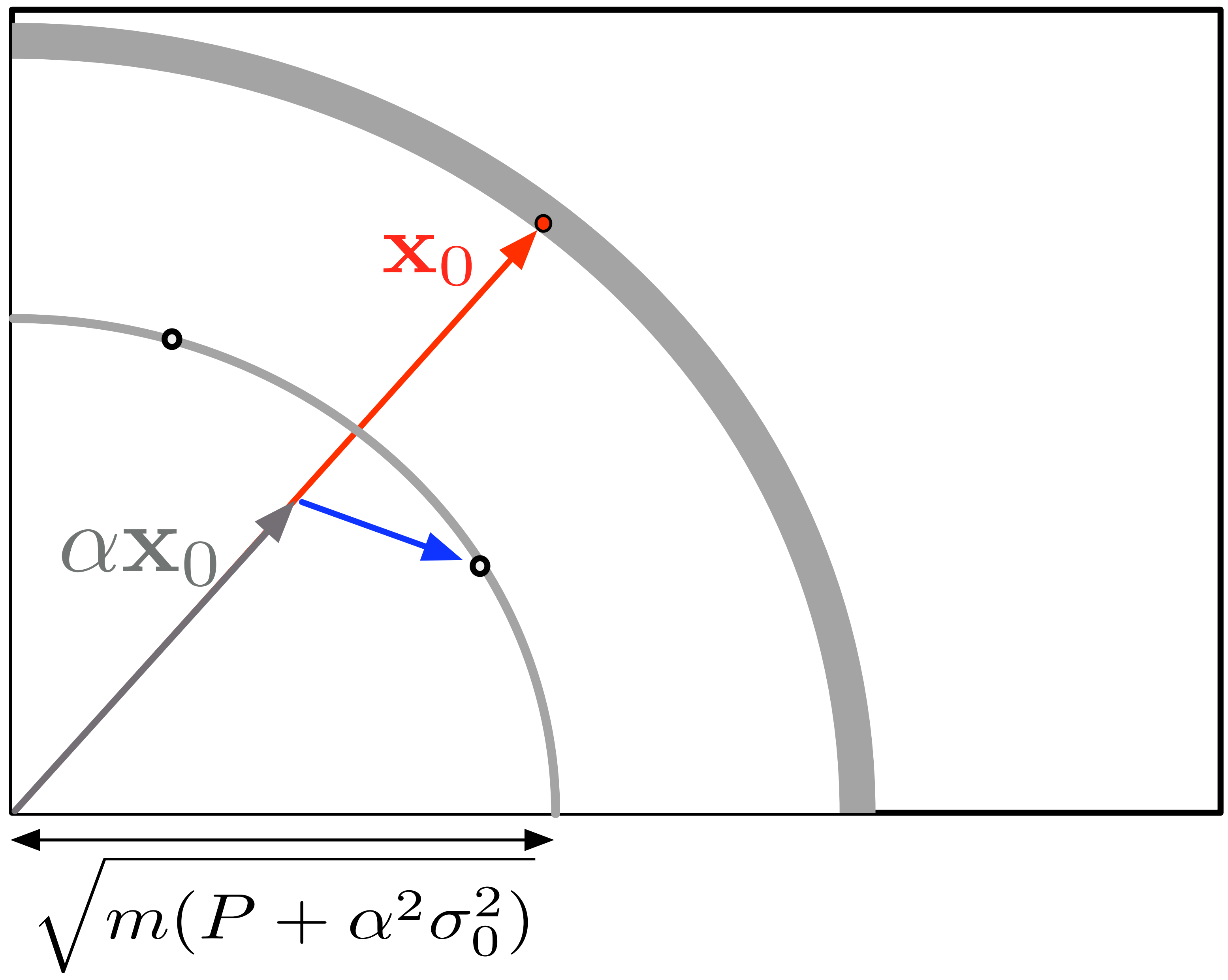

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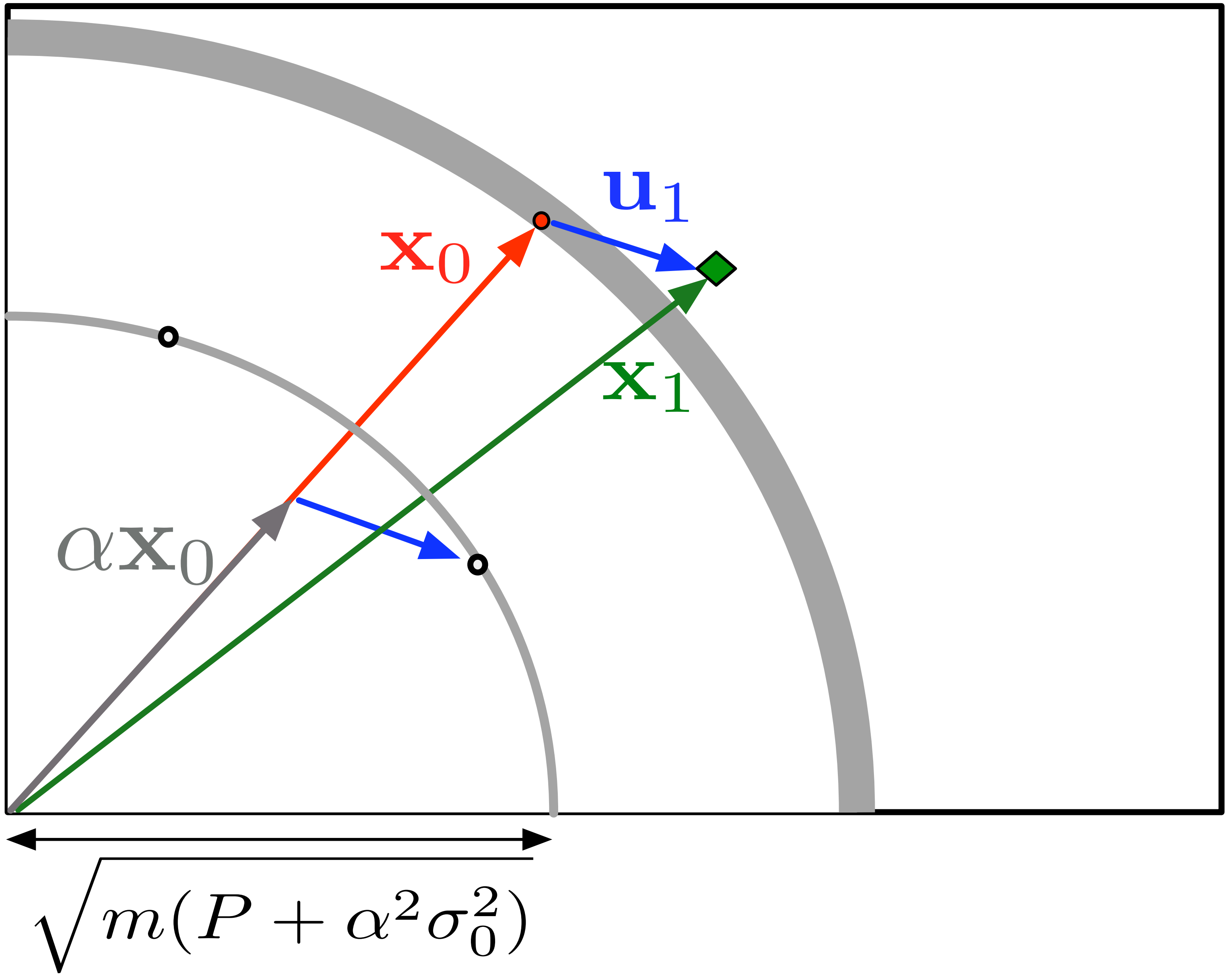

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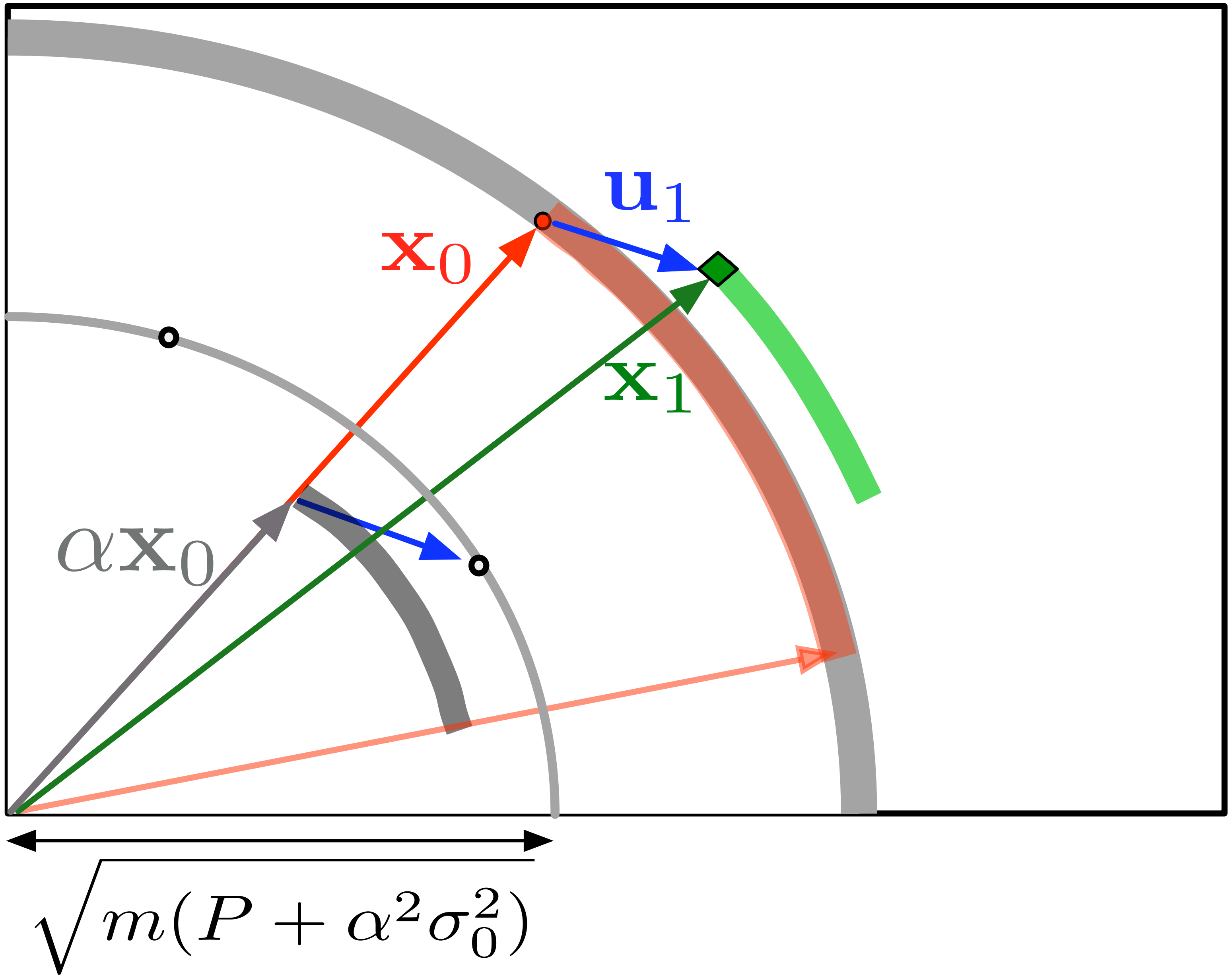

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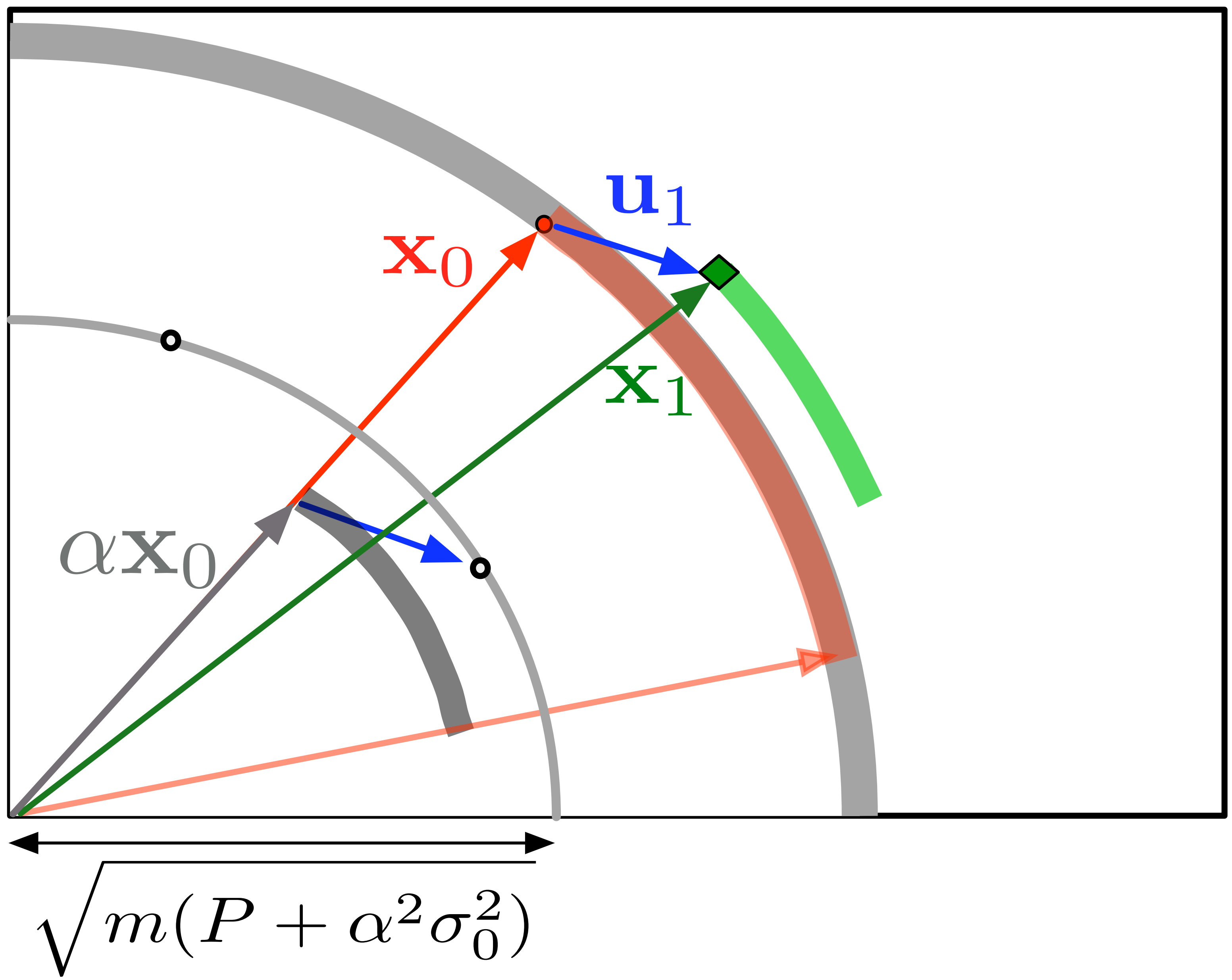
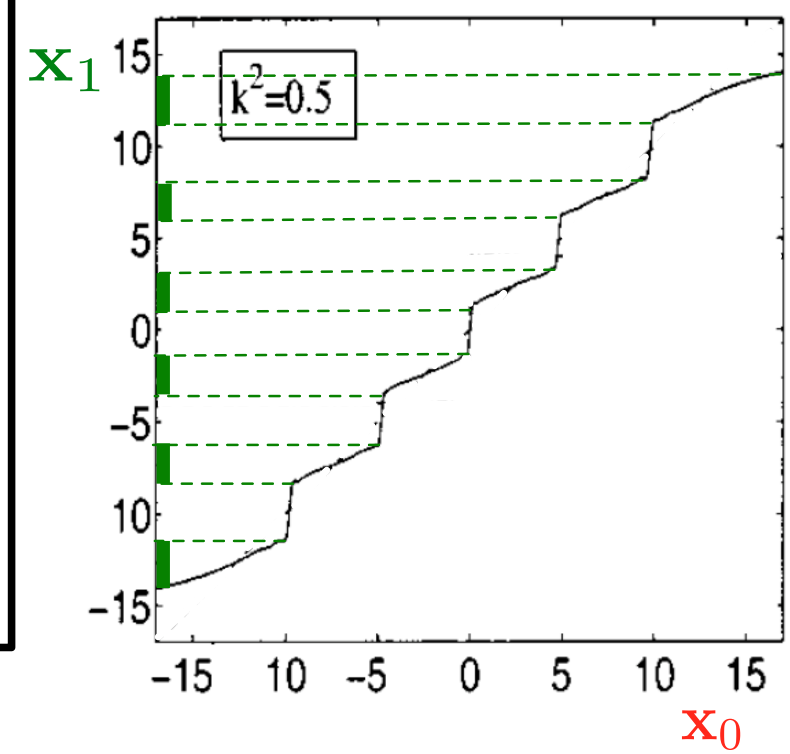
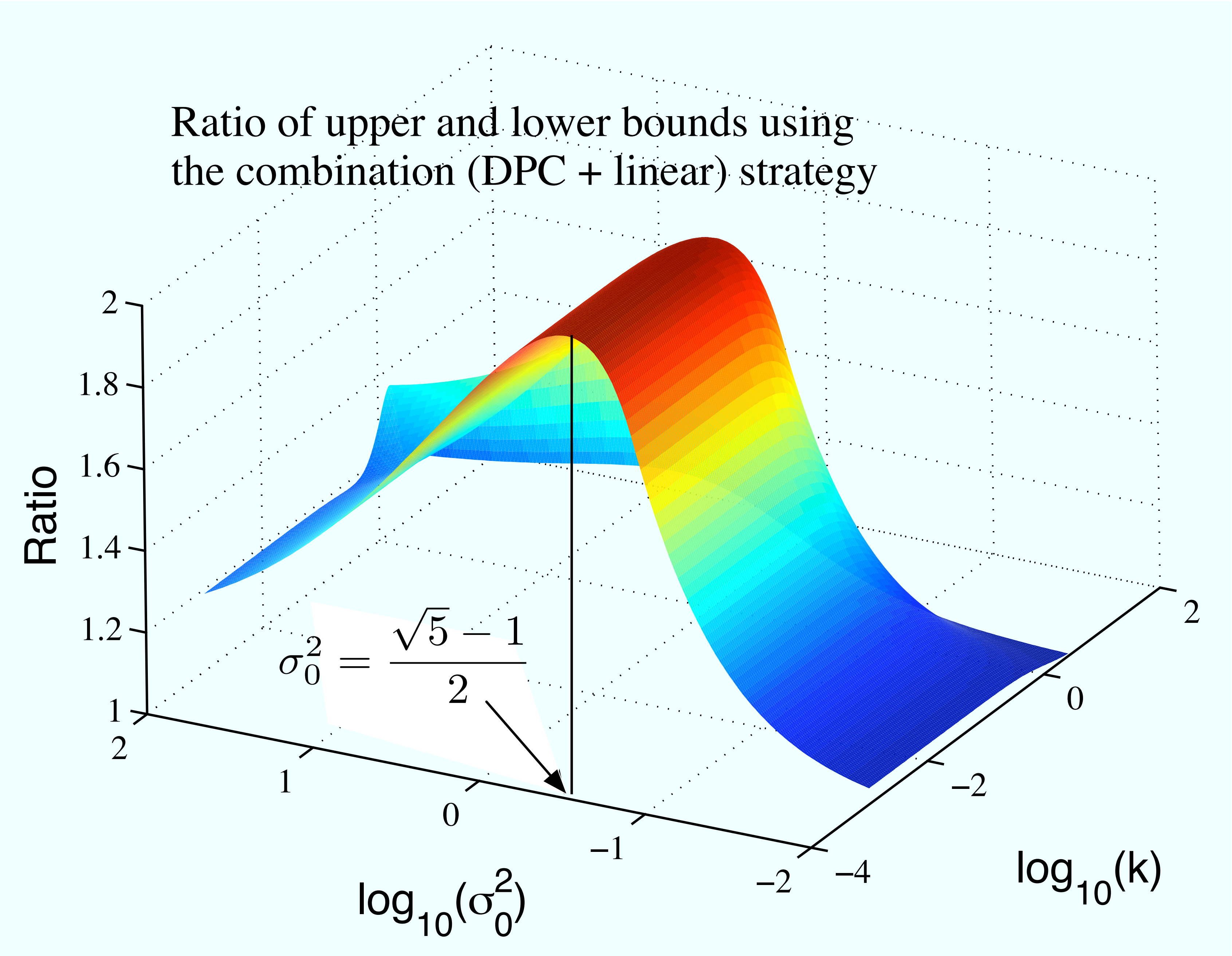




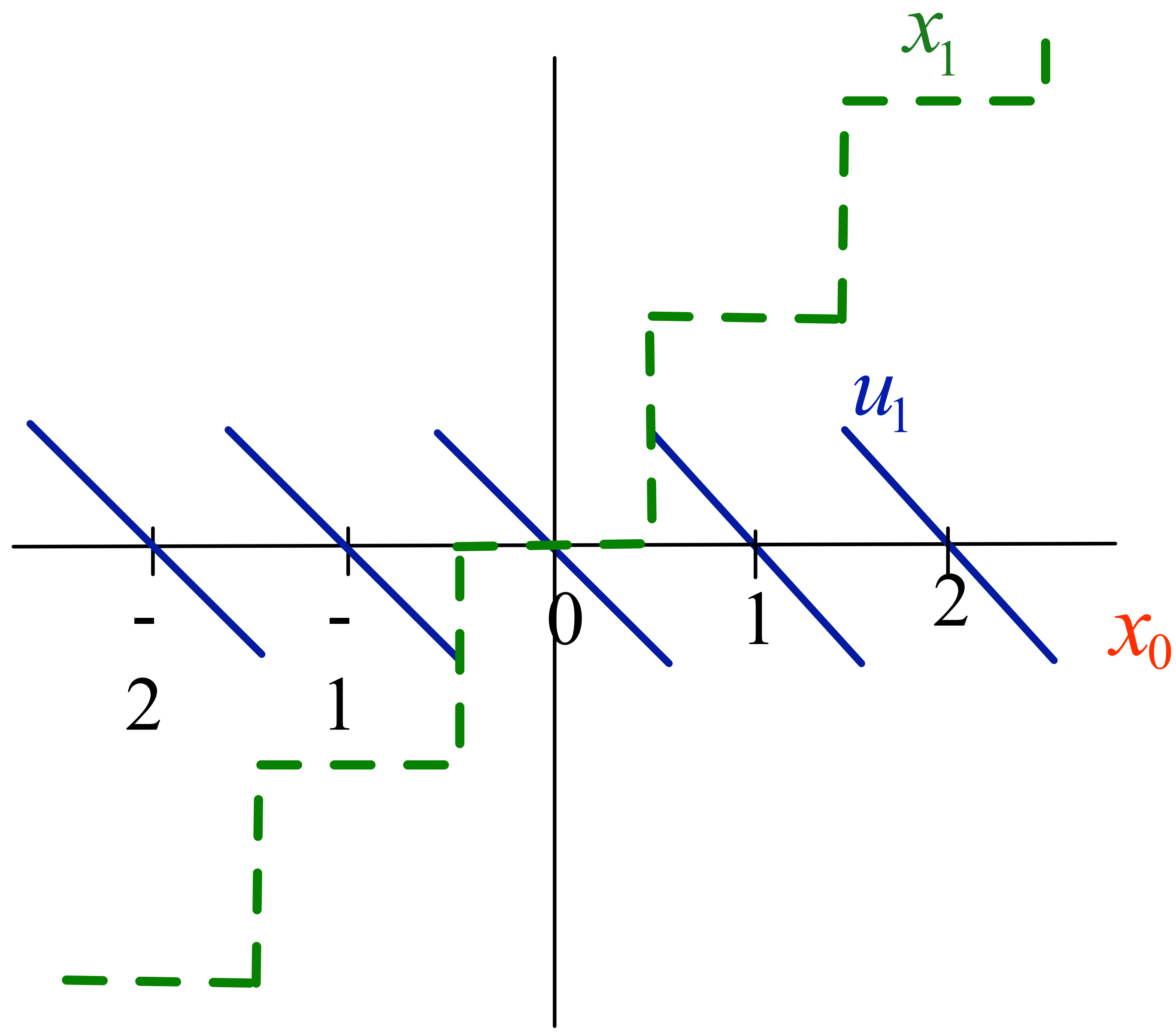
[Baglietto, Parisini, Zoppoli]
[Lee, Lau and Ho]

Optimal cost to within a factor of 2

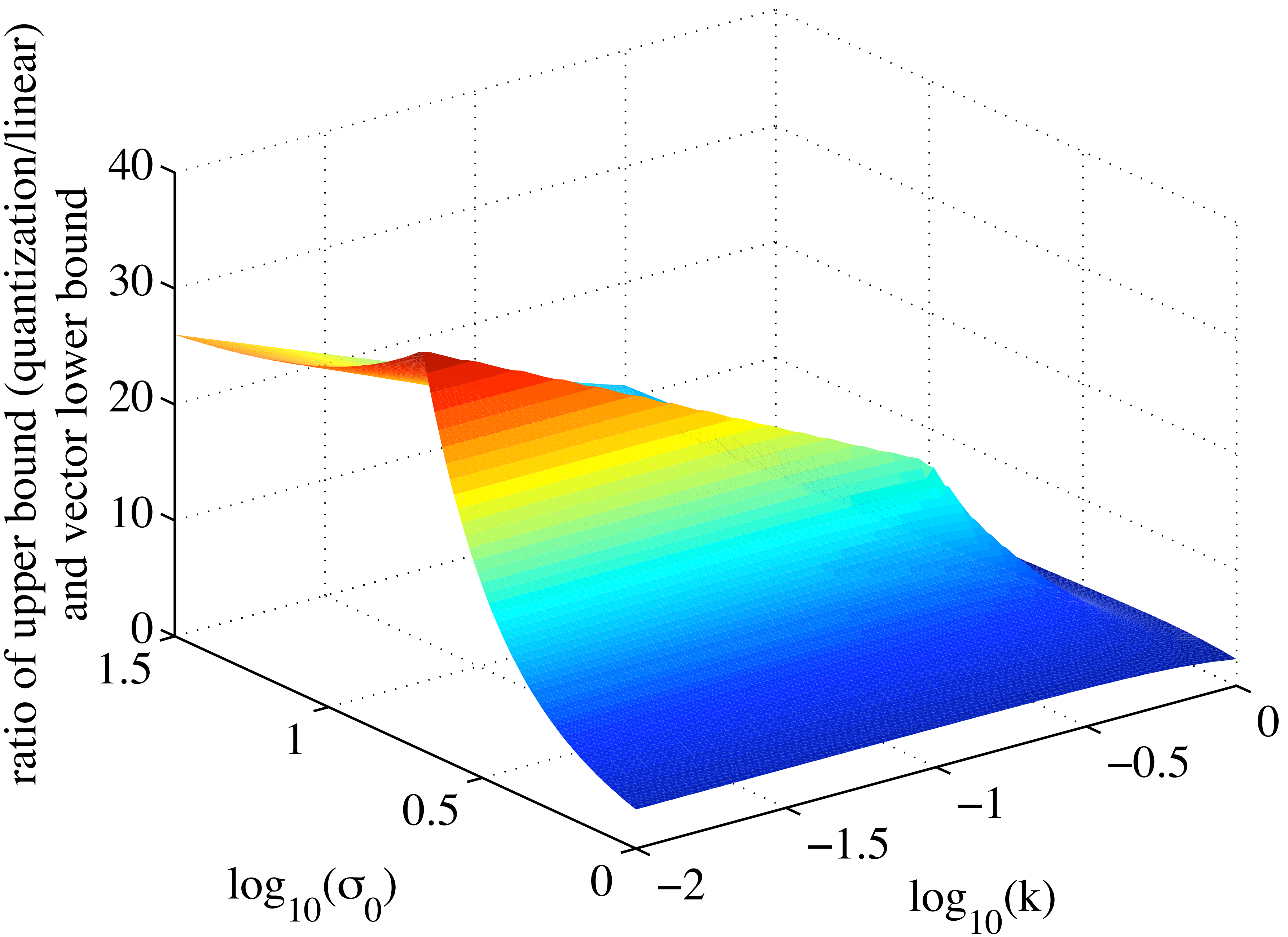


The scalar problem

Quantization-based strategies



Vector lower bound is too loose at finite lengths!



Another look at the vector lower bound

$$\bar{\mathcal{C}}_{\min} \geq \inf_{P \geq 0} k^2 P + \left(\left(\sqrt{\kappa(P)} - \sqrt{P} \right)^+ \right)^2$$

$$\kappa(P) = \frac{\sigma_0^2 {\sigma_G}^2}{(\sigma_0 + \sqrt{P})^2 + {\sigma_G}^2}$$

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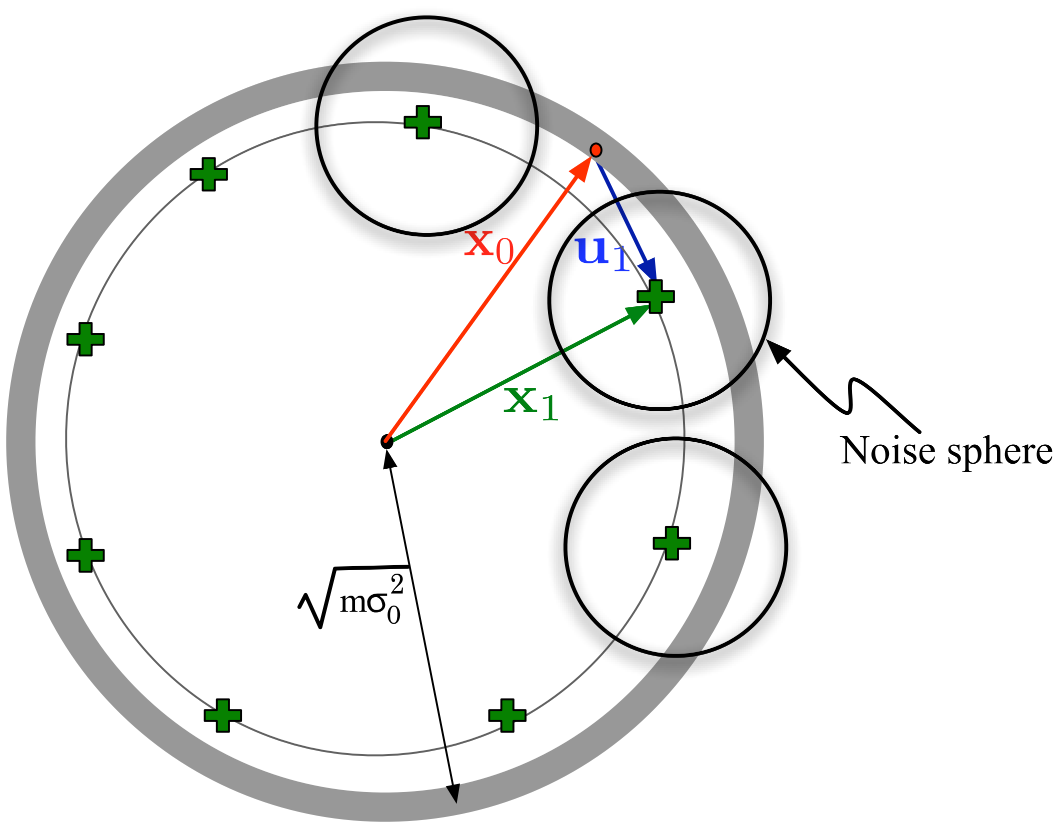

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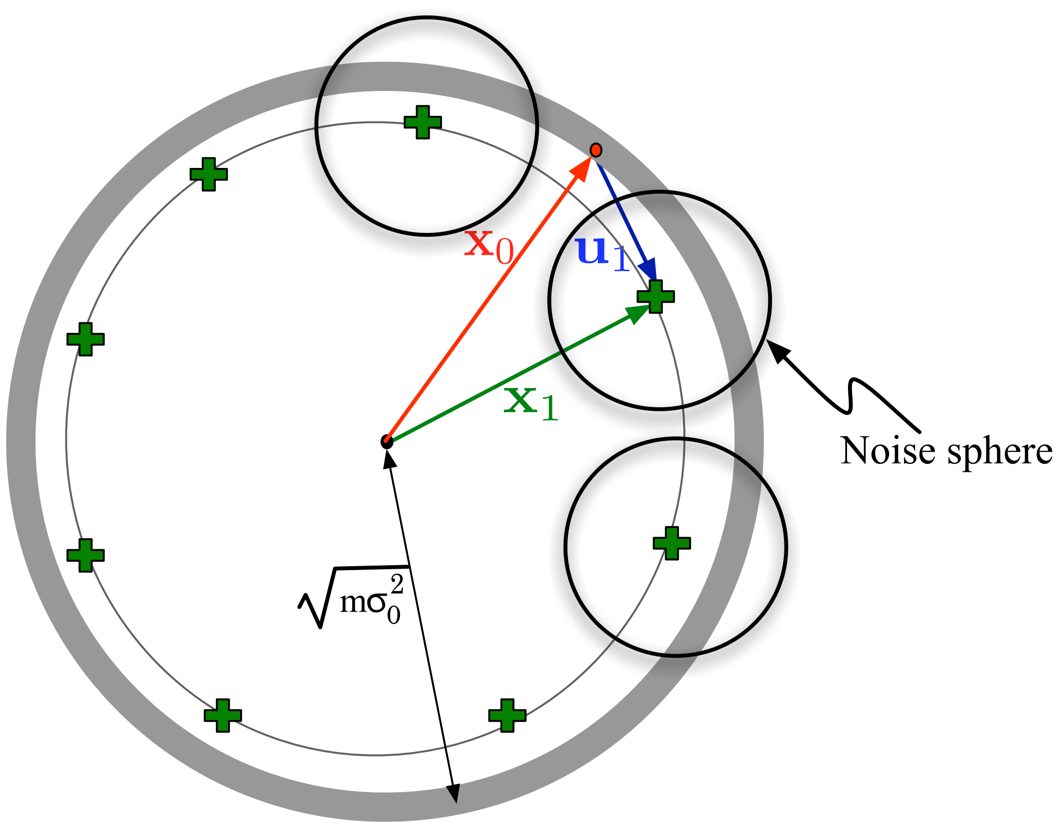
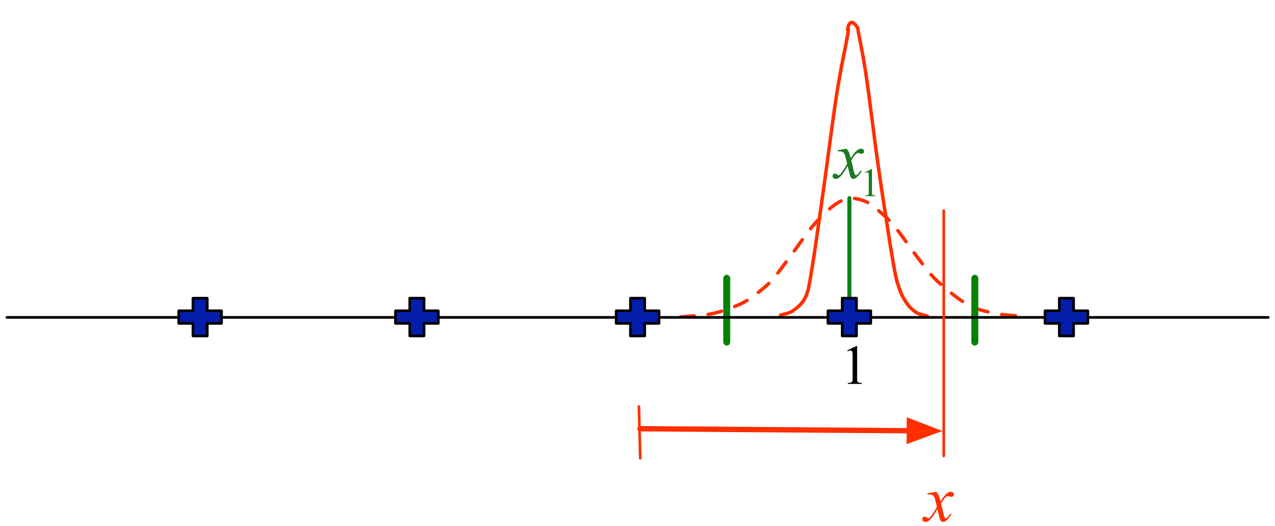
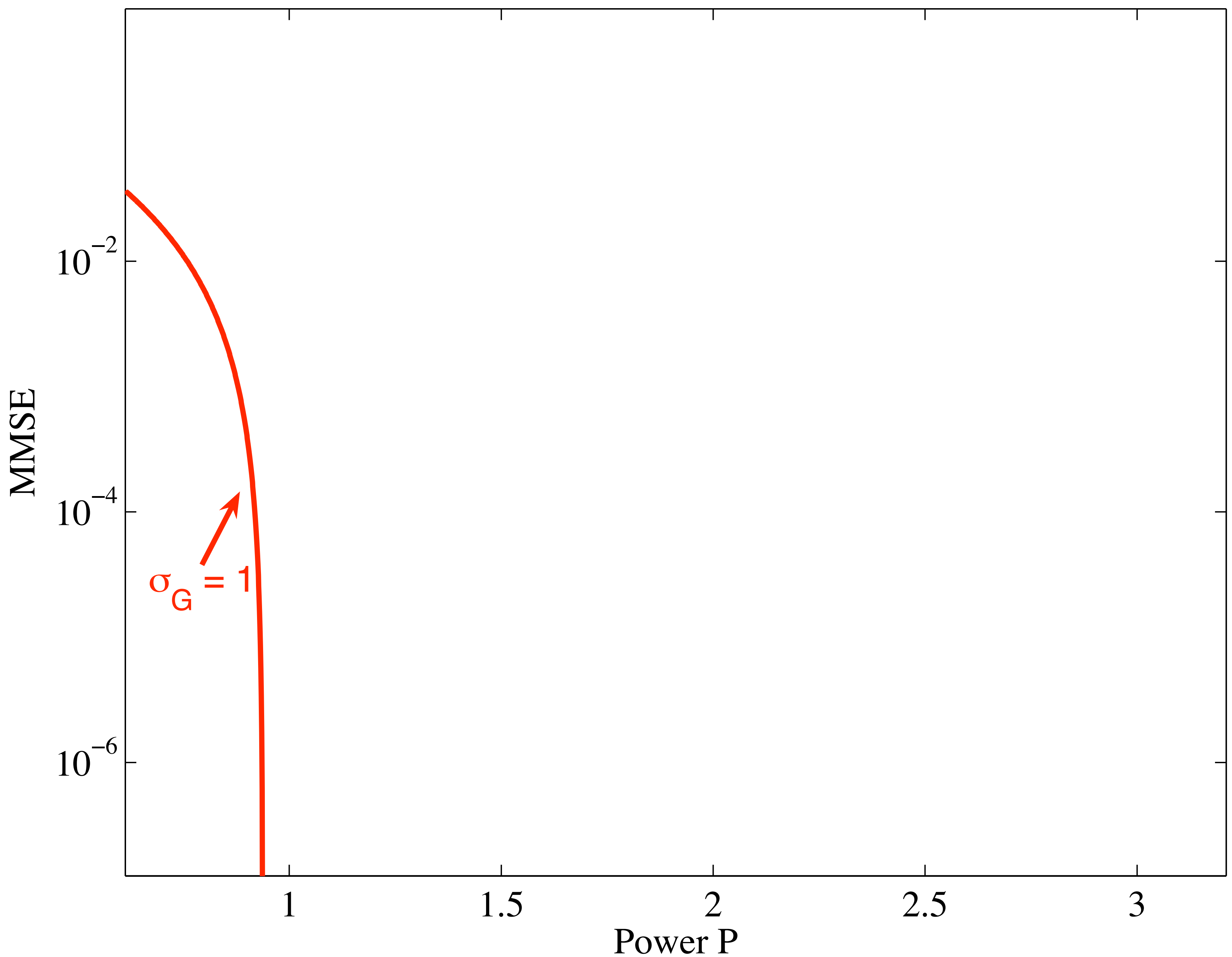




Sphere-packing extension of lower bound



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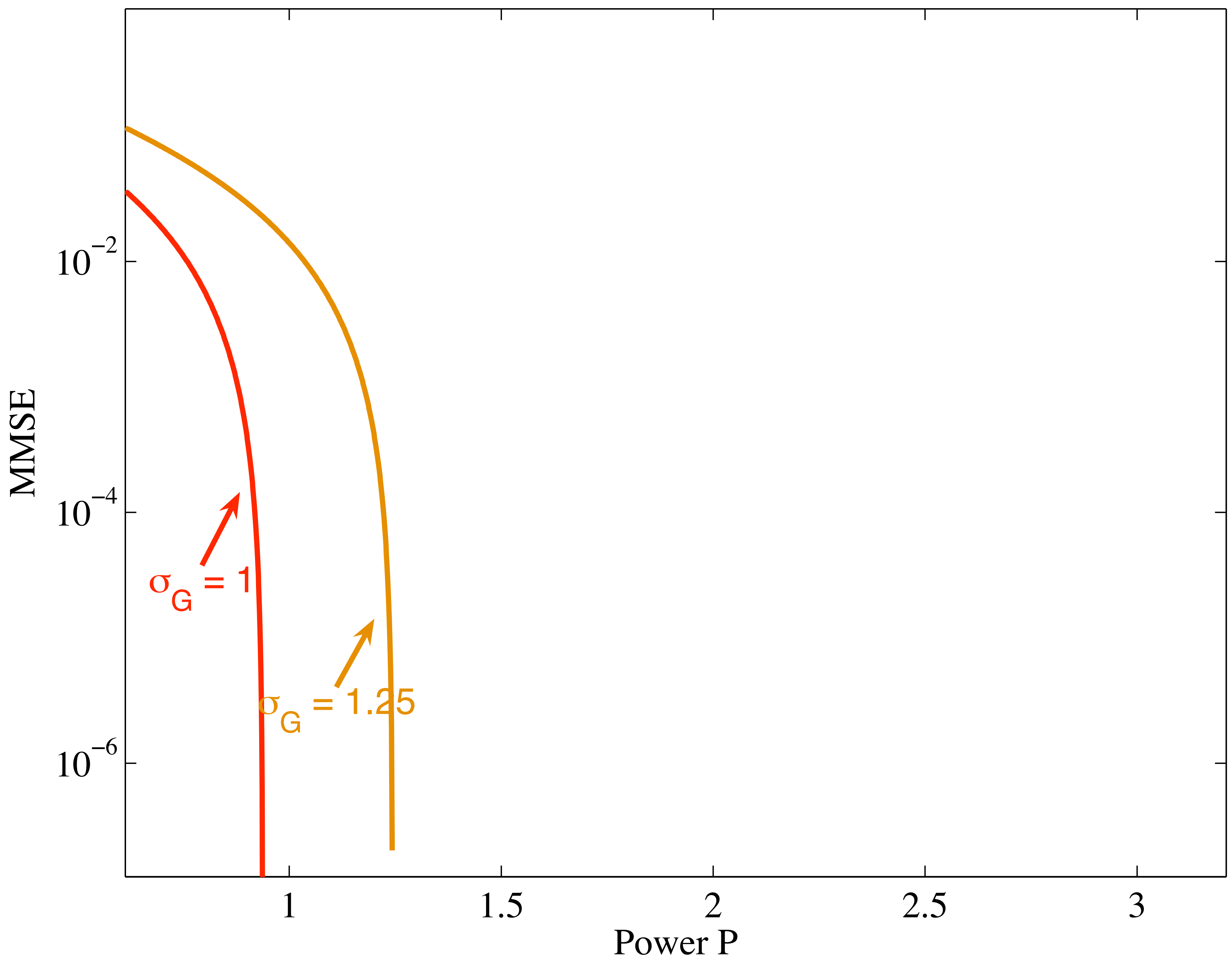

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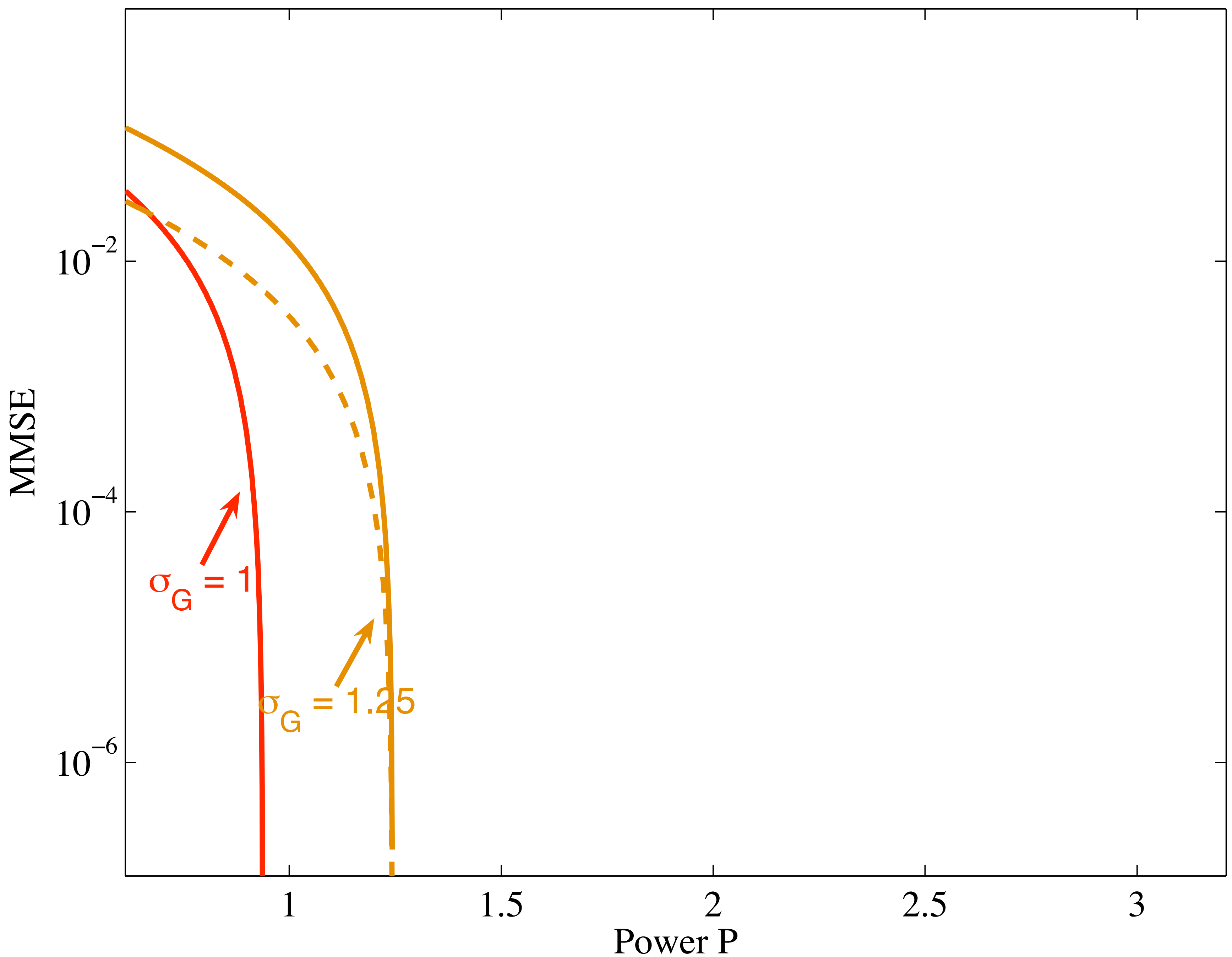

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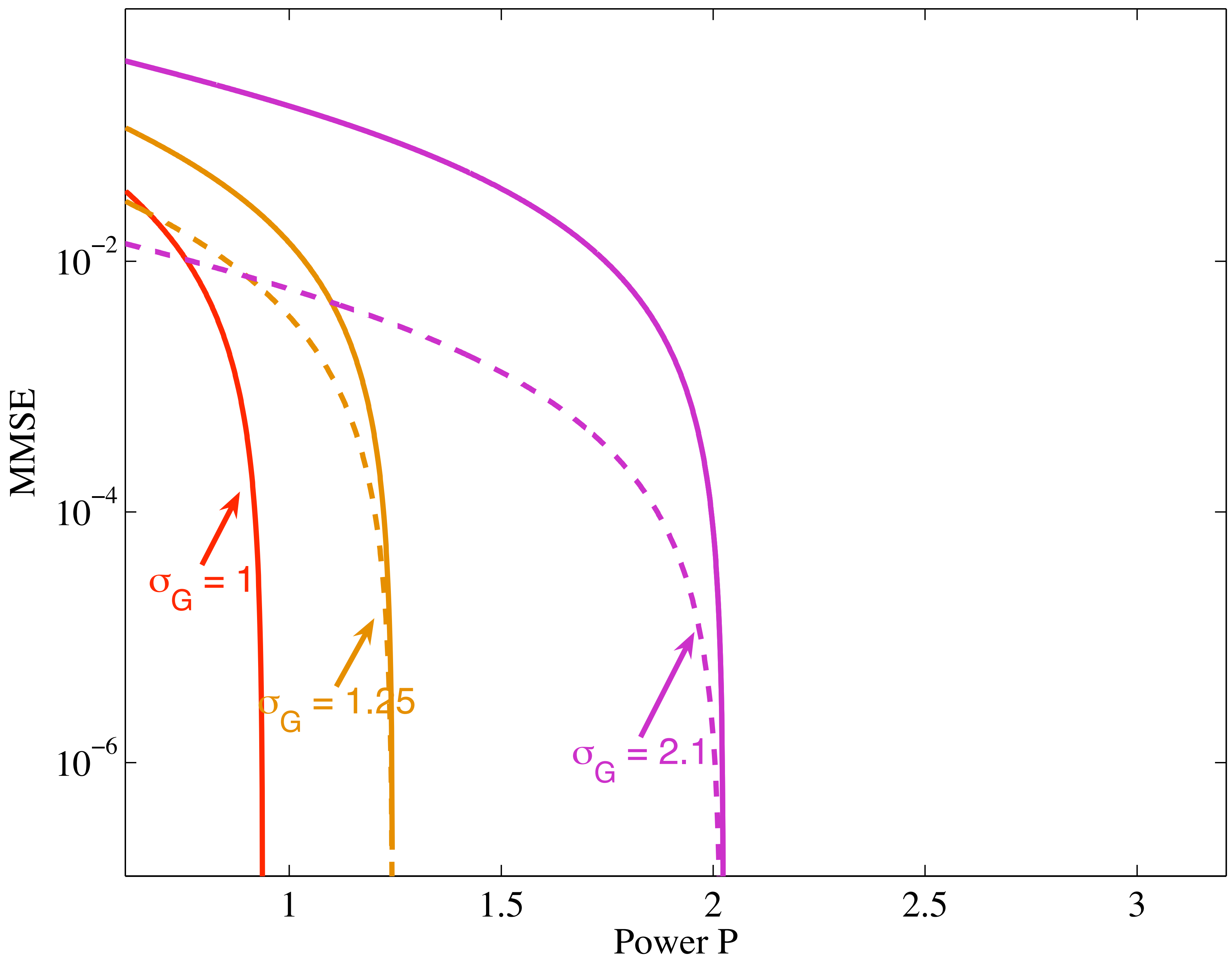

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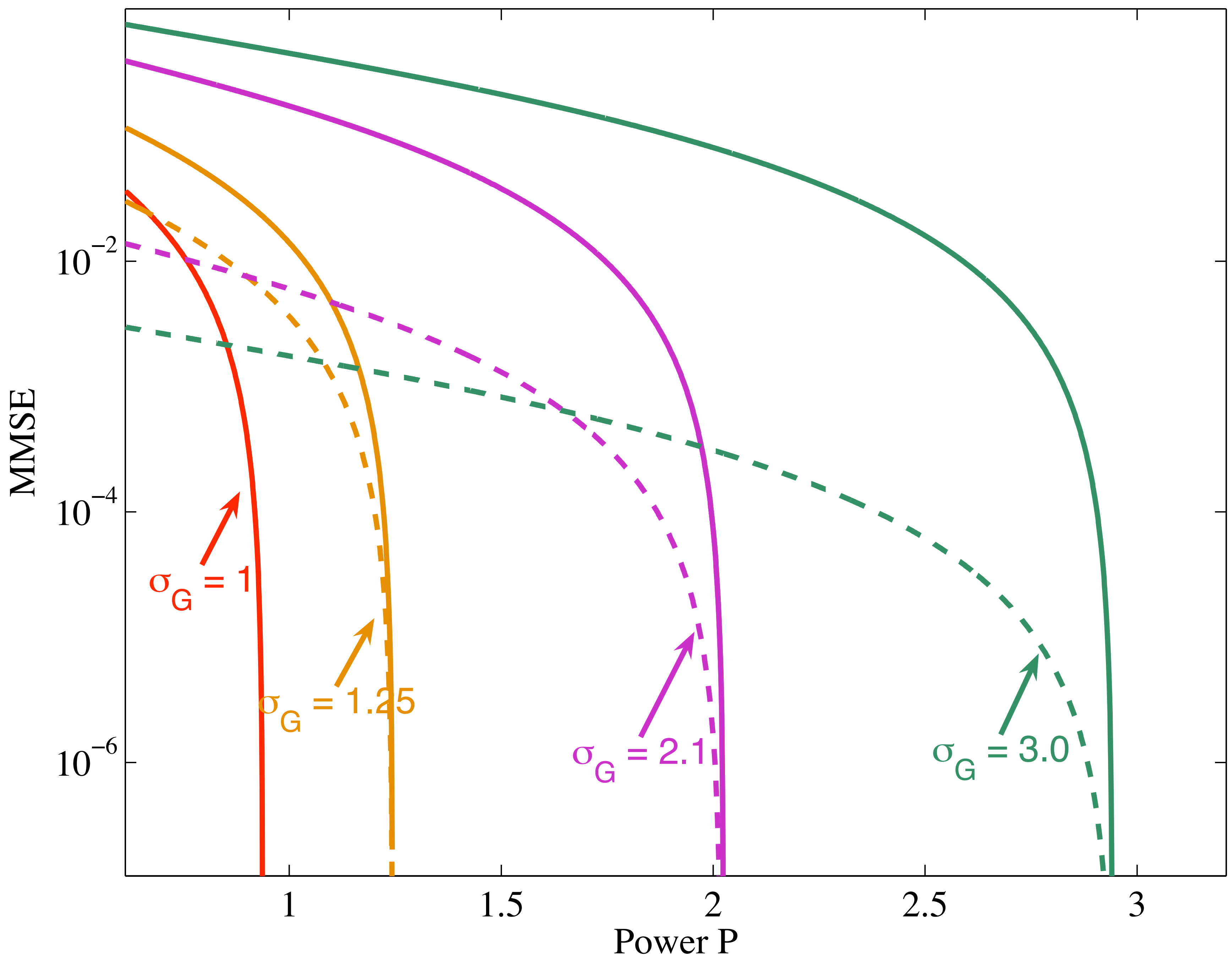

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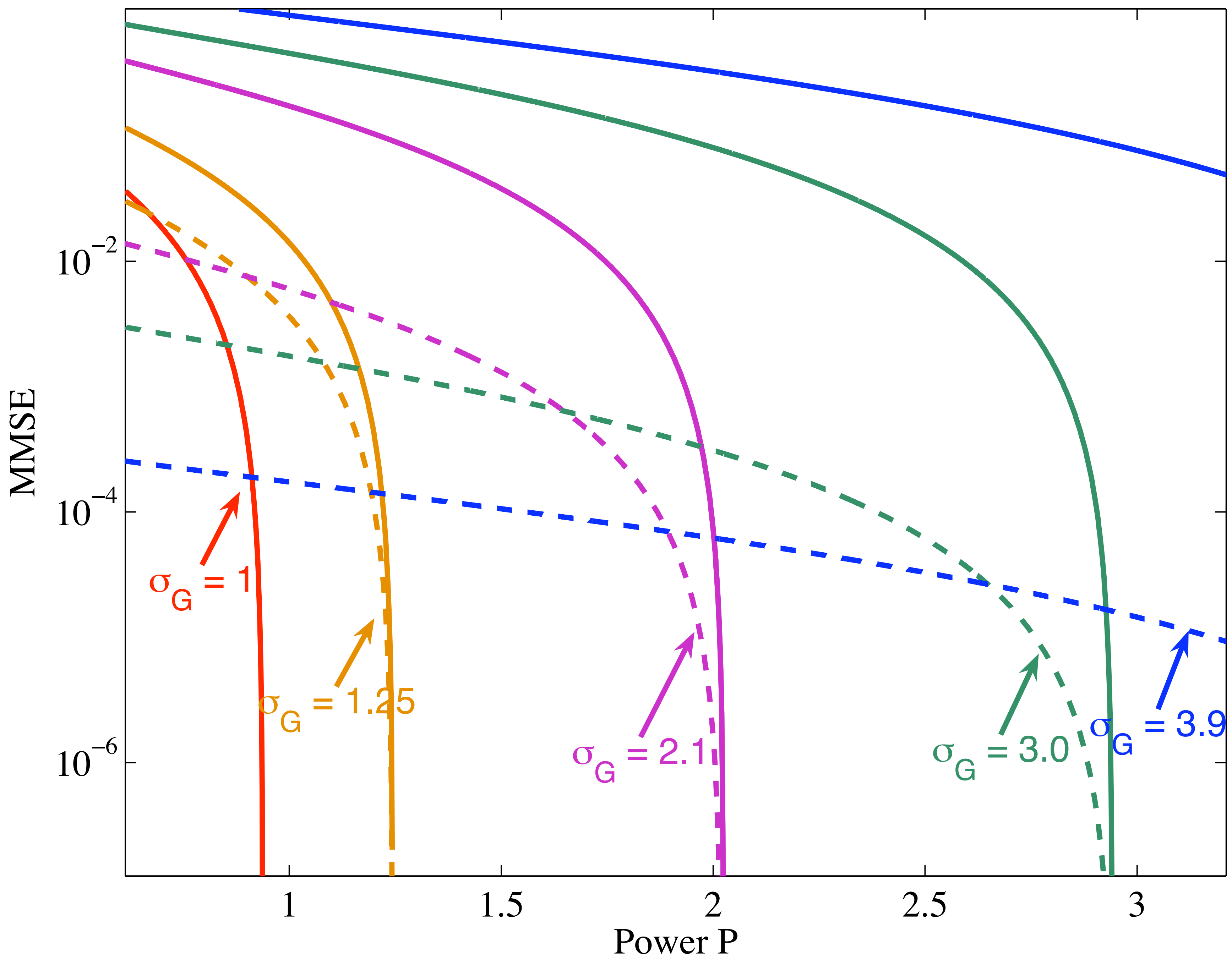

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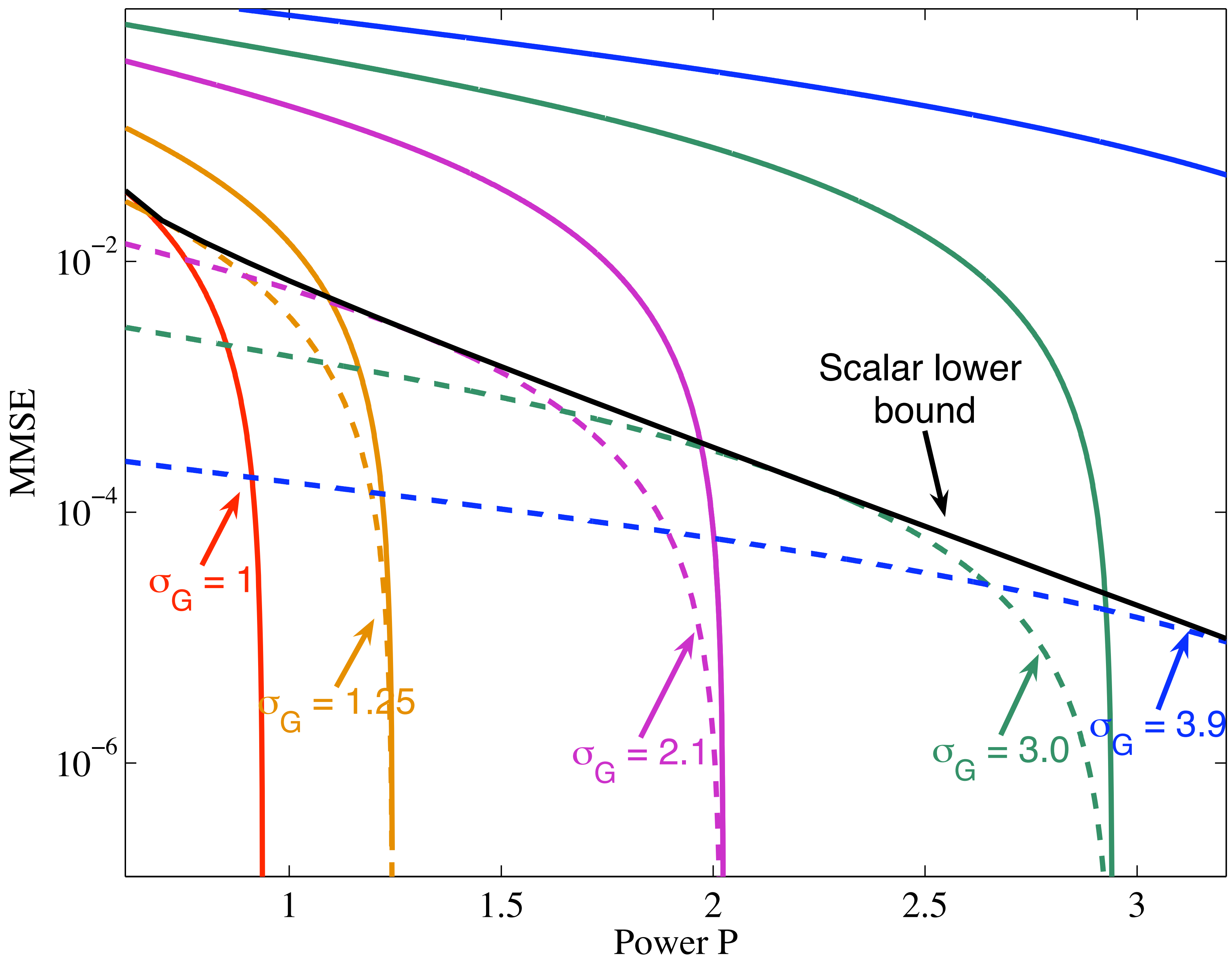
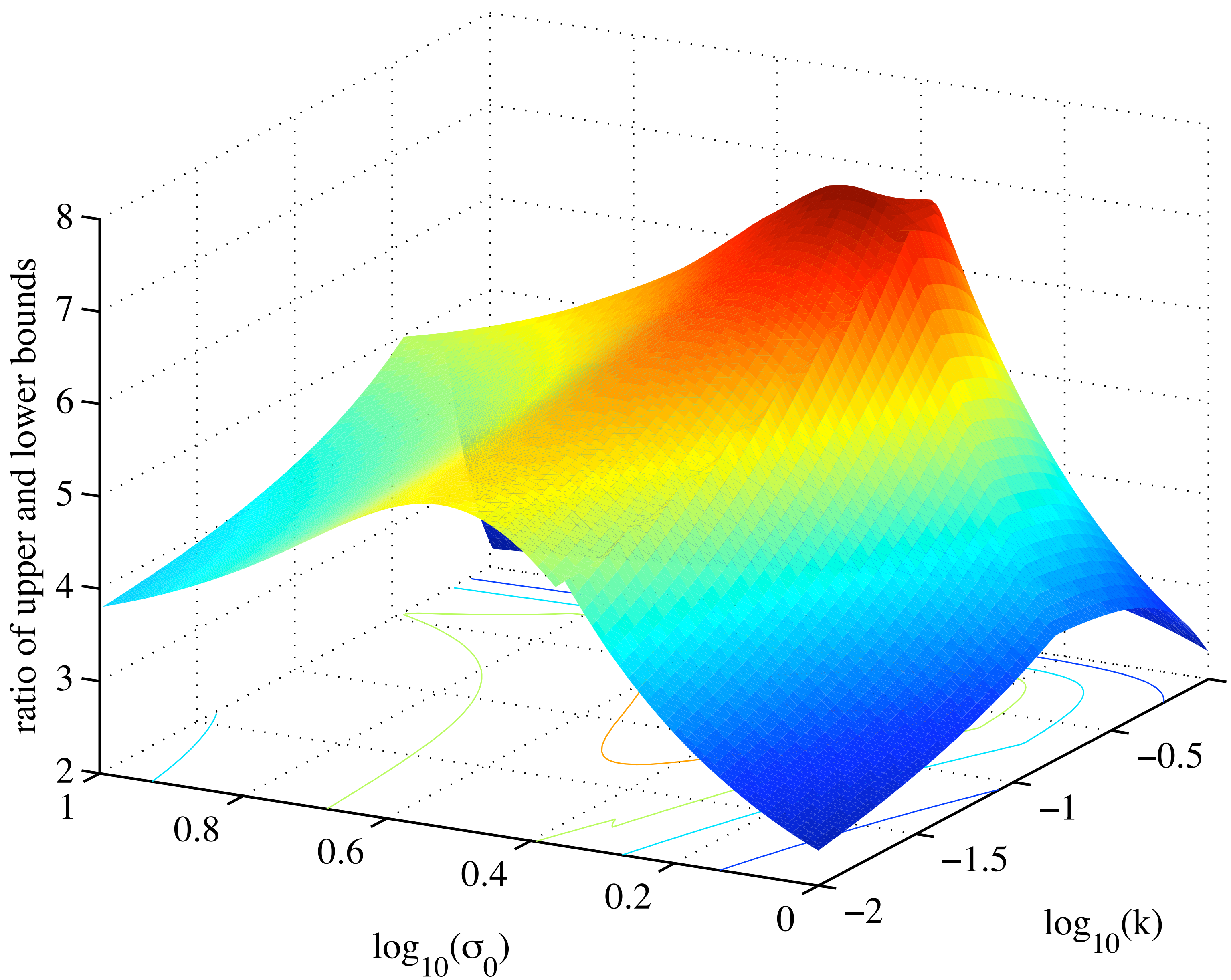
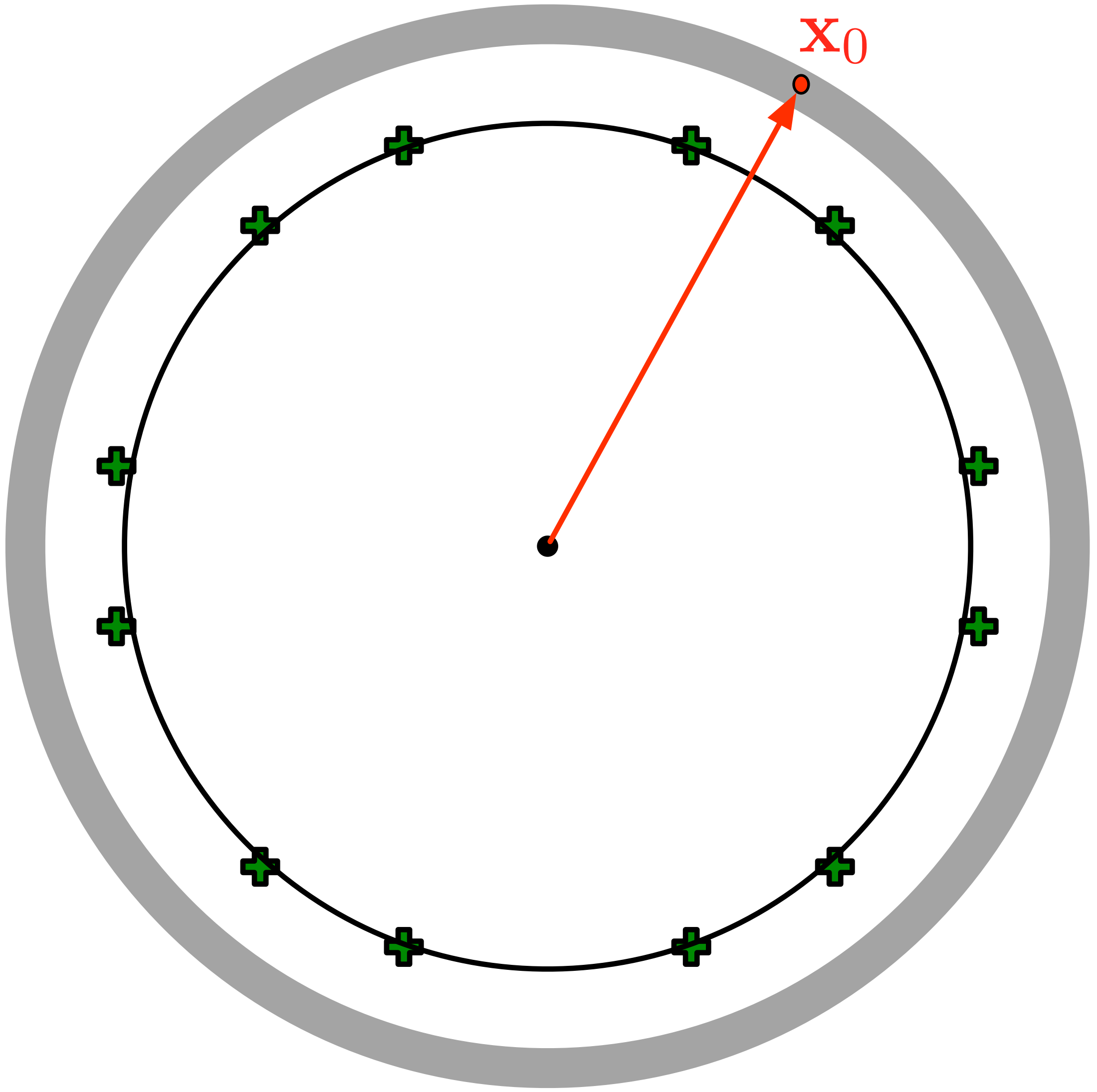


Scalar case : Quantization based strategies are approximately optimal!



Finite vector lengths

“Good” strategies



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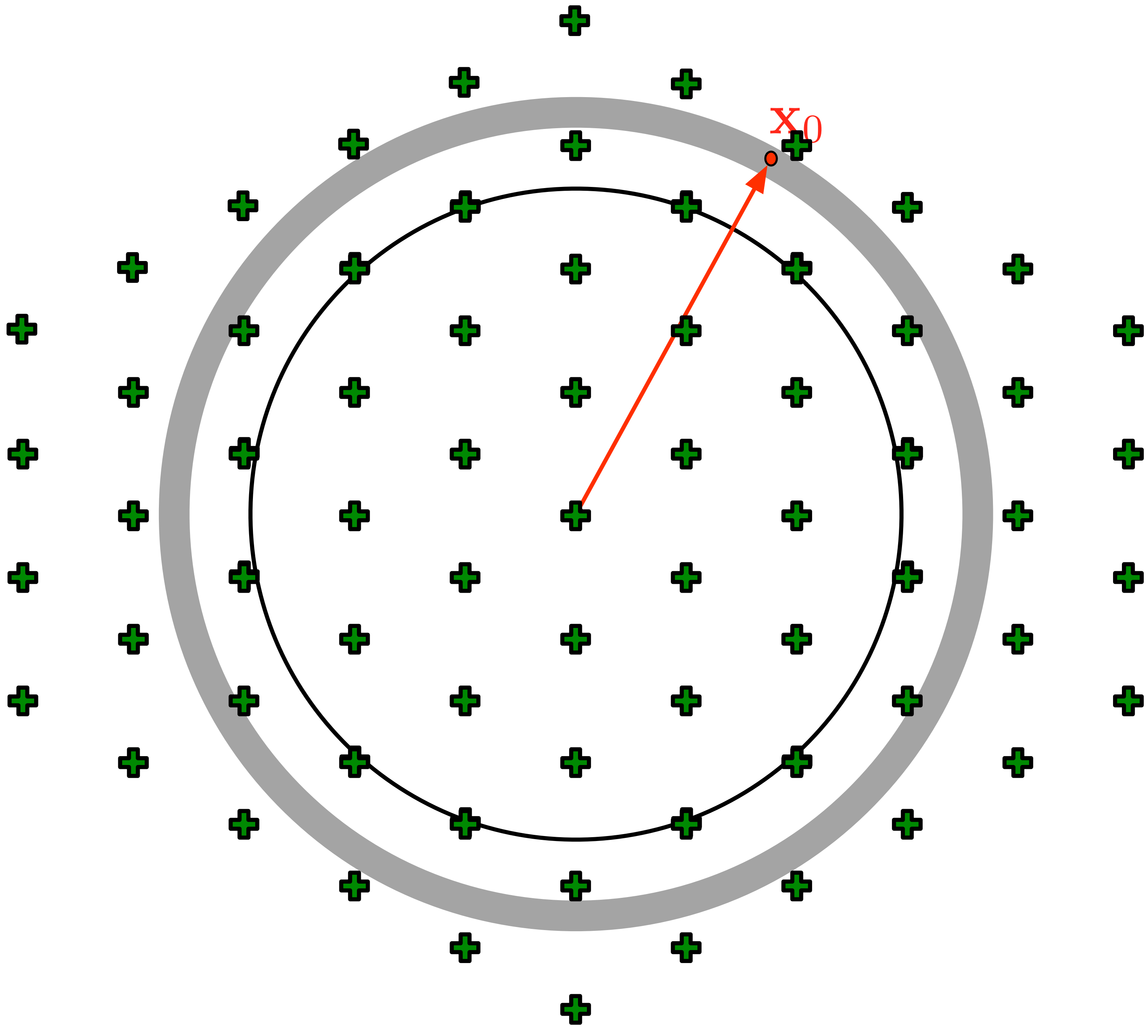

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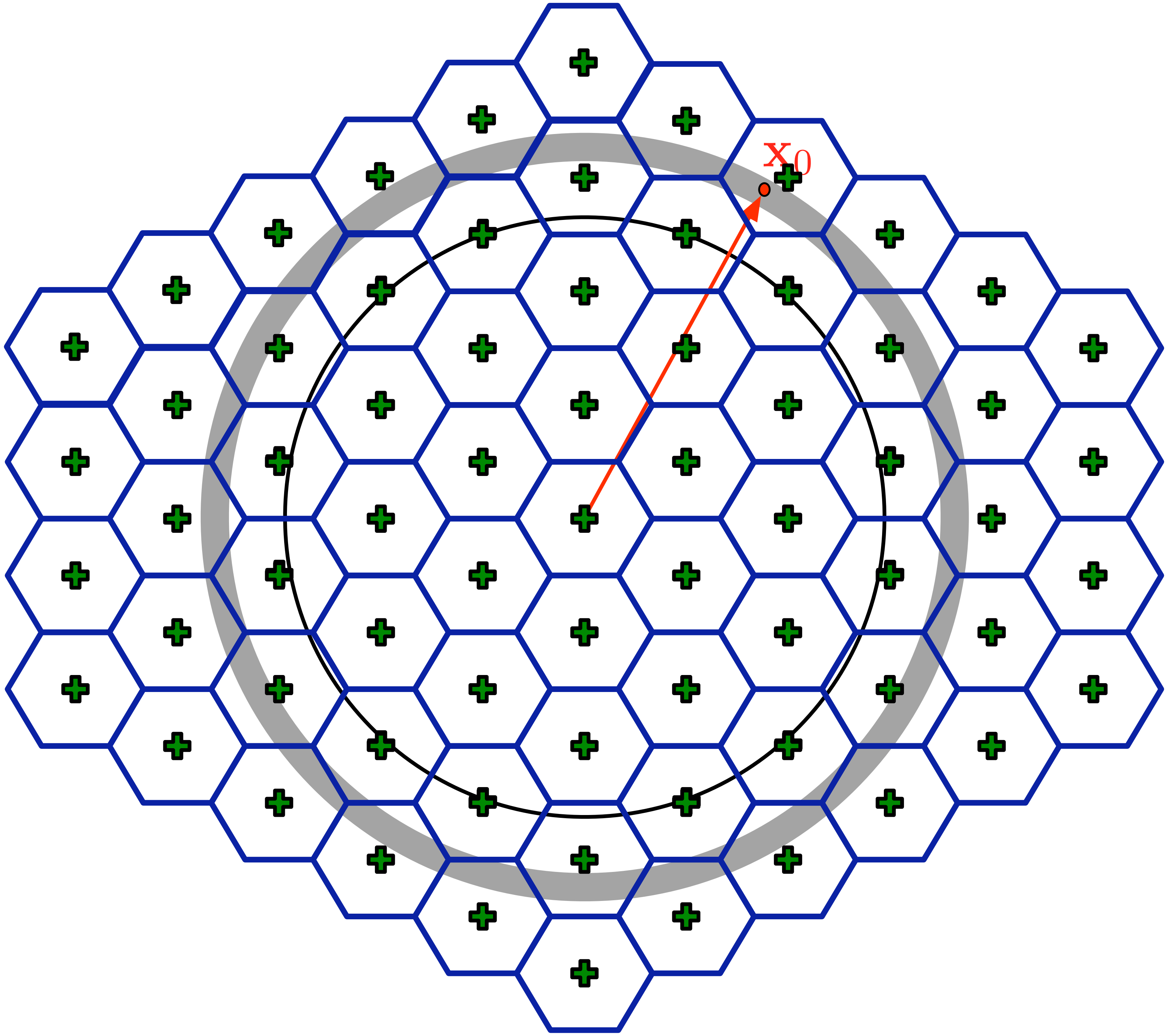
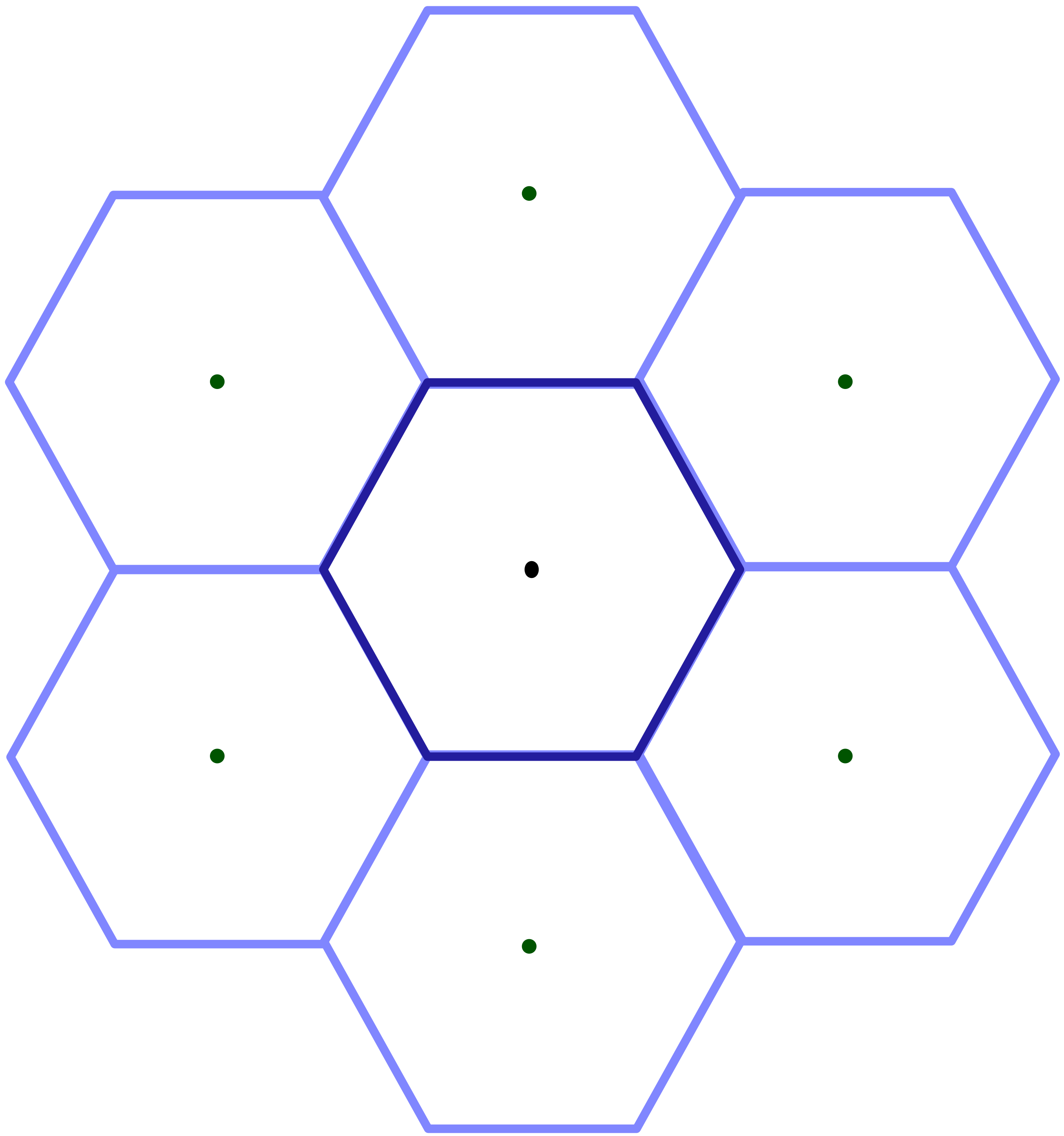


What makes “good” lattices?



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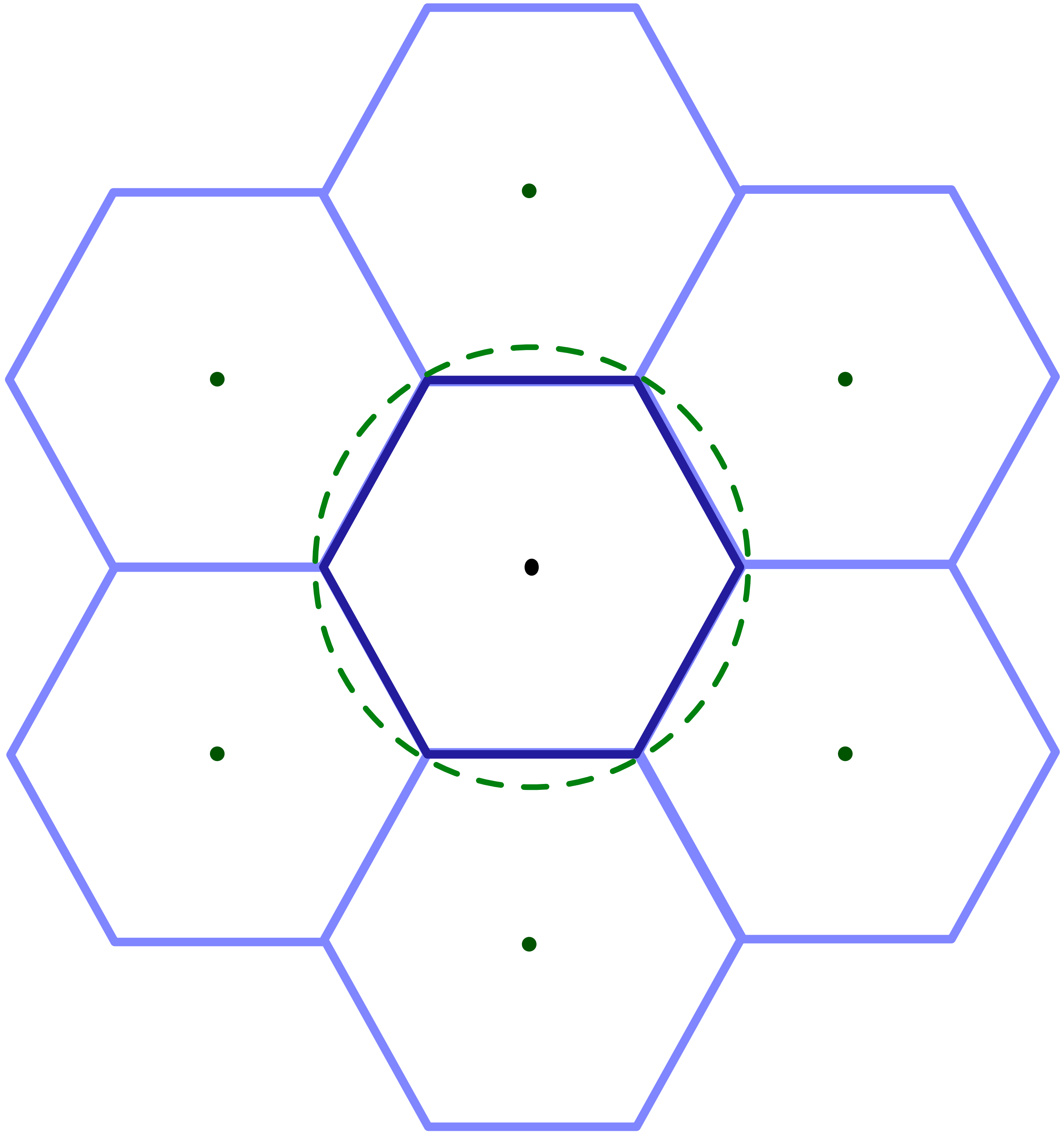

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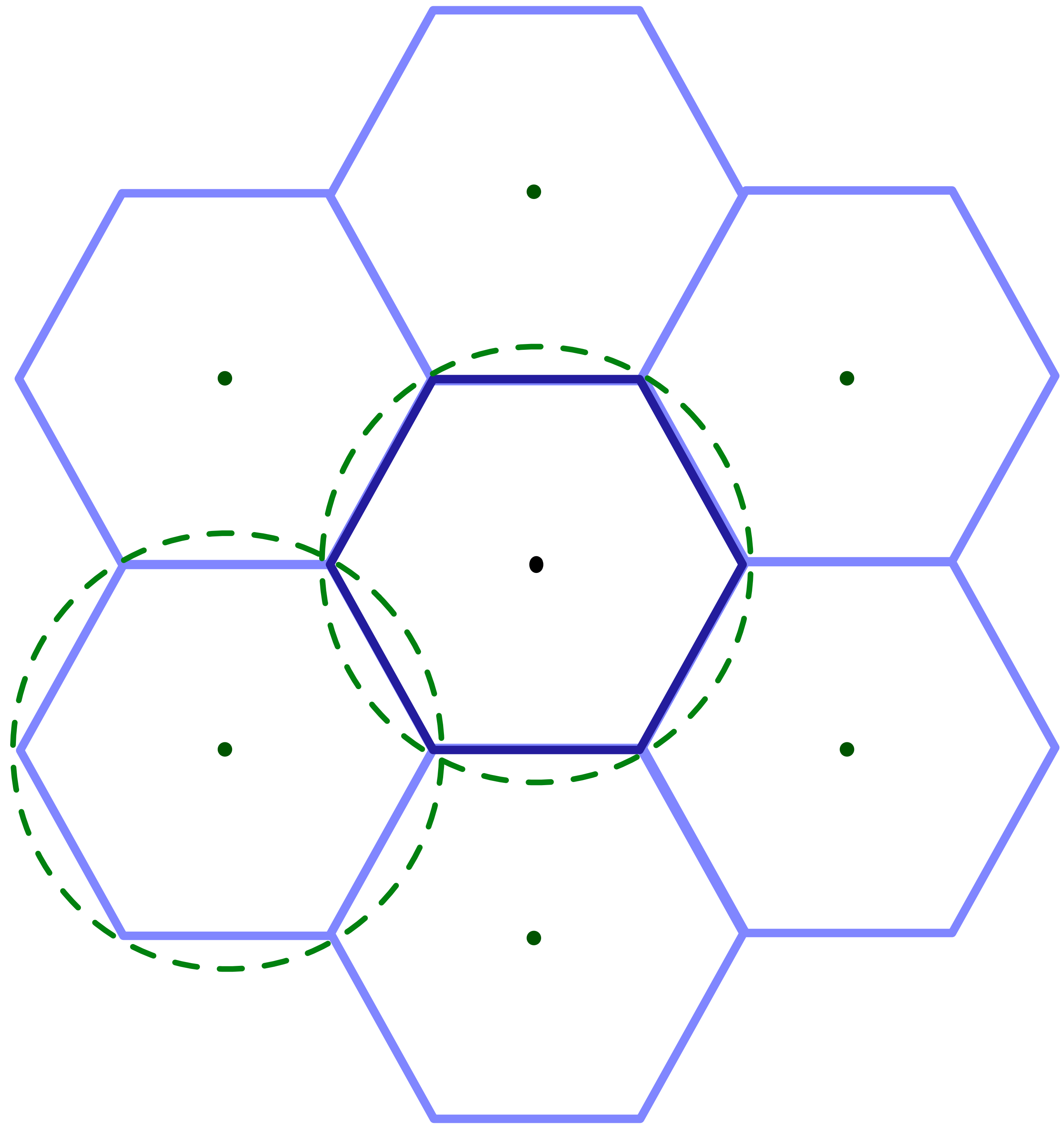

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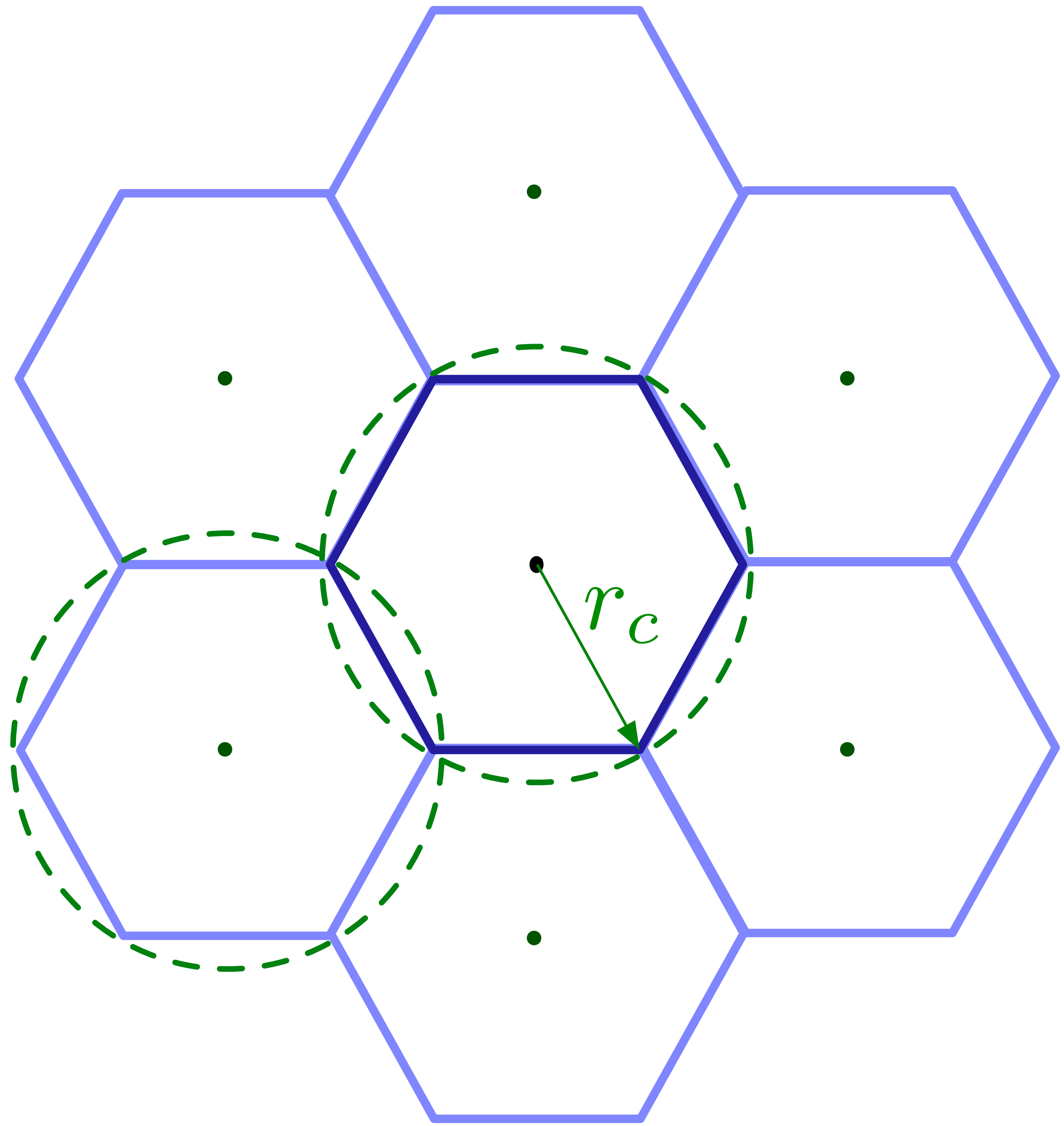

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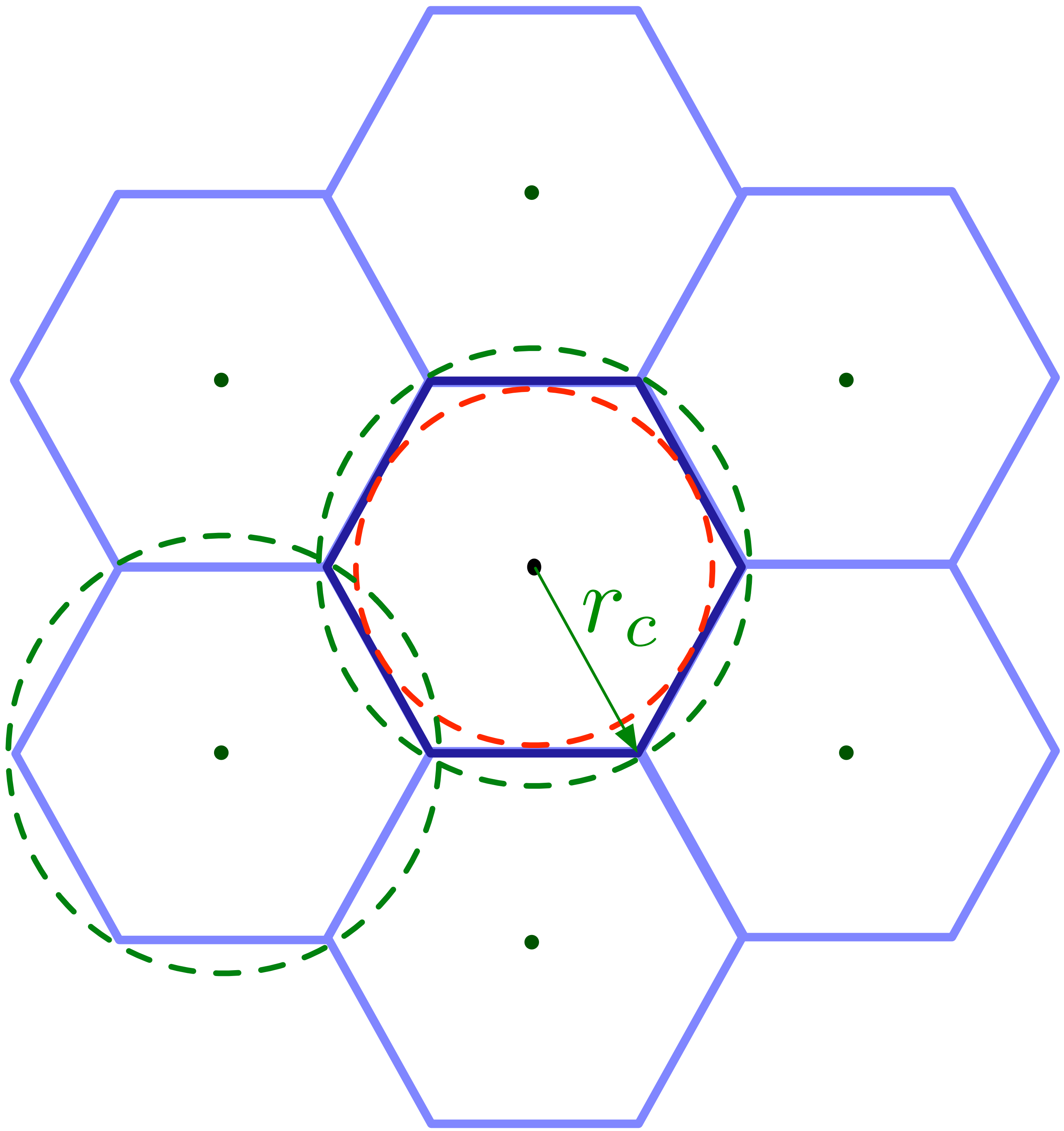

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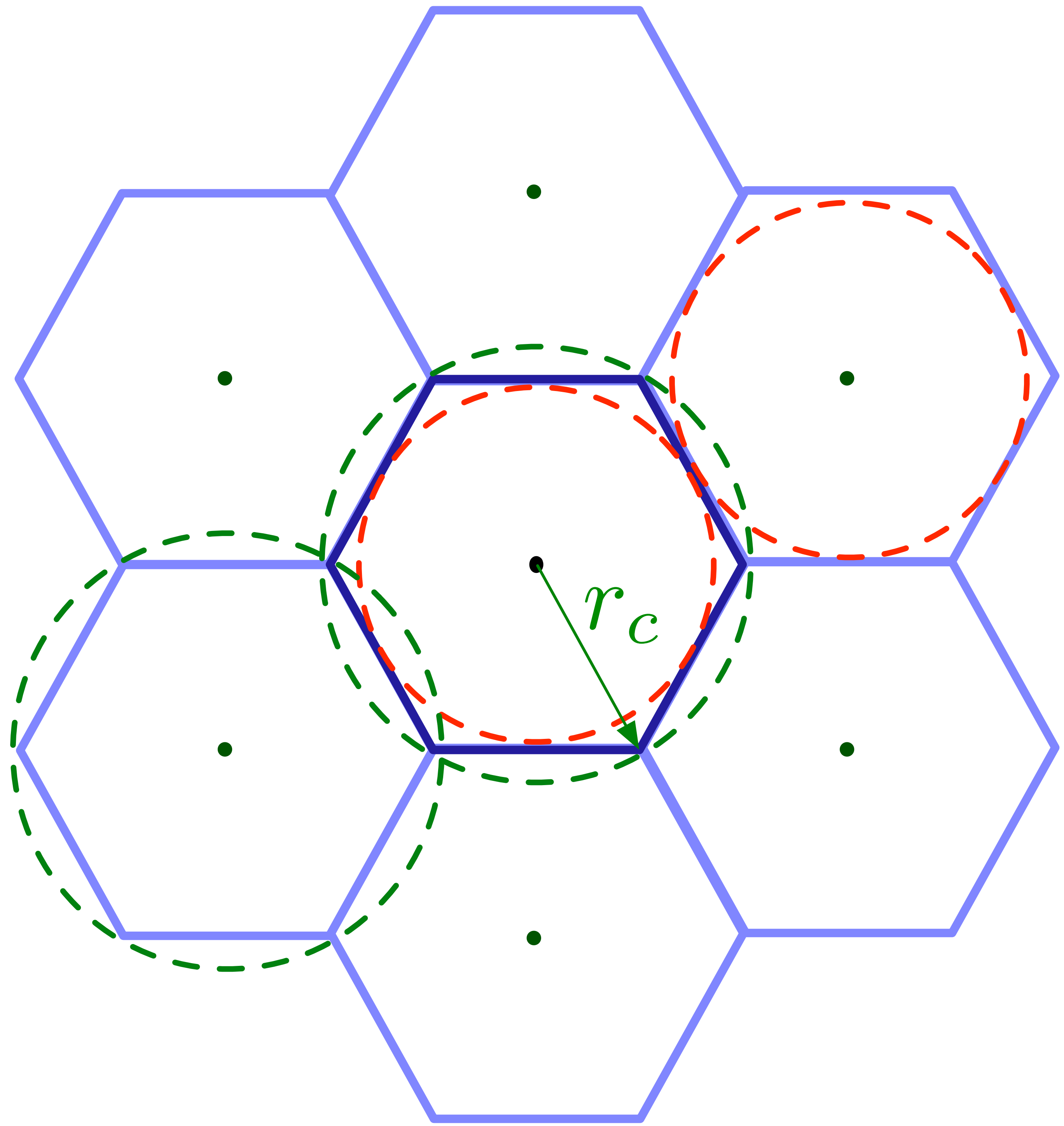

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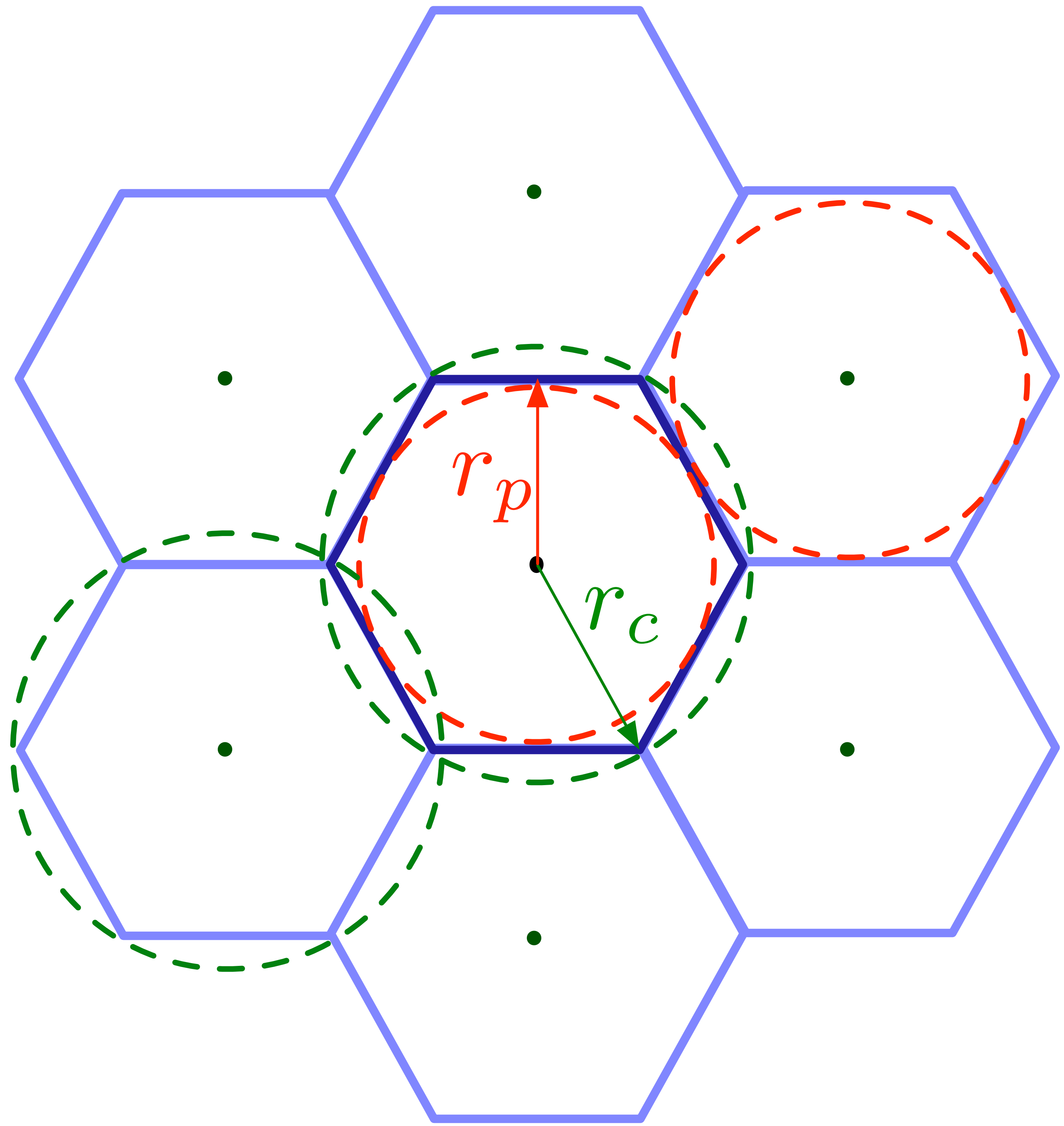

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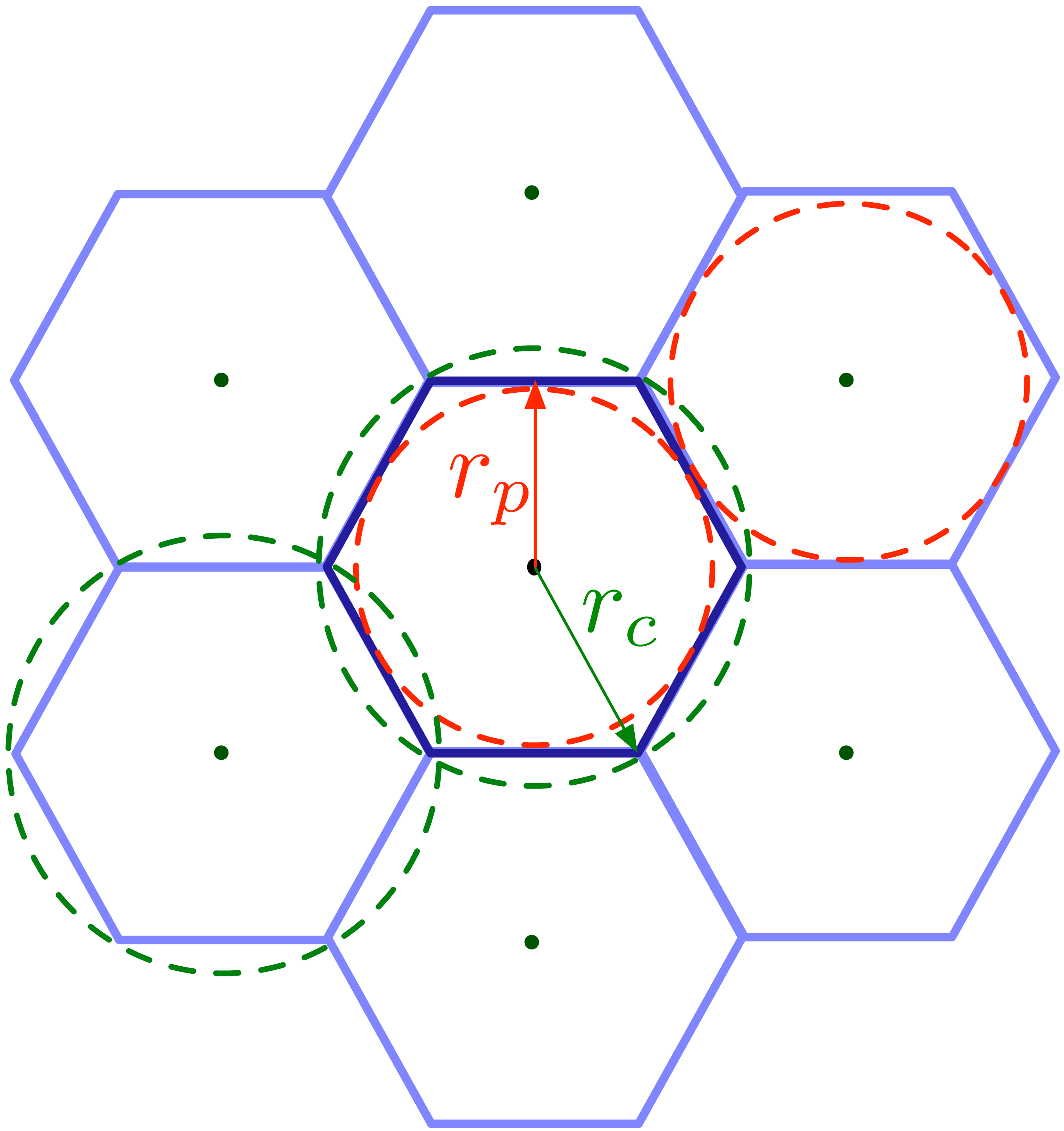
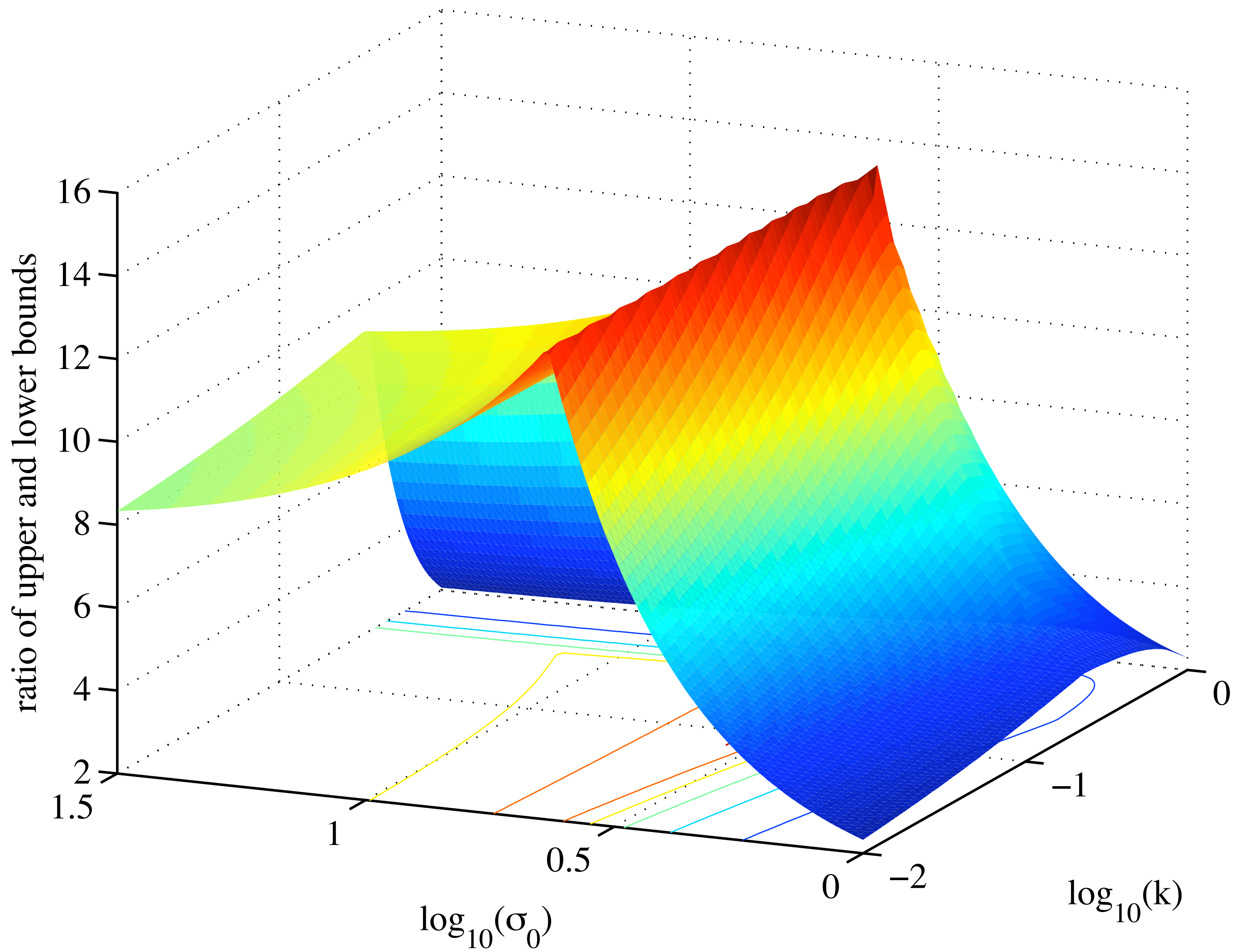


$$\zeta = \frac{r_c}{r_p}$$

2-D case



hexagonal lattice

packing-covering ratio = $\dfrac{2}{\sqrt{3}}$

Lattices are uniformly approximately optimal over dimension size

$$\inf_{P \geq 0} k^2 P + \eta(P, \sigma_0^2) \leq \bar{J} \leq \mu \left(\inf_{P \geq 0} k^2 P + \eta(P, \sigma_0^2) \right)$$

$$\mu \leq 300\zeta^2, \quad \zeta \leq 4$$

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Back-up slides begin